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Strong solution for a high order boundary value problem with integral condition

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Abstract: The present paper is devoted to a proof of the existence and uniqueness of strong solution for a high order boundary value problem with integral condition. The proof is based by a priori estimate and on the density of the range of the operator generated by the studied problem.

Key words: Integral condition, energy inequality, boundary value problem

1. Introduction

In the rectangular domain $Q = (0,1) \times (0,T)$, with $T < \infty$, we consider the differential equation

$$\pounds u = \frac{\partial^4 u}{\partial t^4} + (-1)^\alpha \frac{\partial^\alpha}{\partial x^\alpha}\left(a(x,t)\frac{\partial^\alpha u}{\partial x^\alpha}\right) = f(x,t), \tag{1.1}$$

where $a(x,t)$ satisfy the assumptions

$$0 < a_0 \leq a(x,t) \leq a_1, \tag{1.2}$$

$$c_k^{'} \leq \frac{\partial^k a(x,t)}{\partial x^k} \leq c_k, \quad k = \overline{1,4}, \quad \text{with} \quad c_1^{'} \geq 0, \quad \forall (x,t) \in \overline{Q}, \tag{1.3}$$

subject to the initial conditions

$$u(x,0) = 0, \frac{\partial u(x,0)}{\partial t} = 0, \quad x \in (0,1), \tag{1.4}$$

final conditions

$$\frac{\partial^2 u(x,T)}{\partial t^2} = 0, \frac{\partial^3 u(x,T)}{\partial t^3} = 0, \quad x \in (0,1), \tag{1.5}$$

boundary conditions

$$\frac{\partial^i u(0,t)}{\partial x^i} = 0, \quad \text{for} \quad 0 \leq i \leq \alpha - 1, \quad t \in (0,T), \tag{1.6}$$

$$\frac{\partial^i u(1,t)}{\partial x^i} = 0, \quad \text{for} \quad 0 \leq i \leq \alpha - 2, \quad t \in (0,T), \tag{1.7}$$

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and the integral (nonlocal) condition

$$\int_0^1 u(\xi,t)\, d\xi = 0, \quad t \in (0,T). \tag{1.8}$$

The importance of boundary value problems with integral boundary conditions has been pointed out by Samarski [21]. We remark that integral boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics; see for example [7, 12, 22, 17]. Boundary value problems for parabolic equations with an integral boundary condition are investigated by Batten [1], Bouziani and Benouar [2], Cannon [4, 5], Cannon, et al. [6], Ionkin [15], Kamynin [16], Shi and Shillor [23], Shi [22], Marhoune and Bouzit [19], Denche and Marhoune [8, 9, 10, 11], Yurchuk [24], and many references therein. The problem with an integral one-space-variable condition is studied in Kartynnik [17], and Denche and Marhoune [11]

2. Preliminaries

In this paper, we prove the existence and uniqueness of a strong solution of the problem stated in equation $(1.1)-(1.8)$. The demonstration is based on an a priori estimate and the density of the image of the operator generated by the problem $(1.1)-(1.8)$. This problem can be written in the operator form

$$Lu = F, \tag{2.9}$$

where the operator L is considered from E to F. We consider the domain of definition $D(L)$ such that E is the Banach space consisting of all functions $u \in L^2(Q)$, satisfying equations $(1.1)-(1.8)$, with the finite norm

$$\|u\|_E^2 = \int_Q \frac{(1-x)}{2}\left[\left|\frac{\partial^4 u}{\partial t^4}\right|^2 + \left|\frac{\partial^\alpha}{\partial x^\alpha}\left(a(x,t)\frac{\partial^\alpha u}{\partial x^\alpha}\right)\right|^2 + \left|\frac{\partial^\alpha u}{\partial x^\alpha}\right|^2\right] dxdt, \tag{2.10}$$

and F is the Hilbert space with norm given by

$$\|f\|_F^2 = \int_Q (1-x)^\nu |f|^2\, dxdt, \tag{2.11}$$

where ν is an arbitrary number such that $0 < \nu < 1$. Using the energy inequalities method proposed in [18], we establish an energy inequality

$$\|u\|_E^2 \le C_1 \|Lu\|_F^2 \tag{2.12}$$

and we show that the operator L has the closure $\overline{L}$.

Definition 1 *A solution of the operator equation* $\overline{L}u = F$ *is called a strong solution of the problem* $(1.1)-(1.8)$.

Inequality (2.12) can be extended by

$$\|u\|_E^2 \le C_1 \left\|\overline{L}u\right\|_F^2, \text{ for all } u \in D\left(\overline{L}\right). \tag{2.13}$$

From this inequality, we obtain the uniqueness of a strong solution if it exists, and the equality of sets $R\left(\overline{L}\right)$ and $\overline{R(L)}$. Thus, to prove the existence of a strong solution of the problem in equations (1.1)–(1.8), it remains to prove that the set $R(L)$ is dense in F.

3. An energy inequality and its consequences

Theorem 1 *For any function* $u \in D(L)$ *we have the a priori estimate*

$$\|u\|_E^2 \leq k \|Lu\|_F^2, \tag{3.1}$$

where

$$k = \frac{\exp(cT)\max\left(\left(\frac{2\alpha}{(1-\nu)}\right)^2 + \frac{5}{4}\right)}{\min\left(\frac{1}{4}, \delta\right)} \tag{3.2}$$

and

$$\delta = c_4^{'} - 4cc_3 + 6c^2c_2^{'} - 4c^3c_1 + c^4a_1 > 0, \tag{3.3}$$

with the constant c *satisfying the region*

$$\begin{cases} \sup\left[\frac{1}{a}\frac{\partial a}{\partial t} - \sqrt{\left(\frac{\frac{\partial a}{\partial t}}{a}\right)^2 - \frac{1}{a}\frac{\partial a}{\partial t}}\right] < c < \inf\left[1 + \frac{1}{a}\frac{\partial a}{\partial t} - \sqrt{\left(\frac{\frac{\partial a}{\partial t}}{a}\right)^2 - \frac{1}{a}\frac{\partial a}{\partial t} + \frac{1}{2}}\right], \\ a_0c^3 - c_1c(3c+2) + c_2^{'}(3c+1) - c_3 \geq 0, \\ \delta = c_4^{'} - 4cc_3 + 6c^2c_2^{'} - 4c^3c_1 + c^4a_1 > 0 \end{cases} \tag{3.4}$$

Proof Denote

$$Mu = (1-x)\frac{\partial^4 u}{\partial t^4} + \alpha J\frac{\partial^4 u}{\partial t^4},$$

where

$$Ju = \int_0^x u(\xi, t)\, d\xi.$$

We consider the quadratic formula

$$\mathrm{Re}\int_0^\tau \int_0^1 \exp(-ct)\, \pounds u \overline{Mu} dxdt, \tag{3.5}$$

with the constant c satisfying condition (3.4); obtained by multiplying equation (1.1) by $\exp(-ct)\, \pounds u\overline{Mu}$; and integrating over Q^τ, where $Q^\tau = (0,1)\times(0,\tau)$, with $0 \leq \tau \leq T$, and by taking the real part. Integrating by parts α times in formula (3.5) with the use of boundary conditions in equations $(1.6), (1.7)$, and (1.8), we obtain

$$\mathrm{Re}\int_0^\tau \int_0^1 \exp(-ct)\, \pounds u \overline{Mu} dxdt = \tag{3.6}$$

$$\int_0^\tau \int_0^1 \exp(-ct)(1-x)\left|\frac{\partial^4 u}{\partial t^4}\right|^2 dxdt +$$

$$2\mathrm{Re}\int_0^1 \exp(-c\tau)\left(\frac{\partial^2 a(x,\tau)}{\partial t^2} - 2c\frac{\partial a(x,\tau)}{\partial t} + c^2 a(x,\tau)\right)\frac{(1-x)}{2}\frac{\partial}{\partial t}\left(\frac{\partial^\alpha \overline{u(x,\tau)}}{\partial x^\alpha}\right)\frac{\partial^\alpha u(x,\tau)}{\partial x^\alpha}dx -$$

$$4\int_0^\tau \int_0^1 \exp(-ct)\left(\frac{\partial^2 a}{\partial t^2} - 2c\frac{\partial a}{\partial t} + c^2 a\right)\frac{(1-x)}{2}\left|\frac{\partial}{\partial t}\left(\frac{\partial^\alpha u}{\partial x^\alpha}\right)\right| dxdt -$$

$$\int_0^1 \exp(-c\tau)\left(\frac{\partial^3 a(x,\tau)}{\partial t^3} - 3c\frac{\partial^2 a(x,\tau)}{\partial t^2} + 3c^2\frac{\partial a(x,\tau)}{\partial t} - c^3 a(x,\tau)\right)\frac{(1-x)}{2}\left|\frac{\partial^\alpha u(x,\tau)}{\partial x^\alpha}\right| dx +$$

$$2\int_0^1 \exp(-c\tau)\left(\frac{\partial a(x,\tau)}{\partial t} - ca(x,\tau)\right)\frac{(1-x)}{2}\left|\frac{\partial}{\partial t}\left(\frac{\partial^\alpha u(x,\tau)}{\partial x^\alpha}\right)\right| dx +$$

$$\int_0^\tau \int_0^1 \exp(-ct)\left(\frac{\partial^4 a}{\partial t^4} - 4c\frac{\partial^3 a}{\partial t^3} + 6c^2\frac{\partial^2 a}{\partial t^2} - 4c^3\frac{\partial a}{\partial t} + c^4 a\right)\frac{(1-x)}{2}\left|\frac{\partial^\alpha u}{\partial x^\alpha}\right| dxdt +$$

$$2\int_0^\tau \int_0^1 a\exp(-ct)\frac{(1-x)}{2}\left|\frac{\partial^2}{\partial t^2}\left(\frac{\partial^\alpha u}{\partial x^\alpha}\right)\right| dxdt.$$

By substituting the expression of Mu in formula (3.5), using elementary inequalities and the inequality

$$\int_0^1 \frac{\left|J\frac{\partial^4 u}{\partial t^4}\right|^2}{(1-x)^\nu}dx \le \frac{4}{(1-x)^\nu}\int_0^1 (1-x)\left|\frac{\partial^4 u}{\partial t^4}\right|^2 dx, \quad \text{where } 0<\nu<1, \tag{3.7}$$

yields

$$\begin{aligned}\mathrm{Re}\int_0^\tau \int_0^1 \exp(-ct)\,\pounds u\overline{Mu}dxdt \quad &\le \quad \left(\frac{4\alpha^2}{(1-x)^\nu}+1\right)\int_0^\tau \int_0^1 \exp(-ct)(1-x)^\nu |\pounds u|^2 dxdt \\ &\quad +\frac{1}{2}\int_0^\tau \int_0^1 \exp(-ct)(1-x)\left|\frac{\partial^4 u}{\partial t^4}\right|^2 dxdt. \end{aligned}\tag{3.8}$$

From equation (1.1), we have

$$\begin{aligned}\frac{1}{4}\int_0^\tau \int_0^1 \exp(-ct)\frac{(1-x)}{2}\left|\frac{\partial^\alpha}{\partial x^\alpha}\left(a\frac{\partial^\alpha u}{\partial x^\alpha}\right)\right|^2 dxdt \quad &\le \quad \frac{1}{4}\int_0^\tau \int_0^1 \exp(-ct)(1-x)|\pounds u|^2 dxdt + \\ &\quad \frac{1}{2}\int_0^\tau \int_0^1 \exp(-ct)\frac{(1-x)}{2}\left|\frac{\partial^4 u}{\partial t^4}\right|^2 dxdt.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}&\int_Q \frac{(1-x)}{2}\left[\left|\frac{\partial^4 u}{\partial t^4}\right|^2 + \left|\frac{\partial^\alpha}{\partial x^\alpha}\left(a(x,t)\frac{\partial^\alpha u}{\partial x^\alpha}\right)\right|^2 + \left|\frac{\partial^\alpha u}{\partial x^\alpha}\right|^2\right]dxdt \\ \le \quad &\frac{\exp(cT)\max\left(\left(\frac{2\alpha}{(1-\nu)}\right)^2+\frac{5}{4}\right)}{\min\left(\frac{1}{4},\delta\right)}\int_Q (1-x)^\nu |f|^2 dxdt.\end{aligned}\tag{3.9}$$

□

Lemma 1 *The operator L from E to F admits a closure.*

Proof Suppose that $(u_n) \in D(L)$ is a sequence such that

$$u_n \longrightarrow 0 \text{ in } E, \tag{3.10}$$

and

$$Lu_n \longrightarrow f \text{ in } F, \tag{3.11}$$

We must show that $f = 0$.

Introducing the operator

$$\pounds_0 v = \frac{\partial^4 v}{\partial t^4} + (-1)^\alpha \frac{\partial^\alpha}{\partial x^\alpha}\left(a(x,t)\frac{\partial^\alpha v}{\partial x^\alpha}\right), \tag{3.12}$$

defined on the domain $D(\pounds_0)$ of function $v \in L^2(Q)$ verifying

$$\begin{aligned} v(x,0) &= \frac{\partial v(x,0)}{\partial t} = \frac{\partial^2 v(x,T)}{\partial t} = \frac{\partial^3 v(x,T)}{\partial t} = 0, \\ \frac{\partial^i v(0,t)}{\partial x^i} &= 0, \quad \text{for } 0 \le i \le \alpha - 1, \\ \frac{\partial^i v(1,t)}{\partial x^i} &= 0, \quad \text{for } 0 \le i \le \alpha - 2, \end{aligned} \tag{3.13}$$

we note that $D(\pounds_0)$ is dense in the Hilbert space obtained from the completion of $L^2(Q)$ with respect to the norm

$$\|f\|_F^2 = \int_Q (1-x)^\nu |f|^2 \, dxdt. \tag{3.14}$$

Additionally, since

$$\int_Q (1-x)^\nu f\overline{v} dxdt = \lim_{n\to\infty} \int_Q \pounds u_n \left[(1-x)^\nu \overline{v}\right] dxdt = \lim_{n\to\infty} \int_Q u_n \pounds_0 \left[(1-x)^\nu \overline{v}\right] dxdt = 0, \tag{3.15}$$

this holds for every function $v \in D(\pounds_0)$, and yields $f = 0$.

Theorem 2 *The priori estimate in Theorem 1 can be extended to include all functions* u*, i.e.*

$$\|u\|_E^2 \le k \left\|\overline{L}u\right\|_F^2, \forall u \in D\left(\overline{L}\right), \tag{3.16}$$

Hence we obtain the following corollary.

Corollary 1 *A strong solution of the problem in equations* (1.1) – (1.8) *is unique if it exists, and depends continuously on* f*.*

Corollary 2 *The range* $R(L)$ *of the operator* $\overline{L}$ *is closed in* F*, and* $R\left(\overline{L}\right) = \overline{R(L)}$*.*

□

4. Solvability of the problem

To prove the solvability of problem in equations $(1.1)-(1.8)$, it is sufficient to show that $R(L)$ is dense in F. The proof is based on the following lemma.

Lemma 2 *For all* $\omega \in L^2(Q)$,

$$\int_Q (1-x)\, \pounds u \cdot \overline{\omega} dx dt = 0, \tag{4.1}$$

then $\omega = 0$.

Proof Equality (4.1) can be written as

$$-\int_Q \frac{\partial^4 u}{\partial t^4}(1-x)\overline{\omega} dx dt = (-1)^\alpha \int_Q \frac{\partial^\alpha}{\partial x^\alpha}\left(a(x,t)\frac{\partial^\alpha u}{\partial x^\alpha}\right)(1-x)\overline{\omega} dx dt \tag{4.2}$$

If we introduce the smoothing operators with respect to t [24, 20, 14, 3], $J_\xi^{-1} = \left(I + \xi \frac{\partial}{\partial t}\right)^{-1}$ and $\left(J_\xi^{-1}\right)^*$, then these operators provide the solutions of the respective problems

$$\begin{aligned} \xi \frac{dg_\xi(t)}{dt} + g_\xi(t) &= g(t), \\ g(t)|_{t=0} &= 0, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} -\xi \frac{dg_\xi^*(t)}{dt} + g_\xi^*(t) &= g(t), \\ g(t)|_{t=T} &= 0. \end{aligned} \tag{4.4}$$

The operators also have the following properties: for any $g \in L_2(0,T)$, the function $g_\xi = \left(J_\xi^{-1}\right) g$ and $g_\xi^* = \left(J_\xi^{-1}\right)^* g$ are in $W_2^1(0,T)$ such that $g_\xi|_{t=0} = 0.$ and $g_\xi^*\Big|_{t=T} = 0.$ Moreover, J_ξ^{-1} commutes with $\frac{\partial}{\partial t}$, so $\int_0^T |g_\xi - g|^2 dt \longrightarrow 0$ and $\int_0^T \left|g_\xi^* - g\right|^2 dt \longrightarrow 0$ for $\xi \longrightarrow 0$.

Now, for given $\omega(x,t)$, we introduce the function

$$v(x,t) = -\alpha(1-x)^{\alpha-1}\int_0^x \frac{\omega}{(1-\xi)^\alpha} d\xi + \omega(x,t).$$

Integrating by parts, we obtain

$$(1-x)v + \alpha J v = (1-x)\omega, \text{ and } \int_0^x v(x,t)\, dx = 0. \tag{4.5}$$

Then from equality (4.2), we have

$$-\int_Q \frac{\partial^4 u}{\partial t^4} N \overline{v} dx dt = \int_Q A(t)\, u \overline{v} dx dt, \tag{4.6}$$

where $Nv = (1-x)\,v + \alpha Jv,$ and $A(t)\,u = (-1)^{\alpha}\,\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(a(x,t)\,\frac{\partial^{\alpha} u}{\partial x^{\alpha}}\right).$

Putting $u = \int_0^t \int_0^{\eta} \int_{\delta}^{h} \int_{\xi}^{T} \exp(c\tau)\, v_{\xi}^{*}(\tau)\, d\tau d\xi d\eta dh$ in (4.6), and using (4.4), we obtain

$$-\int_Q \exp(ct)\, v_{\xi}^{*}\overline{Nv}dxdt = \int_Q A(u)\, uv_{\xi}^{*}dxdt - \xi \int_Q A(t)\, u\frac{\partial^4 v_{\xi}^{*}}{\partial t^4}dxdt. \tag{4.7}$$

Integrating by parts each term in the right-hand side of (4.7) and taking the real parts, we have

$$\operatorname{Re}\left(\int_Q A(u)\, uv_{\xi}^{*}dxdt\right) \geq 0, \tag{4.8}$$

$$\operatorname{Re}\left(-\xi \int_Q A(t)\, u\frac{\partial^4 v_{\xi}^{*}}{\partial t^4}dxdt\right) \geq -\xi M, \tag{4.9}$$

where

$$\begin{aligned} M \;=\;& 16\int_Q \frac{(1-x)}{2}\left|\frac{\partial^4 v_{\xi}^{*}}{\partial t^4}\right|^2 dxdt + \int_Q \frac{(1-x)}{2}\left(\frac{\partial^4 a}{\partial t^4}\right)^2 \left|\frac{\partial^{\alpha} u}{\partial t^{\alpha}}\right|^2 dxdt + \\ & 4\int_Q \frac{(1-x)}{2}\left(\frac{\partial^3 a}{\partial t^3}\right)^2 \left|\frac{\partial^{\alpha+1} u}{\partial t^{\alpha+1}}\right|^2 dxdt + \\ & 6\int_Q \frac{(1-x)}{2}\left(\frac{\partial^2 a}{\partial t^2}\right)^2 \left|\frac{\partial^{\alpha+2} u}{\partial t^{\alpha+2}}\right|^2 dxdt + \\ & 4\int_Q \frac{(1-x)}{2}\left(\frac{\partial a}{\partial t}\right)^2 \left|\frac{\partial^{\alpha+3} u}{\partial t^{\alpha+3}}\right|^2 dxdt + \int_Q \frac{(1-x)}{2}a^2 \left|\frac{\partial^{\alpha+4} u}{\partial t^{\alpha+4}}\right|^2 dxdt. \end{aligned} \tag{4.10}$$

Now, using inequalities (4.8) and (4.9) in equation (4.7), we have

$$\operatorname{Re}\left(\int_Q \exp(ct)\, v_{\xi}^{*}\overline{Nv}dxdt\right) \leq 0, \tag{4.11}$$

then for $\xi \longrightarrow 0,$ we obtain

$$\operatorname{Re}\left(\int_Q \exp(ct)\, v\overline{Nv}dxdt\right) \leq 0. \tag{4.12}$$

We conclude that $v = 0,$ hence, $\omega = 0$, which ends the proof of the lemma.

Theorem 3 *The range $R\left(\overline{L}\right)$ of $\overline{L}$ coincides with F.*

□

Proof Since F is a Hilbert space, we have $R\left(\overline{L}\right) = F$ if and only if the relation

$$\int_Q (1-x)^{\nu}\, \pounds u \cdot \overline{f}dxdt = 0 \tag{4.13}$$

for arbitrary function $u \in E$ and $f \in F$, implies that $f = 0$.

Putting $u \in D(L)$ in relation (4.13), taking $\omega = \frac{f}{(1-x)^{\nu-1}}$, and using lemma 7, we obtain $\omega = \frac{f}{(1-x)^{\nu-1}} = 0$, then $f = 0$. □

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