Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2013) 37: 308 – 318
© TÜBİTAK
doi:10.3906/mat-1010-438

On the maximal operators of Vilenkin-Fejér means

George TEPHNADZE*
Institute of Mathematics, Faculty of Exact and Natural Sciences, Tbilisi State University,
Chavchavadze str. 1, Tbilisi 0128, Georgia

Received: 25.10.2010 • **Accepted:** 08.03.2011 • **Published Online:** 19.03.2013 • **Printed:** 22.04.2013

Abstract: The main aim of this paper is to prove that the maximal operator $\widetilde{\sigma}^{*} f := \sup_{n \in P} \frac{|\sigma_n f|}{\log^2(n+1)}$ is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$, where $\sigma_n f$ are Fejér means of bounded Vilenkin-Fourier series.

Key words and phrases: Vilenkin system, Fejér means, martingale Hardy space

1. Introduction

In one-dimensional case the weak type inequality

$$\mu\left(\sigma^{*} f > \lambda\right) \leq \frac{c}{\lambda} \|f\|_1 \qquad (\lambda > 0)$$

can be found in Zygmund [19] for trigonometric series, in Schipp [11] for Walsh series and in Pál and Simon [10] for bounded Vilenkin series. Again in one-dimensional, Fujii [4] and Simon [13] verified that σ^{*} is bounded from H_1 to L_1. Weisz [16] generalized this result and proved the boundedness of σ^{*} from the martingale space H_p to the space L_p for $p > 1/2$. Simon [12] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. The counterexample for $p = 1/2$ is due to Goginava [7], (see also [3]). In the endpoint case, $p = 1/2$, two positive results were showed. Weisz [18] proved that σ^{*} is bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$. For Walsh-Paley system in 2008 Goginava [6] proved that the maximal operator $\widetilde{\sigma}^{*}$ defined by

$$\widetilde{\sigma}^{*} f := \sup_{n \in P} \frac{|\sigma_n f|}{\log^2(n+1)}$$

is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$. He also proved that for any nondecreasing function $\varphi : P_{+} \to [1, \infty)$ satisfying the condition

$$\overline{\lim_{n \to \infty}} \frac{\log^2(n+1)}{\varphi(n)} = +\infty \tag{1}$$

the maximal operator

$$\sup_{n \in P} \frac{|\sigma_n f|}{\varphi(n)}$$

*Correspondence: giorgitephnadze@gmail.com

2000 *AMS Mathematics Subject Classification:* 42C10.

is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$.

For a Walsh-Kaczmarz system an analogical theorem is proved in [9].

The main aim of this paper is to prove that the maximal operator $\widetilde{\sigma}^{*}f$ with respect to Vilenkin system is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$ (see Theorem 1).

We also prove that under the condition (1) the maximal operator

$$\sup_{n\in P}\frac{|\sigma_n f|}{\varphi(n)}$$

is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$. Actually, we prove stronger result (see Theorem 2) than the unboundedness of the maximal operator $\widetilde{\sigma}^{*}f$ from the Hardy space $H_{1/2}$ to the spaces $L_{1/2}$. In particular, we prove that

$$\sup_{n\in P}\left\|\frac{\sigma_n f}{\varphi(n)}\right\|_{L_{1/2}}=\infty.$$

2. Definitions and notation

Let P_+ denote the set of the positive integers , $P:=P_+\cup\{0\}$.

Let $m:=(m_0,m_{1....})$ denote a sequence of the positive integers not less than 2.

Denote by

$$Z_{m_k}:=\{0,1,...m_k-1\}$$

the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j} s.

The direct product μ of the measures

$$\mu_k(\{j\}):=1/m_k \qquad (j\in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m)=1$.

If $\sup_n m_n<\infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded then G_m is said to be an unbounded Vilenkin group. **In this paper we discuss bounded Vilenkin groups only.**

The elements of G_m are represented by sequences

$$x:=(x_0,x_{1,...,}x_{j,...}) \qquad (x_k\in Z_{m_k}).$$

It is easy to give a base for the neighborhood of G_m

$$I_0(x):=G_m,$$

$$I_n(x):=\{y\in G_m\mid y_0=x_0,...y_{n-1}=x_{n-1}\}(x\in G_m,n\in P)$$

Denote $I_n:=I_n(0)$ for $n\in P$ and $\overline{I_n}:=G_m\ \setminus\ I_n$.

Let

$$e_n := (0, ..., 0, x_n = 1, 0, ...) \in G_m \qquad (n \in P).$$

Denote

$$I_N^{k,l} := \begin{cases} I_N(0, ..., 0, x_k \neq 0, 0, ..., 0, x_l \neq 0, x_{l+1,...,}x_{N-1}), k < l < N \\ I_N(0, ..., 0, x_k \neq 0, 0, ..., , 0), l = N \end{cases}$$

and

$$I_N^{k,\alpha,l,\beta} := I_N(0, ..., 0, x_k = \alpha, 0, ..., 0, x_l = \beta, x_{l+1,...,}x_{N-1}), k < l < N.$$

It is evident

$$I_N^{k,l} = \overset{m_k-1}{\underset{\alpha=1}{\cup}} \overset{m_l-1}{\underset{\beta=1}{\cup}} I_N^{k,\alpha,l,\beta} \tag{2}$$

and

$$\bar{I_N} = \left(\bigcup_{k=0}^{N-2} \bigcup_{l=k+1}^{N-1} I_N^{k,l} \right) \cup \left(\bigcup_{k=0}^{N-1} I_N^{k,N} \right). \tag{3}$$

If we define the so-called generalized number system based on m as

$$M_0 := 1, \qquad M_{k+1} := m_k M_k \qquad (k \in P),$$

then every $n \in P$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j M_j$ where $n_j \in Z_{m_j} \quad (j \in P)$ and only a finite number of n_js differ from zero. Let $|n| := \max \ \{j \in P; \ n_j \neq 0\}$.

Denote by $L_1(G_m)$ the usual (one dimensional) Lebesgue space.

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system.

At first define the complex valued function $r_k(x) : G_m \to C$, the generalized Rademacher functions as

$$r_k(x) := \exp(2\pi i x_k / m_k) \qquad \left(i^2 = -1, x \in G_m, k \in P\right).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in P)$ on G_m as

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (n \in P).$$

Specifically, we call this system the Walsh-Paley one if $m \equiv 2$.

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ [1, 14].

Now we introduce analogues of the usual definitions in Fourier-analysis.

If $f \in L_1(G_m)$ we can establish the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet and Fejér kernels with respect to the Vilenkin system ψ in the usual manner:

$$\widehat{f}(k) := \int_{G_m} f \overline{\psi}_k d\mu \ , \ (k \in P),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k \ , (\ n \in P_+, S_0 f := 0),$$

$$\sigma_n f := \frac{1}{n}\sum_{k=0}^{n-1} S_k f \left(n \in P_+\right),$$

$$D_n := \sum_{k=0}^{n-1} \psi_k, \left(n \in P_+\right),$$

$$K_n := \frac{1}{n}\sum_{k=0}^{n-1} D_k \ , \left(n \in P_+\right).$$

Recall that

$$D_{M_n}\left(x\right) = \begin{cases} M_n & \text{if } x \in I_n \\ 0 & \text{if } \ x \notin I_n. \end{cases} \tag{4}$$

It is well known that

$$\sup_n \int_{G_m} \left|K_n\left(x\right)\right| d\mu\left(x\right) \le c < \infty, \tag{5}$$

$$n\left|K_n\left(x\right)\right| \le c\sum_{A=0}^{|n|} M_A \left|K_{M_A}\left(x\right)\right|. \tag{6}$$

The norm (or quasinorm) of the space $L_p(G_m)$ is defined by

$$\|f\|_{L_p} := \left(\int_{G_m} |f(x)|^p \, d\mu(x)\right)^{1/p} \qquad \left(0 < p < \infty\right).$$

The σ-algebra generated by the intervals $\{I_n\left(x\right) : x \in G_m\}$ will be denoted by $\digamma_n$ $\left(n \in P\right)$. Denote by $f = \left(f^{(n)}, n \in P\right)$ a martingale with respect to $\digamma_n$ $\left(n \in P\right)$ (for details, see e.g. [15]). The maximal function of a martingale f is defended by

$$f^* = \sup_{n \in P} \left|f^{(n)}\right|.$$

In case $f \in L_1$, the maximal functions are also given by

$$f^*\left(x\right) = \sup_{n \in P} \frac{1}{\left|I_n\left(x\right)\right|} \left|\int_{I_n(x)} f\left(u\right) \mu\left(u\right)\right|.$$

For $0 < p < \infty$ the Hardy martingale spaces H_p $\left(G_m\right)$ consist of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_{L_p} < \infty.$$

If $f \in L_1$, then it is easy to show that the sequence $\left(S_{M_n}\left(f\right) : n \in P\right)$ is a martingale. If $f = \left(f^{(n)}, n \in P\right)$ is martingale then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}\left(i\right) := \lim_{k \to \infty} \int_{G_m} f^{(k)}\left(x\right) \overline{\Psi}_i\left(x\right) d\mu\left(x\right).$$

The Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}(f) : n \in P)$ obtained from f .

For the martingale f we consider maximal operators

$$\sigma^* f = \sup_{n \in P} |\sigma_n f| ,$$

$$\widetilde{\sigma}^* f := \sup_{n \in P} \frac{|\sigma_n f|}{\log^2(n+1)}.$$

A bounded measurable function a is p-atom, if there exist a dyadic interval I, such that

$$\begin{cases} a) & \int_I a d\mu = 0, \\ b) & \|a\|_\infty \leq \mu(I)^{-1/p}, \\ c) & \operatorname{supp}(a) \subset I. \end{cases}$$

3. Formulation of main results

Theorem 1 *The maximal operator*

$$\overset{\sim}{\sigma}^* f := \sup_{n \in P} \frac{|\sigma_n f|}{\log^2(n+1)}$$

is bounded from the Hardy space $H_{1/2}(G_m)$ to the space $L_{1/2}(G_m)$.

Theorem 2 *Let $\varphi : P_+ \to [1, \infty)$ be a nondecreasing function satisfying the condition*

$$\overline{\lim_{n\to\infty}} \frac{\log^2(n+1)}{\varphi(n)} = +\infty. \tag{7}$$

Then there exists a martingale $f \in H_{1/2}$, such that

$$\sup_{n \in P} \left\| \frac{\sigma_n f}{\varphi(n)} \right\|_{L_{1/2}} = \infty.$$

Corollary 1 *The maximal operator*

$$\sup_{n \in P} \frac{|\sigma_n f|}{\varphi(n)}$$

is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$.

4. Auxiliary propositions

Lemma 1 *[17] Suppose that an operator T is sublinear and for some $0<p\leq 1$*

$$\int_{\bar{I}} |Ta|^p \, d\mu \leq c_p < \infty$$

for every p-atom a, where I denotes the support of the atom. If T is bounded from L_∞ to L_∞, then

$$\|Tf\|_{L_p(G_m)} \leq c_p \|f\|_{H_p(G_m)}.$$

Lemma 2 *[2, 8] Let $2<A\in P_+,\ k\leq s<A$ and $q_A=M_{2A}+M_{2A-2}+...+M_2+M_0$. Then*

$$q_{A-1}\left|K_{q_{A-1}}(x)\right| \geq \frac{M_{2k}M_{2s}}{4}$$

for

$$x\in I_{2A}\left(0,...,x_{2k}\neq 0,0,...,0,x_{2s}\neq 0,x_{2s+1},...x_{2A-1}\right);$$

$$k=0,1,...,A-3. \qquad s=k+2,k+3,...,A-1.$$

Lemma 3 *[5] Let $A>t,\ t,A\in P,\ z\in I_t\backslash\ I_{t+1}$. Then*

$$K_{M_A}(z)=\begin{cases} 0 & \text{if } z-z_te_t\notin I_A, \\ \frac{M_t}{1-r_t(z)} & \text{if } z-z_te_t\in I_A. \end{cases}$$

Lemma 4 *Let $x\in I_N^{k,l}$, $\qquad k=0,...,N-1,\ l=k+1,...,N$. Then*

$$\int_{I_N} |K_n(x-t)|\, d\mu(t) \leq \frac{cM_lM_k}{M_N^2} \quad \text{when } n\geq M_N.$$

Proof. Let $x\in I_N^{k,\alpha,l,\beta}$. Then applying lemma 3 we have

$$K_{M_A}(x)=0 \text{ when } A>l.$$

Hence we can suppose that $A\leq l$. Let $k<A\leq l$. Then we have

$$|K_{M_A}(x)| = \frac{M_k}{|1-r_k(x)|} \leq \frac{m_kM_k}{2\pi\alpha}. \tag{8}$$

Let $A\leq k<l$. Then it is easy to show that

$$|K_{M_A}(x)| \leq cM_k. \tag{9}$$

Combining (8) and (9), from (2) we have

$$\left|K_{M_A}(x)\right| \leq cM_k, \text{ when } x \in I_N^{k,l}$$

and if we apply (6) we conclude that

$$n\left|K_n(x)\right| \leq c\sum_{A=0}^{l-1} M_A M_k \leq cM_k M_l. \tag{10}$$

Let $x \in I_N^{k,l}$, for some $0 \leq k < l \leq N-1$. Since $x - t \in I_N^{k,l}$ and $n \geq M_N$ from (10) we obtain

$$\int_{I_N} \left|K_n(x-t)\right| d\mu(t) \leq \frac{cM_k M_l}{M_N^2}. \tag{11}$$

Let $x \in I_N^{k,N}$, then applying (6) we have

$$\int_{I_N} n\left|K_n(x-t)\right| d\mu(t) \leq \sum_{A=0}^{|n|} M_A \int_{I_N} \left|K_{M_A}(x-t)\right| d\mu(t). \tag{12}$$

Let

$$\begin{cases} x = \left(0, ..., 0, x_k \neq 0, 0, ...0, x_N, x_{N+1}, x_q, ..., x_{|n|-1}, , ...\right) \\ t = \left(0, ..., 0, x_N, ...x_{q-1}, t_q \neq x_q, t_{q+1}, ..., t_{|n|-1}, ...\right), \ q = N, ..., |n|-1. \end{cases}$$

Then using Lemma 3 in (12) it is easy to show that

$$\begin{aligned} \int_{I_N} \left|K_n(x-t)\right| d\mu(t) &\leq \frac{c}{n}\sum_{A=0}^{q-1} M_A \int_{I_N} M_k d\mu(t) \\ &\leq \frac{cM_k M_q}{nM_N} \leq \frac{cM_k}{M_N}. \end{aligned} \tag{13}$$

Let

$$\begin{cases} x = \left(0, ..., 0, x_m \neq 0, 0, ..., 0, x_N, x_{N+1}, x_q, ..., x_{|n|-1}, ...\right), \\ t = \left(0, 0, ..., x_N, ..., x_{|n|-1}, t_{|n|}, ...\right). \end{cases}$$

If we apply Lemma 3 in (12), we obtain

$$\begin{aligned} &\int_{I_N} \left|K_n(x-t)\right| d\mu(t) \\ \leq\ &\frac{c}{n}\sum_{A=0}^{|n|-1} M_A \int_{I_N} M_k d\mu(t) \leq \frac{cM_k}{M_N}. \end{aligned} \tag{14}$$

Combining (11), (13) and (14) we complete the proof of lemma 4.

5. Proof of the theorems

Proof of Theorem 1. By Lemma 1, the proof of Theorem 1 will be complete, if we show that

$$\int_{\overline{I}_N} \left(\sup_{n\in P} \frac{|\sigma_n a|}{\log^2 (n+1)} \right)^{1/2} d\mu \leq c < \infty$$

for every 1/2-atom a, where I denotes the support of the atom. The boundedness of maximal operator $\sup\limits_{n\in P} \frac{\sigma_n f}{log^2(n+1)}$ from L_∞ to L_∞ follows from (5).

Let a be an arbitrary 1/2-atom with support I and $\mu(I) = M_N^{-1}$. We may assume that $I = I_N$. It is easy to see that $\sigma_n(a) = 0$ when $n \leq M_N$. Therefore we can suppose that $n > M_N$.

Since $\|a\|_\infty \leq cM_N^2$, we can write

$$\begin{aligned} &\frac{|\sigma_n (a)|}{\log^2 (n+1)} \\ \leq\ & \frac{1}{\log^2 (n+1)} \int_{I_N} |a(t)| \, |K_n(x-t)| \, d\mu(t) \\ \leq\ & \frac{\|a\|_\infty}{\log^2 (n+1)} \int_{I_N} |K_n(x-t)| \, d\mu(t) \\ \leq\ & \frac{cM_N^2}{\log^2 (n+1)} \int_{I_N} |K_n(x-t)| \, d\mu(t). \end{aligned}$$

Let $x \in I_N^{k,l}$, $0 \leq k < l \leq N$. Then from Lemma 4 we get

$$\frac{|\sigma_n (a)|}{\log^2 (n+1)} \leq \frac{cM_N^2}{N^2} \frac{M_l M_k}{M_N^2} = \frac{cM_l M_k}{N^2} \tag{15}$$

Combining (3) and (15) we obtain

$$\begin{aligned} &\int_{\overline{I_N}} |\widetilde{\sigma}^* a(x)|^{1/2} d\mu(x) \\ =\ & \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{x_j=0, j\in\{l+1,\dots,N-1\}}^{m_j-1} \int_{I_N^{k,l}} \left|\widetilde{\sigma}^* a(x)\right|^{1/2} d\mu(x) \\ &+ \sum_{k=0}^{N-1} \int_{I_N^{k,N}} |\widetilde{\sigma}^* a(x)|^{1/2} d\mu(x) \\ \leq\ & c \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{m_{l+1} \dots m_{N-1}}{M_N} \frac{\sqrt{M_l M_k}}{N} \\ &+ c \sum_{k=0}^{N-1} \frac{1}{M_N} \frac{\sqrt{M_N M_k}}{N} \\ \leq\ & c \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{\sqrt{M_k}}{N\sqrt{M_l}} + c \sum_{k=0}^{N-1} \frac{1}{\sqrt{M_N}} \frac{\sqrt{M_k}}{N} \leq c < \infty. \end{aligned}$$

Which completes the proof of Theorem 1. □

Proof of Theorem 2. Let $\{\lambda_k; k \in P_+\}$ be an increasing sequence of the positive integers such that

$$\lim_{k\to\infty} \frac{\log^2(\lambda_k)}{\varphi(\lambda_k)} = \infty.$$

It is evident that for every λ_k there exists positive integers $m_k^{'}$ such that $q_{m_k'} \leq \lambda_k < q_{m_k'+1} < M^5 \; q_{m_k'}$, $M := \sup_k m_k$. Since $\varphi(n)$ is a nondecreasing function we have

$$\overline{\lim_{k\to\infty}} \frac{\left(m_k^{'}\right)^2}{\varphi\left(q_{m_k^{'}}\right)} \geq c \lim_{k\to\infty} \frac{\log^2(\lambda_k)}{\varphi(\lambda_k)} = \infty; \tag{16}$$

let $\{n_k; k \in P_+\} \subset \{m_k'; k \in P_+\}$ such that

$$\lim_{k\to\infty} \frac{n_k^2}{\varphi(q_{n_k})} = \infty$$

and

$$f_{n_k}(x) = D_{M_{2n_k+1}}(x) - D_{M_{2n_k}}(x).$$

It is evident

$$\widehat{f}_{n_k}(i) = \begin{cases} 1, \text{ if } i = M_{2n_k}, ..., M_{2n_k+1} - 1; \\ 0, \text{otherwise.} \end{cases}$$

Then we can write

$$S_i f_{n_k}(x) = \begin{cases} D_i(x) - D_{M_{2n_k}}(x), \text{ if } i = M_{2n_k}, ..., M_{2n_k+1} - 1 \\ f_{n_k}(x), \text{if } i \geq M_{2n_k+1} \\ 0, \quad \text{otherwise}. \end{cases} \tag{17}$$

From (4) we get

$$\begin{aligned} \|f_{n_k}\|_{H_{1/2}} &= \left\|\sup_{n\in P} S_{M_n} f_{n_k}\right\|_{L_{1/2}} \qquad (18) \\ &= \left\|D_{M_{2n_k+1}} - D_{M_{2n_k}}\right\|_{L_{1/2}} \\ &= \left(\int_{I_{2n_k}\setminus I_{2n_k+1}} M_{2n_k}^{1/2} d\mu(x) + \int_{I_{2n_k+1}} \left(M_{2n_k+1} - M_{2n_k}\right)^{1/2} d\mu(x)\right)^2 \\ &= \left(\frac{m_{2n_k} - 1}{M_{2n_k+1}} M_{2n_k}^{1/2} + \frac{\left(m_{2n_k} + 1\right)^{1/2}}{M_{2n_k+1}} M_{2n_k}^{1/2}\right)^2 \\ &\leq \frac{c}{M_{2n_k}}. \end{aligned}$$

By (17) we can write:

$$\frac{\left|\sigma_{q_{n_k}} f_{n_k}(x)\right|}{\varphi\left(q_{n_k}\right)} = \frac{1}{\varphi\left(q_{n_k}\right) q_{n_k}} \left|\sum_{j=0}^{q_{n_k}-1} S_j f_{n_k}(x)\right|$$

$$= \frac{1}{\varphi\left(q_{n_k}\right) q_{n_k}} \left|\sum_{j=M_{2n_k}}^{q_{n_k}-1} S_j f_{n_k}(x)\right|$$

$$= \frac{1}{\varphi\left(q_{n_k}\right) q_{n_k}} \left|\sum_{j=M_{2n_k}}^{q_{n_k}-1} \left(D_j\left(x\right) - D_{M_{2n_k}}\left(x\right)\right)\right|$$

$$= \frac{1}{\varphi\left(q_{n_k}\right) q_{n_k}} \left|\sum_{j=0}^{q_{n_k-1}-1} \left(D_{j+M_{2n_k}}\left(x\right) - D_{M_{2n_k}}\left(x\right)\right)\right|$$

Since

$$D_{j+M_{2n_k}}\left(x\right) - D_{M_{2n_k}}\left(x\right) = \psi_{M_{2n_k}} D_j,\ j = 1, 2, .., M_{2n_k} - 1,$$

we obtain

$$\begin{aligned} \frac{\left|\sigma_{q_{n_k}} f_{n_k}(x)\right|}{\varphi\left(q_{n_k}\right)} &= \frac{1}{\varphi\left(q_{n_k}\right) q_{n_k}} \left|\sum_{j=0}^{q_{n_k-1}-1} D_j\left(x\right)\right| \\ &= \frac{1}{\varphi\left(q_{n_k}\right)} \frac{q_{n_k-1}}{q_{n_k}} \left|K_{q_{n_k-1}}\left(x\right)\right|. \end{aligned}$$

Let $x \in I_{2n_k}^{2s,2l}$. Then from Lemma 2 we have

$$\frac{\left|\sigma_{q_{n_k}} f_{n_k}(x)\right|}{\varphi\left(q_{n_k}\right)} \geq \frac{cM_{2s}M_{2l}}{M_{2n_k}\varphi\left(q_{n_k}\right)}.$$

Hence we can write:

$$\int_{G_m} \left(\frac{\left|\sigma_{q_{n_k}} f_{n_k}(x)\right|}{\varphi\left(q_{n_k}\right)}\right)^{1/2} d\mu\left(x\right)$$

$$\geq \sum_{s=0}^{n_k-3} \sum_{l=s+1}^{n_k-1} \sum_{x_{2l+1}=0}^{m_{2l+1}} ... \sum_{x_{2n_k-1}=0}^{m_{2n_k-1}} \int_{I_{2n_k}^{2s,2l}} \left(\frac{\left|\sigma_{q_{n_k}} f_{n_k}(x)\right|}{\varphi\left(q_{n_k}\right)}\right)^{1/2} d\mu\left(x\right)$$

$$\geq c \sum_{s=0}^{n_k-3} \sum_{l=s+1}^{n_k-1} \frac{m_{2l+1}...m_{2n_k-1}}{M_{2n_k}} \frac{\sqrt{M_{2s}M_{2l}}}{\sqrt{\varphi\left(q_{n_k}\right) M_{2n_k}}} \geq$$

$$c \sum_{s=0}^{n_k-3} \sum_{l=s+1}^{n_k-1} \frac{\sqrt{M_{2s}}}{\sqrt{M_{2l}M_{2n_k}\varphi\left(q_{n_k}\right)}} \geq \frac{cn_k}{\sqrt{M_{2n_k}\varphi\left(q_{n_k}\right)}}$$

Then from (18) we have

$$\frac{\left(\int_{G_m}\left(\frac{\left|\sigma_{q_{n_k}}f_{n_k}(x)\right|}{\varphi(n_k)}\right)^{1/2}d\mu\left(x\right)\right)^2}{\left\|f_{n_k(x)}\right\|_{H_{1/2}}} \geq \frac{cn_k^2}{M_{2n_k}\varphi\left(q_{n_k}\right)}M_{2n_k}$$

$$\geq \frac{cn_k^2}{\varphi\left(q_{n_k}\right)} \to \infty \qquad \text{when} \qquad k\to\infty.$$

Theorem 2 is proved. □

References

[1] Agaev, G. N., Vilenkin, N. Ya., Dzhafarly, G. M., Rubinshtein, A. I.: Multiplicative systems of functions and harmonic analysis on zero-dimensional groups, Baku, Ehim, 1981 (in Russian).

[2] Blahota, I., Gát, G., Goginava, U.: Maximal operators of Fejer means of double Vilenkin-Fourier series, Colloq. Math. 107, no. 2, 287–296, (2007).

[3] Blahota, I., Gát, G., Goginava, U.: Maximal operators of Fejér means of Vilenkin-Fourier series, JIPAM. J. Inequal. Pure Appl. Math. 7, 1–7, (2006).

[4] Fujii, N. J.: A maximal inequality for H_1 functions on generalized Walsh-Paley group, Proc. Amer. Math. Soc. 77, (1979), 111–116.

[5] Gát, G.: Cesàro means of integrable functions with respect to unbounded Vilenkin systems, J. Approx. Theory 124, no. 1, 25–43, (2003).

[6] Goginava, U.: Maximal operators of Fejér-Walsh means, Acta Sci. Math. (Szeged) 74, no. 3-4, 615–624, (2008).

[7] Goginava, U.: The maximal operator of the Fejér means of the character system of the p-series field in the Kaczmarz rearrangement, Publ. Math. Debrecen 71, no. 1-2, 43–55, (2007).

[8] Goginava, U.: Maximal operators of Fejér means of double Walsh-Fourier series, Acta Math. Hungar. 115, no. 4, 333–340, (2007).

[9] Goginava, U., Nagy, K.: On the maximal operator of Walsh-Kaczmarz-Fejer means, Czechoslovak Math. J. (to appear).

[10] Pál, J., Simon, J.: On a generalization of the concept of derivate, Acta Math. Hung., 29, 155–164, (1977).

[11] Schipp, F.: Certain rearrangements of series in the Walsh series, Mat. Zametki, 18, 193–201, (1975).

[12] Simon, P.: Cesàro summability wish respect to two-parameter Walsh systems, Monatsh. Math.,131, 321–334, (2000).

[13] Simon, P.: Investigations with respect to the Vilenkin system, Annales Univ. Sci. Budapest Eotv., Sect. Math., 28, 87–101, (1985).

[14] Vilenkin, N.Ya.: A class of complete orthonormal systems, Izv. Akad. Nauk. U.S.S.R., Ser. Mat., 11, 363–400, (1947).

[15] Weisz, F.: Martingale Hardy spaces and their applications in Fourier Analysis, Springer, Berlin-Heidelberg-New York, 1994.

[16] Weisz, F.: Cesro summability of one and two-dimensional Fourier series, Anal. Math. 5 (1996), 353-367.re Appl. Math. 7, 1–7, (2006).

[17] Weisz, F.: Summability of multi-dimensional Fourier series and Hardy space, Kluwer Academic, Dordrecht, 2002.

[18] Weisz, F.: Weak type inequalities for the Walsh and bounded Ciesielski systems. Anal. Math. 30, no. 2, 147–160, (2004).

[19] Zygmund, A.: Trigonometric Series, Vol. 1, Cambridge Univ. Press, 1959.