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Sasakian Finsler manifolds

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Abstract: In this study, almost contact Finsler structures on vector bundle are defined and the condition of normality in terms of the Nijenhuis torsion N_ϕ of almost contact Finsler structure is obtained. It is shown that for a K-contact structure on Finsler manifold $\nabla_X \xi = -\frac{1}{2}\phi X$ and the flag curvature for plane sections containing ξ are equal to $\frac{1}{4}$. By using the Sasakian Finsler structure, the curvatures of a Finsler connection ∇ on V are obtained. We prove that a locally symmetric Finsler manifold with K-contact Finsler structure has a constant curvature $\frac{1}{4}$. Also, the Ricci curvature on Finsler manifold with K-contact Finsler structure is given. As a result, Sasakian structures in Riemann geometry and Finsler condition are generalized.

As a conclusion we can state that Riemannian Sasakian structures are compared to Sasakian Finsler structures and it is proven that they are adaptable.

Key words: Finsler connection, vector bundle, almost contact manifold, Sasakian manifold, nonlinear connection, Ricci tensor

1. Introduction

Let $V(M) = \{V, \pi, M\}$ be a vector bundle of total space V with a $(n+m)$-dimensional C^∞ manifold and with a base space M that is an n-dimensional C^∞-manifold. The projection map $\pi : V \to M, u \in V \mapsto \pi(u) = x \in M$, where $u = (x, y)$, and $y \in R^m = \pi^{-1}(x)$ the fibre of $V(M)$ over x.

A non-linear connection N on the total space V of $V(M)$ is a differentiable distribution $N : V \to T_u(V),\ u \in V \mapsto N_u \in T_u(V)$ such that

$$T_u(V) = N_u \oplus V_u^v \ \text{ where } \ V_u^v = \{X \in T_u(V) : \pi_*(X) = 0\}. \tag{1.1}$$

N_u the *horizontal distribution* and V^v is the *vertical distribution*. Thus for all $X \in T_u(V)$ can be separated by its components

$$X = X^H + X^V \ \text{ where } X^H \in N_u,\ \ X^V \in V_u^v. \tag{1.2}$$

Let x^i, i=1,2,...,n and y^a, a=1,2,...,m be the coordinates of x and y such that (x^i, y^a) are the coordinates of $u \in V$. The local base of N_u is

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^a(x,y)\frac{\partial}{\partial y^a} \tag{1.3}$$

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and that of V_u^v is where $N_i^a(x,y)$ are the coefficients of N. Their dual bases are $\left(dx^i, \delta y^a\right)$ where

$$\delta y^a = dy^a + N_i^a\left(x,y\right)dx^i. \tag{1.4}$$

Let $X = X^i(x,y)\frac{\delta}{\delta x^i} + \tilde{X}^a(x,y)\frac{\partial}{\partial y^a}, \forall X \in T_u(V)$. Then

$$X^H = X^i(x,y)\frac{\delta}{\delta x^i}, X^V = \tilde{X}^a(x,y)\frac{\partial}{\partial y^a}, \quad \tilde{X}^a = X^a + N_i^a X^i. \tag{1.5}$$

Let ω be a *1*-form $\omega = \tilde{\omega}_i(x,y)dx^i + \omega_a(x,y)\delta y^a$. Then

$$\omega^H = \tilde{\omega}_i dx^i,\ \tilde{\omega}_i = \omega_i - N_i^a(x,y)\omega_a;\ \ \omega^V = \omega_a \delta y^a \tag{1.6}$$

which gives

$$\omega^H(X^V) = 0, \omega^V(X^H) = 0 \text{ where } \omega = \omega^H + \omega^V. \tag{1.7}$$

The Finsler tensor field of type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ on V has the following local form [4]:

$$T = T_{j_1,\ldots,j_q,b_1,\ldots,b_s}^{i_1,\ldots,i_p,a_1,\ldots,a_r}(x,y)\frac{\delta}{\delta x^{i_1}} \otimes \ldots \otimes \frac{\delta}{\delta x^{i_p}} \otimes dx^{a_1} \otimes \ldots \otimes dx^{a_r} \otimes \frac{\partial}{\partial y^{j_1}} \otimes \ldots \otimes \frac{\partial}{\partial y^{j_q}} \otimes \delta y^{b_1} \otimes \ldots \otimes \delta y^{b_s}. \tag{1.8}$$

Definition 1.1 *A Finsler connection on V is a linear connection $\nabla = F\Gamma$ on V with the property that the horizontal linear space N_u, $u \in V$ of the distribution N is parallel with respect to ∇ and the vertical spaces V_u^v, $u \in V$ are also parallel relative to ∇ [3].*

A linear connection ∇ on V is a Finsler connection on V if and only if

$$\left(\nabla_X Y^H\right)^V = 0, \left(\nabla_X Y^V\right)^H = 0, \forall X, Y \in T_u\left(V\right). \tag{1.9}$$

A linear connection ∇ on V is a Finsler connection on V if and only if [4]

$$\nabla_X Y = \left(\nabla_X Y^H\right)^H + \left(\nabla_X Y^V\right)^V, \forall X, Y \in T_u\left(V\right), \tag{1.10a}$$

$$\nabla_X \omega = \left(\nabla_X \omega^H\right)^H + \left(\nabla_X \omega^V\right)^V, \forall \omega \in T_u^*\left(V\right) \ \textit{and}\ X \in T_u\left(V\right). \tag{1.10b}$$

Remark 1.1 Let ∇ on V is a Finsler connection on V. We get immediately that [6]

$$Y \in V_u^v \Rightarrow \forall X \in T_u\left(V\right); \nabla_X Y \in V_u^v,\ Y \in N_u \Rightarrow \forall X \in T_u\left(V\right); \nabla_X Y \in N_u. \tag{1.11}$$

For a Finsler connection ∇ on V, there is an associated pair of operators; h- and v-covariant derivation in the algebra of Finsler tensor fields. For each $X \in T_u\left(V\right)$, set

$$\nabla_X^H Y = \nabla_{X^H} Y, \nabla_X^H f = X^H\left(f\right), \forall Y \in T_u\left(V\right), \forall f \in F\left(V\right). \tag{1.12}$$

If $\omega \in T_u^*\left(V\right)$, we define

$$\left(\nabla_X^H \omega\right)\left(Y\right) = X^H\left(\omega\left(Y\right)\right) - \omega\left(\nabla_X^H Y\right), \forall Y \in T_u\left(V\right). \tag{1.13}$$

So, we may extend the action of the operator ∇_X^H to any Finsler tensor field by asking these questions: does ∇_X^H preserve the type of Finsler tensor fields, is it $\mathbb{R}$-linear, does it satisfy the Leibniz rule with respect to tensor product and does it commute with all contractions? We keep the notation ∇_X^H for this operator on the algebra of Finsler tensor fields. We call it the *operator of h-covariant derivation.*

In a similar way, for every vector field $X \in T_u(V)$ set

$$\nabla_X^V Y = \nabla_{X^V} Y, \nabla_X^V f = X^V(f), \forall Y \in T_u(V), \forall f \in F(V). \tag{1.14}$$

If $\omega \in T_u^*(V)$, we define

$$(\nabla_X^V \omega)(Y) = X^V(\omega(Y)) - \omega\left(\nabla_X^V Y\right), \forall Y \in T_u(V). \tag{1.15}$$

We extend the action of ∇_X^V to any Finsler tensor field in a similar way, as for ∇_X^H. We obtain an operator on the algebra of Finsler tensor fields on V; this will be denoted also by ∇_X^V and will be called the *operator of v-covariant derivation* [1].

Definition 1.2 *Let $\omega \in T_u^*(V)$ be a differential q-form on V, ∇ is a linear connection on V and T is the torsion tensor of ∇. Then its exterior differential $d\omega$ is also defined as [4]:*

$$\begin{aligned}(d\omega)(X_1, ..., X_{q+1}) &= \sum_{i=1}^{q+1} (-1)^{i+1} (\nabla_X \omega)\left(X_1, ..., \tilde{X}_i, ..., X_{q+1}\right), \forall X_i \in T_u(V) \\ &- \sum_{1\le i\le j\le q+1} (-1)^{i+j} \omega\left(T(X_i, X_j), X_1, ..., \tilde{X}_i, ..., \tilde{X}_j, ..., X_{q+1}\right).\end{aligned} \tag{1.16}$$

Proposition 1.1 *If $\omega \in T_u^*(V)$ is a 1-form and ∇ is a Finsler connection on V, then its exterior differential is given by [3]*

$$\begin{aligned}(d\omega)\left(X^H, Y^H\right) &= \left(\nabla_X^H \omega\right)\left(Y^H\right) - \left(\nabla_Y^H \omega\right)\left(X^H\right) + \omega\left(T\left(X^H, Y^H\right)\right), \\ (d\omega)\left(X^V, Y^H\right) &= \left(\nabla_X^V \omega\right)\left(Y^H\right) - \left(\nabla_Y^H \omega\right)\left(X^V\right) + \omega\left(T\left(X^V, Y^H\right)\right), \\ (d\omega)\left(X^V, Y^V\right) &= \left(\nabla_X^V \omega\right)\left(Y^V\right) - \left(\nabla_Y^V \omega\right)\left(X^V\right) + \omega\left(T\left(X^V, Y^V\right)\right), \forall X, Y \in T_u(V).\end{aligned} \tag{1.17}$$

In the canonical coordinates$\left(x^i, y^a\right)$, there exists a well determined set of differentiable functions on V. $F^i_{\ jk}(x,y), F^a_{bk}(x,y); C^i_{ja}(x,y); C^a_{bc}(x,y)$ such that

$$\begin{aligned}\nabla^H_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j} &= F^i_{jk}(x,y) \frac{\delta}{\delta x^i},\ \nabla^H_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^b} = F^a_{bk}(x,y) \frac{\partial}{\partial y^a}, \\ \nabla^V_{\frac{\partial}{\partial y^a}} \frac{\delta}{\delta x^j} &= C^i_{ja}(x,y) \frac{\delta}{\delta x^i},\ \nabla^V_{\frac{\partial}{\partial y^c}} \frac{\partial}{\partial y^b} = C^a_{bc}(x,y) \frac{\partial}{\partial y^a}\end{aligned}$$

where $F^i_{\ jk}(x,y)$, $F^a_{bk}(x,y)$ are called coefficients of h-connections ∇^H and $C^a_{bc}(x,y)$, $C^i_{ja}(x,y)$ are called coefficients of v-connections∇^V.

The torsion tensor field T of a Finsler-connection is characterised by five Finsler tensor fields:

$$\left[T\left(X^H, Y^H\right)\right]^H,\ \left[T\left(X^H, Y^H\right)\right]^V,\ \left[T\left(X^H, Y^V\right)\right]^H,\ \left[T\left(X^H, Y^V\right)\right]^V,\ \left[T\left(X^V, Y^V\right)\right]^V.$$

Proposition 1.2 *If the Finsler connection on V is without torsion then we have [3]*

$$T\left(X^H, Y^H\right) = 0, T\left(X^H, Y^V\right) = 0, T\left(X^V, Y^V\right) = 0, \forall X, Y \in T_u(V).$$

2. Almost contact Finsler structure on vector bundle

Let ϕ be an almost contact structure on V given by the tensor field of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ with the properties

$$\begin{aligned} &1.\ \phi.\phi = -\mathrm{I}_n + \eta^H \otimes \xi^H + \eta^V \otimes \xi^V \\ &2.\ \phi\, \xi^H = 0, \phi\, \xi^V = 0 \\ &3.\ \eta^H\left(\xi^H\right) + \eta^V\left(\xi^V\right) = 1 \\ &4.\ \eta^H\left(\phi\, X^H\right) = 0, \eta^V\left(\phi\, X^H\right) = 0, \eta^H\left(\phi\, X^V\right) = 0, \eta^V\left(\phi\, X^V\right) = 0, \end{aligned} \tag{2.1}$$

where η is 1-form and ξ is vector field [2].

Proposition 2.1 *If ϕ is an almost contact Finsler structure on V, there exists a unique decomposition of ϕ in the Finsler tensor fields,*

$$\phi = \phi^1 + \phi^2 + \phi^3 + \phi^4 = \begin{pmatrix} \phi^1 & \phi^2 \\ \phi^3 & \phi^4 \end{pmatrix} \tag{2.2}$$

where

$$\begin{aligned} &\phi^1(\omega, X) = \phi\left(\omega^H, X^H\right), \phi^2(\omega, X) = \phi\left(\omega^H, X^V\right), \\ &\phi^3(\omega, X) = \phi\left(\omega^V, X^H\right), \phi^4(\omega, X) = \phi\left(\omega^V, X^V\right) \forall X \in T_u(V), \forall \omega \in T_u^*(V). \end{aligned} \tag{2.3}$$

We can write

$$\phi\left(X^H\right) = \phi^1\left(X^H\right) = \phi^3\left(X^H\right), \phi\left(X^V\right) = \phi^2\left(X^V\right) = \phi^4\left(X^V\right). \tag{2.4}$$

Let G be the Finsler metric structure on V which is symmetric, positive definite and non-degenerate on V. The metric-structure G on V is decomposed as:

$$G = G^H + G^V \tag{2.5}$$

where G^H is of type $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$, symmetric, positive definite and non-degenerate on N_u and G^V is of type $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, symmetric, positive definite and non-degenerate on V_u^v i.e. for $X, Y \in T_u(V)$

$$G(X, Y) = G^H(X, Y) + G^V(X, Y) \tag{2.6}$$

where $G^H(X, Y) = G\left(X^H, Y^H\right), G^V(X, Y) = G\left(X^V, Y^V\right)$..

Now, if the Finsler metric structure G on V satisfies

$$\begin{aligned} &G(\phi X, \phi Y) = G(X, Y) - \eta(X)\, \eta(Y), \\ &G^H(\phi X, \phi Y) = G^H(X, Y) - \eta^H\left(X^H\right) \eta^H\left(Y^H\right), \\ &G^V(\phi X, \phi Y) = G^V(X, Y) - \eta^V\left(X^V\right) \eta^V\left(Y^V\right), \end{aligned} \tag{2.7}$$

which is equivalent to

$$\begin{aligned} &G^H(X, \xi) = \eta^H(X), G^V(X, \xi) = \eta^V(X), \\ &G^H(\phi X, \phi Y) = -G^H\left(\phi^2 X, Y\right), G^V(\phi X, \phi Y) = -G^V\left(\phi^2 X, Y\right), \end{aligned} \tag{2.8}$$

then $(\phi,\ \eta,\ \xi,\ G)$ *is called almost contact metrical Finsler structure on V [5]. Now, we define*

$$\Omega\left(X,Y\right)=G\left(X,\phi Y\right),\ \Omega\left(X^{H},Y^{H}\right)=G^{H}\left(X,\phi Y\right),\ \Omega\left(X^{V},Y^{V}\right)=G^{V}\left(X,\phi Y\right) \tag{2.9}$$

and call it the fundamental 2-form.

Proposition 2.2 *The fundamental 2-form, defined above, satisfies [5]*

$$\begin{aligned}&\Omega\left(\phi X^{H},\phi Y^{H}\right)=\Omega\left(X^{H},Y^{H}\right),\Omega\left(\phi X^{V},\phi Y^{V}\right)=\Omega\left(X^{V},Y^{V}\right),\\&\Omega\left(X^{H},Y^{H}\right)=-\Omega\left(Y^{H},X^{H}\right),\Omega\left(X^{V},Y^{V}\right)=-\Omega\left(Y^{V},X^{V}\right)\forall X,Y\in T_{u}\left(V\right).\end{aligned} \tag{2.10}$$

Proposition 2.3 *Let* ∇ *be a Finsler connection on V and* Ω *be the fundamental 2-form which satisfies* $\Omega\left(X,Y\right)=d\eta\left(X,Y\right)$ *i.e.*

$$\begin{aligned}\Omega\left(X^{H},Y^{H}\right)&=\left(\nabla_{X}^{H}\eta\right)\left(Y^{H}\right)-\left(\nabla_{Y}^{H}\eta\right)\left(X^{H}\right)+\eta\left(T\left(X^{H},Y^{H}\right)\right),\\\Omega\left(X^{V},Y^{H}\right)&=\left(\nabla_{X}^{V}\eta\right)\left(Y^{H}\right)-\left(\nabla_{Y}^{H}\eta\right)\left(X^{V}\right)+\eta\left(T\left(X^{V},Y^{H}\right)\right),\\\Omega\left(X^{V},Y^{V}\right)&=\left(\nabla_{X}^{V}\eta\right)\left(Y^{V}\right)-\left(\nabla_{Y}^{V}\eta\right)\left(X^{V}\right)+\eta\left(T\left(X^{V},Y^{V}\right)\right).\end{aligned} \tag{2.11}$$

Then, the almost contact metrical Finsler structure is called almost Sasakian Finsler structure and the Finsler connection ∇ *satisfying* (2.11) *is called almost Sasakian Finsler connection on V [5].*

Theorem 2.1 *Let* Ω *be the fundamental 2-form and almost Sasakian Finsler connection* ∇ *on V is torsion free. Then [5]*

$$\begin{aligned}\Omega\left(X^{H},Y^{H}\right)&=\left(\nabla_{X}^{H}\eta\right)\left(Y^{H}\right)-\left(\nabla_{Y}^{H}\eta\right)\left(X^{H}\right),\\\Omega\left(X^{V},Y^{H}\right)&=\left(\nabla_{X}^{V}\eta\right)\left(Y^{H}\right)-\left(\nabla_{Y}^{H}\eta\right)\left(X^{V}\right),\\\Omega\left(X^{V},Y^{V}\right)&=\left(\nabla_{X}^{V}\eta\right)\left(Y^{V}\right)-\left(\nabla_{Y}^{V}\eta\right)\left(X^{V}\right),\forall X,Y\in T_{u}\left(V\right).\end{aligned} \tag{2.12}$$

Proof From Proposition 1.2 and equations in (2.11), we have (2.12). □

Definition 2.1 *An almost Sasakian Finsler structure on V is said to be a Sasakian Finsler structure if the 1-form* η *is a killing vector field, i.e.*

$$\begin{aligned}&\left(\nabla_{X}^{H}\eta\right)\left(Y^{H}\right)+\left(\nabla_{Y}^{H}\eta\right)\left(X^{H}\right)=0,\left(\nabla_{X}^{V}\eta\right)\left(Y^{H}\right)+\left(\nabla_{Y}^{H}\eta\right)\left(X^{V}\right)=0,\\&\left(\nabla_{X}^{V}\eta\right)\left(Y^{V}\right)+\left(\nabla_{Y}^{V}\eta\right)\left(X^{V}\right)=0\forall X,Y\in T_{u}\left(V\right).\end{aligned} \tag{2.13}$$

The Finsler connection ∇ *on V is torsion free, which is called Sasakian Finsler connection [5].*

Theorem 2.2 *Let* ∇ *be the torsion free Finsler connection together with a Sasakian Finsler structure on V and* Ω *is to be the fundamental 2-form; then*

$$\begin{aligned}\Omega\left(X^{H},Y^{H}\right)&=2\left(\nabla_{X}^{H}\eta\right)\left(Y^{H}\right)=-2\left(\nabla_{Y}^{H}\eta\right)\left(X^{H}\right),\\\Omega\left(X^{H},Y^{V}\right)&=2\left(\nabla_{X}^{H}\eta\right)\left(Y^{V}\right)=-2\left(\nabla_{Y}^{V}\eta\right)\left(X^{H}\right),\\\Omega\left(X^{V},Y^{V}\right)&=2\left(\nabla_{X}^{V}\eta\right)\left(Y^{V}\right)=-2\left(\nabla_{Y}^{V}\eta\right)\left(X^{V}\right),\forall X,Y\in T_{u}\left(V\right).\end{aligned} \tag{2.14}$$

Proof From (2.12) and (2.13) we have (2.14) [5]. □

Example 2.1 *Let* $V(M) = \{V, \pi, M\}$ *be a vector bundle with the total space* $V = R^{10}$ *is a 10-dimensional* C^{∞}*-manifold and the base space* $M = R^5$ *is a 5-dimensional* C^{∞}*-manifold. Let* x^i $, 1 \leq i \leq 5$ *and* $y^a, 1 \leq a \leq 5$ *be the coordinates of* $u = (x, y) \in V$*, that is* $u = (x^1, x^2, x^3, x^4, x^5, y^1, y^2, y^3, y^4, y^5) \in V$*. The local base of* N_u *is* $\left(\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2}, \frac{\delta}{\delta x^3}, \frac{\delta}{\delta x^4}, \frac{\delta}{\delta x^5}\right)$ *and the local base of* V_u^v *is* $\left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3}, \frac{\partial}{\partial y^4}, \frac{\partial}{\partial y^5}\right)$ *such that* $T_u(V) = N_u \oplus V_u^v$*. Then*

$$
\begin{aligned}
X^H &= X^1 \tfrac{\delta}{\delta x^1} + X^2 \tfrac{\delta}{\delta x^2} + X^3 \tfrac{\delta}{\delta x^3} + X^4 \tfrac{\delta}{\delta x^4} + X^5 \tfrac{\delta}{\delta x^5}, \\
X^V &= \tilde{X}^1 \tfrac{\partial}{\partial y^1} + \tilde{X}^2 \tfrac{\partial}{\partial y^2} + \tilde{X}^3 \tfrac{\partial}{\partial y^3} + \tilde{X}^4 \tfrac{\partial}{\partial y^4} + \tilde{X}^5 \tfrac{\partial}{\partial y^5} \ni X^H \in N_u, X^V \in V_u^v.
\end{aligned}
$$

Let η *be a 1-form,* $\eta = \eta_i dx^i + \tilde{\eta}_a \delta y^a$ *then* $\eta^H = \eta_1 dx^1 + \eta_2 dx^2 + \eta_3 dx^3 + \eta_4 dx^4 + \eta_5 dx^5$ *and* $\eta^V = \tilde{\eta}_1 \delta y^1 + \tilde{\eta}_2 \delta y^2 + \tilde{\eta}_3 \delta y^3 + \tilde{\eta}_4 \delta y^4 + \tilde{\eta}_5 \delta y^5$ *where* $\eta = \eta^H + \eta^V$ *and* $\eta^H(X^V) = 0$, $\eta^V(X^H) = 0$.

We put $\eta^H = \frac{1}{3}\left(dx^5 - x^3 dx^1 - x^4 dx^2\right)$ *and* $\eta^V = \frac{1}{3}\left(\delta y^5 - y^3 \delta y^1 - y^4 \delta y^2\right)$.

The structure vector field ξ *is given by* $\xi = 3\left(\frac{\delta}{\delta x^5} + \frac{\partial}{\partial y^5}\right)$ *and* ξ *is decomposed as* $\xi^H = 3\frac{\delta}{\delta x^5}$ *and* $\xi^V = 3\frac{\partial}{\partial y^5}$.

The tensor field ϕ^H *of type* $(1,1)$ *and* ϕ^V *of type* $(1,1)$ *by a matrix form is given by*

$$
\phi^H = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -x^4 & x^3 & 0 & 0 & 0 \end{bmatrix}, \phi^V = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -y^4 & y^3 & 0 & 0 & 0 \end{bmatrix}.
$$

We can see that $\eta^H(\xi^H) = 1$, $\phi^H(\xi^H) = 0$, $\eta^V(\xi^V) = 1$, $\phi^V(\xi^V) = 0$, $\eta^H(\xi^V) = 0$, $\eta^V(\xi^H) = 0$, $(\phi^H)^2 X^H = -X^H + \eta^H(X^H)\xi^H$, $(\phi^V)^2 X^V = -X^V + \eta^V(X^V)\xi^V$ *and hence* (ϕ, ξ, η) *is almost contact Finsler structure on* R^{10}.

3. Integrability tensor field of the almost contact Finsler structure

The *integrability tensor field* of the almost contact Finsler structure on V is given by [4] $\tilde{N}(X,Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X,Y] + d\eta^H(X,Y)\xi^H + d\eta^V(X,Y)\xi^V, \forall X, Y \in T_u(V)$.

We define four tensors $N^{(1)}, N^{(2)}, N^{(3)}$ and $N^{(4)}$, respectively by $\forall X^H, Y^H, \xi^H \in N_u$ and $\forall X^V, Y^V, \xi^V \in V_u^v$

$$
N^{(1)}(X^H, Y^H) = N_\phi(X^H, Y^H) + d\eta^H(X^H, Y^H)\xi^H, \tag{3.1a}
$$

$$
N^{(2)}(X^H, Y^H) = \left(L^H_{\phi X}\eta^H\right)(Y^H) - \left(L^H_{\phi Y}\eta^H\right)(X^H), \tag{3.1b}
$$

$$
N^{(3)}(X^H) = \left(L^H_\xi \phi\right)(X^H), \ N^{(4)}(X^H) = \left(L^H_\xi \eta^H\right)(X^H), \tag{3.2a}
$$

$$N^{(1)}\left(X^V,Y^V\right)=N_\phi\left(X^V,Y^V\right)+d\eta^V\left(X^V,Y^V\right)\xi^V, \tag{3.2b}$$

$$N^{(2)}\left(X^V,Y^V\right)=\left(L^V_{\phi X}\eta^V\right)\left(Y^V\right)-\left(L^V_{\phi Y}\eta^V\right)\left(X^V\right), \tag{3.2c}$$

$$N^{(3)}\left(X^V\right)=\left(L^V_\xi\phi\right)\left(X^V\right),\ N^{(4)}\left(X^V\right)=\left(L^V_\xi\eta^V\right)\left(X^V\right), \tag{3.3a}$$

$$N^{(1)}\left(X^V,Y^H\right)=N_\phi\left(X^V,Y^H\right)+d\eta^V\left(X^V,Y^H\right)\xi^V+d\eta^H\left(X^V,Y^H\right)\xi^H, \tag{3.3b}$$

$$N^{(2)}\left(X^V,Y^H\right)=\left(L^V_{\phi X}\eta^H\right)\left(Y^H\right)+\left(L^V_{\phi X}\eta^V\right)\left(Y^H\right)-\left(L^H_{\phi Y}\eta^H\right)\left(X^V\right)-\left(L^H_{\phi Y}\eta^V\right)\left(X^V\right), \tag{3.3c}$$

$$N^{(3)}\left(X^V\right)=\left(L^H_\xi\phi\right)\left(X^V\right),\ N^{(4)}\left(X^V\right)=\left(L^H_\xi\eta^V\right)\left(X^V\right), \tag{3.3d}$$

$$N^{(3)}\left(Y^H\right)=\left(L^V_\xi\phi\right)\left(Y^H\right),\ N^{(4)}\left(Y^H\right)=\left(L^V_\xi\eta^H\right)\left(Y^H\right). \tag{3.3e}$$

It is clear that the almost contact Finsler structure $(\phi,\ \xi,\ \eta)$ is normal if and only if these four tensors vanish.

Lemma 3.1 *If* $N^{(1)}=0$*, then* $N^{(2)}=N^{(3)}=N^{(4)}=0$.

Proof If $N^{(1)}=0$, then for $X^H,Y^H,\xi^H\in N_u$, from (3.1.a) we have

$$\left[\xi^H,X^H\right]+\phi\left[\xi^H,\phi X^H\right]-\xi^H\left(\eta^H\left(X^H\right)\right)\xi^H=0. \tag{3.4}$$

Applying η^H to (3.4), we see that

$$N^{(4)}\left(X^H\right)=\left(L^H_\xi\eta^H\right)\left(X^H\right)=\xi^H\left(\eta^H\left(X^H\right)\right)-\eta^H\left[\xi^H,X^H\right]=0.$$

From this equation, we also have

$$\eta^H\left[\xi^H,\phi X^H\right]=0. \tag{3.5}$$

On the other hand, applying ϕ to (3.4), we get

$$N^{(3)}\left(X^H\right)=\left(L^H_\xi\phi\right)X^H=\phi\left[X^H,\xi^H\right]-\left[\phi X^H,\xi^H\right]=0. \tag{3.6}$$

Finally, from $N^{(1)}=0$, by using (3.6), we derive

$$\begin{aligned}0=&-\left[\phi X^H,Y^H\right]-\left[X^H,\phi Y^H\right]+\phi\left[X^H,Y^H\right]-\phi\left[\phi X^H,\phi Y^H\right]\\&-\phi Y^H\left(\eta^H\left(X^H\right)\xi^H\right)+\phi\left(Y^H\left(\eta^H\left(X^H\right)\right)\xi^H\right)+\phi X^H\left(\eta^H\left(Y^H\right)\right)\xi^H.\end{aligned} \tag{3.7}$$

Applying η^H to (3.7), we get $N^{(2)}\left(X^H,Y^H\right)=0$. Similarly, $\forall X^V,Y^V,\xi^V\in V^v_u$, if $N^{(1)}\left(X^V,Y^V\right)=0$, then $N^{(2)}\left(X^V,Y^V\right)=0$, $N^{(3)}\left(X^V\right)=0, N^{(4)}\left(X^V\right)=0$.

If $N^{(1)}\left(X^V,Y^H\right)=0$, from (3.3.a) we obtain

$$N^{(1)}\left(X^V,\xi^H\right)=\left[\xi^H,X^V\right]-\phi\left[\phi X^V,\xi^H\right]-\xi^H\left(\eta^V\left(X^V\right)\right)\xi^V=0. \tag{3.8}$$

Applying η^V and η^H to (3.8), we get (3.9):

$$\eta^V\left[\xi^H,X^V\right]=\xi^H\left(\eta^V\left(X^V\right)\right),\eta^H\left[\xi^H,X^V\right]=0. \tag{3.9}$$

Using (3.9) in (3.3.c), we obtain

$$N^{(4)}\left(X^V\right)=\left(L_\xi^H\eta^V\right)\left(X^V\right)=\xi^H\left(\eta^V\left(X^V\right)\right)-\eta^V\left[\xi^H,X^V\right]=0.$$

Applying ϕ to (3.8), we get

$$N^{(3)}\left(X^V\right)=\ \left(L_\xi^H\phi\right)X^V=\ \ \left[\xi^H,\phi X^V\right]\ +\ \ \phi\left[X^V,\xi^H\right]=0.$$

On the other hand, replacing X by ξ in (3.3.a), we obtain

$$\left[Y^H,\xi^V\right]-\phi\left[\xi^V,\phi Y^H\right]+\xi^V\left(\eta^H\left(Y^H\right)\right)\xi^H=0. \tag{3.10}$$

Applying η^H and η^V to (3.10), we get

$$\eta^H\left[\xi^V,Y^H\right]=\xi^V\left(\eta^H\left(Y^H\right)\right),\ \eta^V\left[\xi^V,Y^H\right]=0. \tag{3.11}$$

Using (3.11) in (3.3.d), we obtain

$$N^{(4)}\left(Y^H\right)=\left(L_\xi^V\eta^H\right)\left(Y^H\right)=\xi^V\left(\eta^H\left(Y^H\right)\right)-\eta^H\left[\xi^V,Y^H\right]=0.$$

Applying ϕ to (3.10) and by using (3.11), we obtain

$$N^{(3)}\left(Y^H\right)=\left(L_\xi^V\phi\right)\left(Y^H\right)=\left[\xi^V,\phi Y^H\right]+\phi\left[Y^H,\xi^V\right]=0.$$

By using (3.11), from (3.3.a), we calculate

$$\begin{aligned}0&=N_\phi\left(\phi X^V,Y^H\right)+d\eta^V\left(\phi X^V,Y^H\right)\xi^V+d\eta^H\left(\phi X^V,Y^H\right)\xi^H\\&=\left[Y^H,\phi X^V\right]+\left[\phi Y^H,X^V\right]+\phi\left[X^V,Y^H\right]-\phi\left[\phi X^V,\phi Y^H\right]-\phi Y^H\left(\eta^V\left(X^V\right)\right)\xi^V+\phi X^V\left(\eta^H\left(Y^H\right)\right)\xi^H.\end{aligned} \tag{3.12}$$

Applying η^V to (3.12), from (3.3.b), we obtain

$$\begin{aligned}0=N^{(2)}\left(X^V,Y^H\right)&=\phi X^V\left(\eta^H\left(Y^H\right)\right)-\phi Y^H\left(\eta^V\left(X^V\right)\right)-\eta^V\left[\phi X^V,Y^H\right]+\eta^H\left[\phi Y^H,X^V\right]\\&+\eta^V\left[\phi Y^H,X^V\right]-\eta^H\left[\phi X^V,Y^H\right]\ .\end{aligned}$$

□

Proposition 3.1 *The almost contact Finsler structure on V is normal if and only if*

$$N_\phi+d\eta^H\otimes\xi^H+d\eta^V\otimes\xi^V=0. \tag{3.13}$$

Let $(\phi,\ \eta,\ \xi,\ G)$ be almost metrical Finsler structure on V with contact metric. If the structure vector field ξ is a Killing vector field with respect to G, the contact structure on V is called a K-contact Finsler structure and V is called a K-contact Finsler manifold.

Lemma 3.2 *Let $(\phi,\ \eta,\ \xi,\ G)$ be a contact metrical Finsler structure on V. Then $N^{(2)}$ and $N^{(4)}$ vanish. Moreover, $N^{(3)}$ vanishes if and only if ξ is a Killing vector field with respect to G.*

Proof We have
$d\eta^H\left(\phi X^H,\phi Y^H\right)=\Omega\left(\phi X^H,\phi Y^H\right)=G\left(\phi X^H,\phi^2 Y^H\right)=-G\left(X^H,\phi^3 Y^H\right)=G\left(X^H,\phi Y^H\right)=d\eta^H\left(X^H,Y^H\right)$ from which $d\eta^H\left(\phi X^H,Y^H\right)+d\eta^H\left(X^H,\phi Y^H\right)=0$. This is equivalent to $N^{(2)}\left(X^H,Y^H\right)=0$.

On the other hand, we have $0 = G\left(X^H,\phi\xi^H\right) = d\eta^H\left(X^H,\xi^H\right) = X^H\eta^H\left(\xi^H\right)-\xi^H\eta^H\left(X^H\right)-\eta^H\left[X^H,\xi^H\right]$. Thus we obtain $\xi^H\eta^H\left(X^H\right)-\eta^H\left(\left[\xi^H,X^H\right]\right)=0$. Therefore, we have $L_\xi^H\eta^H=0$ hence $N^{(4)}\left(X^H\right)=0$.

We mention that $\left(L_\xi^H G\right)\left(X^H,\xi^H\right)=\xi^H\left(\eta^H\left(X^H\right)\right)-\eta^H\left[\xi^H,X^H\right]=\left(L_\xi^H\eta^H\right)X^H=0$. Simply, it is clear that $L_\xi^H d\eta^H=0$ and consequently, $\left(L_\xi^H d\eta^H\right)\left(X^H,Y^H\right)=\left(L_\xi^H\Omega\right)\left(X^H,Y^H\right)=0$ from which

$$\begin{aligned}0&=\xi^H G\left(X^H,\phi Y^H\right)-G\left(\left[\xi^H,X^H\right],\phi Y^H\right)-G\left(X^H,\phi\left[\xi^H,Y^H\right]\right)\\&=\left(L_\xi^H G\right)\left(X^H,\phi Y^H\right)+G\left(X^H,\left(L_\xi^H\phi\right)Y^H\right)=\left(L_\xi^H G\right)\left(X^H,\phi Y^H\right)+G\left(X^H,N^{(3)}\left(Y^H\right)\right).\end{aligned}$$

Thus ξ^H is a Killing vector field if and only if $N^{(3)}\left(Y^H\right)=0$. Similarly, we consider that $N^{(2)}\left(X^V,Y^V\right)=0$ and $N^{(4)}\left(X^V\right)=0$. Moreover, $N^{(3)}\left(X^V\right)=0$ if and only if ξ^V is a Killing vector field with respect to G^V. □

Lemma 3.3 *For an almost contact metric Finsler structure* $(\phi,\ \eta,\ \xi,\ G)$ *on V, we have*

$$\begin{aligned}&2G\left(\left(\nabla_X\phi\right)Y,Z\right)=d\Omega\left(X,\phi Y,\phi Z\right)-d\Omega\left(X,Y,Z\right)+G\left(N^{(1)}\left(Y,Z\right),\phi X\right)\\&+N^{(2)}\left(Y,Z\right)\eta\left(X\right)+d\eta\left(\phi Y,X\right)\eta\left(Z\right)-d\eta\left(\phi Z,X\right)\eta\left(Y\right).\end{aligned}\tag{3.14}$$

Proof The Finsler connection ∇ with respect to G is given by

$$\begin{aligned}&2G^H\left(\nabla_X^H Y^H,Z^H\right)=X^H G^H\left(Y^H,Z^H\right)+Y^H G^H\left(X^H,Z^H\right)-Z^H G\left(X^H,Y^H\right)+G^H\left(\left[X^H,Y^H\right],Z^H\right)\\&+G^H\left(\left[Z^H,X^H\right],Y^H\right)-G^H\left(\left[Y^H,Z^H\right],X^H\right),\end{aligned}\tag{3.15}$$

$$\begin{aligned}&2G^V\left(\nabla_X^V Y^V,Z^V\right)=X^V G^V\left(Y^V,Z^V\right)+Y^V G^V\left(X^V,Z^V\right)-Z^V G\left(X^V,Y^V\right)+G^V\left(\left[X^V,Y^V\right],Z^V\right)\\&+G^V\left(\left[Z^V,X^V\right],Y^V\right)-G^V\left(\left[Y^V,Z^V\right],X^V\right)\end{aligned}\tag{3.16}$$

$$2G^H\left(\nabla_X^V Y^H,Z^H\right)=X^V G^H\left(Y^H,Z^H\right)+G^H\left(\left[X^V,Y^H\right]^H,Z^H\right)+G^H\left(\left[Z^H,X^V\right]^H,Y^H\right)\tag{3.17}$$

$$2G^V\left(\nabla_X^H Y^V,Z^V\right)=X^H G^V\left(Y^V,Z^V\right)+G^V\left(\left[X^H,Y^V\right]^V,Z^V\right)+G^V\left(\left[Z^V,X^H\right]^V,Y^V\right).\tag{3.18}$$

Furthermore, we have

$$\begin{aligned}&d\Omega\left(X^H,Y^H,Z^H\right)=X^H\Omega\left(Y^H,Z^H\right)+Y^H\Omega\left(Z^H,X^H\right)+Z^H\Omega\left(X^H,Y^H\right)\\&-\Omega\left(\left[X^H,Y^H\right],Z^H\right)-\Omega\left(\left[Z^H,X^H\right],Y^H\right)-\Omega\left(\left[Y^H,Z^H\right],X^H\right),\end{aligned}\tag{3.19}$$

$$\begin{aligned}&d\Omega\left(X^V,Y^V,Z^V\right)=X^V\Omega\left(Y^V,Z^V\right)+Y^V\Omega\left(Z^V,X^V\right)+Z^V\Omega\left(X^V,Y^V\right)\\&-\Omega\left(\left[X^V,Y^V\right],Z^V\right)-\Omega\left(\left[Z^V,X^V\right],Y^V\right)-\Omega\left(\left[Y^V,Z^V\right],X^V\right),\end{aligned}\tag{3.20}$$

$$d\Omega\left(X^V,Y^H,Z^H\right)=X^V\Omega\left(Y^H,Z^H\right)-\Omega\left(\left[X^V,Y^H\right]^H,Z^H\right)-\Omega\left(\left[Z^H,X^V\right]^H,Y^H\right), \tag{3.21}$$

$$d\Omega\left(X^V,Y^V,Z^H\right)=Z^H\Omega\left(X^V,Y^V\right)-\Omega\left(\left[Z^H,X^V\right]^V,Y^V\right)-\Omega\left(\left[Y^V,Z^H\right]^V,X^V\right), \tag{3.22}$$

$$d\Omega\left(X^H,Y^V,Z^H\right)=Y^V\Omega\left(Z^H,X^H\right)-\Omega\left(\left[X^H,Y^V\right]^H,Z^H\right)-\Omega\left(\left[Y^V,Z^H\right]^H,X^H\right), \tag{3.23}$$

$$d\Omega\left(X^H,Y^V,Z^V\right)=X^H\Omega\left(Y^V,Z^V\right)-\Omega\left(\left[X^H,Y^V\right]^V,Z^V\right)-\Omega\left(\left[Z^V,X^H\right]^V,Y^V\right), \tag{3.24}$$

$$d\Omega\left(X^V,Y^H,Z^V\right)=Y^H\Omega\left(Z^V,X^V\right)-\Omega\left(\left[X^V,Y^H\right]^V,Z^V\right)-\Omega\left(\left[Y^H,Z^V\right]^V,X^V\right), \tag{3.25}$$

$$d\Omega\left(X^H,Y^H,Z^V\right)=Z^V\Omega\left(X^H,Y^H\right)-\Omega\left(\left[Z^V,X^H\right]^H,Y^H\right)-\Omega\left(\left[Y^H,Z^V\right]^H,X^H\right). \tag{3.26}$$

By using (2.9), from (3.15) we get

$$\begin{aligned}&2G^H\left(\left(\nabla^H_X\phi\right)Y^H,Z^H\right)=\phi Y^HG\left(X^H,Z^H\right)-Z^H\Omega\left(X^H,Y^H\right)+G^H\left(\left[X^H,\phi Y^H\right],Z^H\right)\\&+\Omega\left(\left[Z^H,X^H\right],Y^H\right)-G^H\left(\left[\phi Y^H,Z^H\right],X^H\right)+Y^H\Omega\left(X^H,Z^H\right)-\phi Z^HG\left(X^H,Y^H\right)\\&+\Omega\left(\left[X^H,Y^H\right],Z^H\right)G\left(\left[\phi Z^H,X^H\right],Y^H\right)-G^H\left(\left[Y^H,\phi Z^H\right],X^H\right).\end{aligned}$$

Also from (3.19), we calculate

$$\begin{aligned}&d\Omega\left(X^H,\phi Y^H,\phi Z^H\right)=X^H\Omega\left(Y^H,Z^H\right)+\phi Y^HG\left(Z^H,X^H\right)-\phi Y^H\left(\eta^H\left(Z^H\right)\eta^H\left(X^H\right)\right)\\&-\phi Z^HG\left(X^H,Y^H\right)+\phi Z^H\left(\eta^H\left(X^H\right)\eta^H\left(Y^H\right)\right)+G\left(\left[X^H,\phi Y^H\right],Z^H\right)\\&-\eta^H\left[X^H,\phi Y^H\right]\eta^H\left(Z^H\right)+G\left(\left[\phi Z^H,X^H\right],Y^H\right)-\eta^H\left[\phi Z^H,X^H\right]\eta^H\left(Y^H\right)\\&-\Omega\left(\left[\phi Y^H,\phi Z^H\right],X^H\right).\end{aligned} \tag{3.27}$$

Also from (3.1.a) by using (2.9), we obtain

$$\begin{aligned}&G\left(N^{(1)}\left(Y^H,Z^H\right),\phi X^H\right)=-\Omega\left(\left[Y^H,Z^H\right],X^H\right)+\Omega\left(\left[\phi Y^H,\phi Z^H\right],X^H\right)-G\left(\left[\phi Y^H,Z^H\right],X^H\right)\\&+\eta^H\left[\phi Y^H,Z^H\right]\eta^H\left(X^H\right)-G\left(\left[Y^H,\phi Z^H\right],X^H\right)+\eta^H\left[Y^H,\phi Z^H\right]\eta^H\left(X^H\right).\end{aligned} \tag{3.28}$$

From (3.1.b), we have

$$\begin{aligned}&N^{(2)}\left(Y^H,Z^H\right)\eta^H\left(X^H\right)=\phi Y^H\left(\eta^H\left(Y^H\right)\right)\eta^H\left(X^H\right)-\phi Z^H\left(\eta^H\left(Y^H\right)\right)\eta^H\left(X^H\right)\\&-\eta^H\left[\phi Y^H,Z^H\right]\eta^H\left(X^H\right)-\eta^H\left[Y^H,\phi Z^H\right]\eta^H\left(X^H\right).\end{aligned} \tag{3.29}$$

By using (3.27), (3.28) and (3.29), we have the equation.

Similarly by using (3.2.a), (3.2.b), (2.9), (3.16) and (3.20), we get

$$\begin{aligned}&2G\left(\left(\nabla^V_X\phi\right)Y^V,Z^V\right)=d\Omega\left(X^V,\phi Y^V,\phi Z^V\right)-d\Omega\left(X^V,Y^V,Z^V\right)+G\left(N^{(1)}\left(Y^V,Z^V\right),\phi X^V\right)\\&+N^{(2)}\left(Y^V,Z^V\right)\eta^V\left(X^V\right)+d\eta^V\left(\phi Y^V,X^V\right)\eta^V\left(Z^V\right)-d\eta^V\left(\phi Z^V,X^V\right)\eta^V\left(Y^V\right).\end{aligned}$$

By using (2.9), (3.1.a), (3.1.b), (3.17) and (3.21), we calculate

$$\begin{aligned}&d\Omega\left(X^V,\phi Y^H,\phi Z^H\right)-\Omega\left(X^V,Y^H,Z^H\right)+d\eta^H\left(\phi Y^H,X^V\right)\eta^H\left(Z^H\right)-d\eta^H\left(\phi Z^H,X^V\right)\eta^H\left(Y^H\right)\\&=G^H\left(\left[X^V,\phi Y^H\right]^H,Z^H\right)+G^H\left(\left[\phi Z^H,X^V\right]^H,Y^H\right)+\Omega\left(\left[X^V,Y^H\right]^H,Z^H\right)+\Omega\left(\left[Z^H,X^V\right]^H,Y^H\right)\\&=2G\left(\left(\nabla^V_X\phi\right)Y^H,Z^H\right).\end{aligned}$$

By using (2.9) and (3.18), (3.24), (3.2.a) and (3.2.b), we obtain

$$\begin{aligned}
&d\Omega\left(X^H,\phi Y^V,\phi Z^V\right)-d\Omega\left(X^H,Y^V,Z^V\right)+G^V\left(N^{(1)}\left(Y^V,Z^V\right),\phi X^H\right)\\
&+N^{(2)}\left(Y^V,Z^V\right)\eta^V\left(X^H\right)+d\eta^V\left(\phi Y^V,X^H\right)\eta^V\left(Z^V\right)-d\eta^V\left(\phi Z^V,X^H\right)\eta^V\left(Y^V\right)\\
&+d\eta^H\left(\phi Y^V,X^H\right)\eta^H\left(Z^V\right)-d\eta^H\left(\phi Z^V,X^H\right)\eta^H\left(Y^V\right)\\
&=\ G\left(\left[X^H,\phi Y^V\right]^V,Z^V\right)-\eta^V\left(Z^V\right)\eta^V\left[X^H,\phi Y^V\right]^V+G\left(\left[\phi Z^V,X^H\right]^V,Y^V\right)\\
&\ \ -\eta^V\left(Y^V\right)\eta^V\left[\phi Z^V,X^H\right]^V+\Omega\left(\left[X^H,Y^V\right]^V,Z^V\right)+\Omega\left(\left[Z^V,X^H\right]^V,Y^V\right)\\
&\ \ -\eta^V\left[\phi Y^V,X^H\right]^V\eta^V\left(Z^V\right)+\eta^V\left[\phi Z^V,X^H\right]^V\eta^V\left(Y^V\right)\\
&=2G^V\left(\left(\nabla^H_X\phi\right)Y^V,Z^V\right).
\end{aligned}$$

□

Lemma 3.4 *For a contact metric Finsler structure* $(\phi,\ \eta,\ \xi,\ G)$ *of* V *with* $\Omega=\ d\eta$ *and* $N^{(2)}=\ 0$*, we get*$2G\left(\left(\nabla_X\phi\right)Y,Z\right)=\ G\left(N^{(1)}\left(Y,Z\right),\phi X\right)+d\eta\left(\phi Y,X\right)\eta\left(Z\right)-d\eta\left(\phi Z,X\right)\eta\left(Y\right)$*. Especially we have* $\nabla_\xi\phi=0$.

Proof The first equation is trivial by the assumption. We prove that $\nabla_\xi\phi=0$.

From $N^{(2)}=0$ we have $d\eta^H\left(X^H,\xi^H\right)=0$ and $d\eta^V\left(X^V,\xi^V\right)=0$. Thus the first equation implies that $\nabla^H_\xi\phi=0$ and $\nabla^V_\xi\phi=0$. □

Proposition 3.2 *Let* $(\phi,\ \eta,\ \xi,\ G)$ *be a contact metrical Finsler structure on* V*. Then* $(\phi,\ \eta,\ \xi,\ G)$ *is a K-contact Finsler structure if and only if* $N^{(3)}$ *vanishes.*

Proposition 3.3 *Let* $(\phi,\ \eta,\ \xi,\ G)$ *be contact metrical Finsler structure on* V*. Then* $(\phi,\ \eta,\ \xi,\ G)$ *is a K-contact structure if and only if*

$$\nabla_X\xi^H=-\frac{1}{2}\phi X^H,\nabla_X\xi^V=-\frac{1}{2}\phi X^V. \tag{3.30}$$

Proof If the structure vector field ξ is a Killing vector field with respect to G, then we have

$$L^H_\xi G^H=0,L^V_\xi G^V=0. \tag{3.31}$$

$\forall X^H,Y^H,\xi^H\in N_u$ and $\forall X^V,Y^V,\xi^V\ \in\ V^v_u$ from (3.31), we can get

$$G\left(\nabla^H_X\xi^H,Y^H\right)=-G\left(X^H,\nabla^H_Y\xi^H\right),G\left(\nabla^V_X\xi^V,Y^V\right)=-G\left(X^V,\nabla^V_Y\xi^V\right). \tag{3.32}$$

Replacing Y^H by ξ^H and Z^H by Y^H in (3.15), we have

$$\begin{aligned}
2G\left(\nabla^H_X\xi^H,Y^H\right)=&\ X^H\eta^H\left(Y^H\right)+\xi^HG\left(X^H,Y^H\right)-Y^H\eta^H\left(X^H\right)\\
&+G\left(\left[X^H,\xi^H\right],Y^H\right)-\eta^H\left([X,Y]^H\right)-G\left(\left[\xi^H,Y^H\right],X^H\right).
\end{aligned} \tag{3.33}$$

Replacing Y^H by ξ^H , X^H by Y^H and Z^H by X^H in (3.15), we can get

$$\begin{aligned}
2G\left(\nabla^H_Y\xi^H,X^H\right)=&\ Y^H\eta^H\left(X^H\right)+\xi^HG\left(X^H,Y^H\right)-X^H\eta^H\left(Y^H\right)\\
&+G\left(\left[Y^H,\xi^H\right],X^H\right)+\eta^H\left([X,Y]^H\right)-G\left(\left[\xi^H,X^H\right],Y^H\right).
\end{aligned} \tag{3.34}$$

Using (3.33) and (3.34), we get $G\left(\nabla^H_X\xi^H, Y^H\right) - G\left(X^H, \nabla^H_Y\xi^H\right) = d\eta^H\left(X^H, Y^H\right)$.

Since ξ^H is a Killing vector field with respect to G^H, using (3.32), we obtain $d\eta^H\left(X^H, Y^H\right) = 2G\left(\nabla^H_X\xi^H, Y^H\right) = G\left(X^H, \phi Y^H\right) = -G\left(\phi X^H, Y^H\right)$ and $\nabla^H_X\xi^H = -\frac{1}{2}\phi X^H$.

Similarly for $X^V, Y^V, \xi^V \in V^v_u$, from (3.16) and (3.32), we get $\nabla^V_X\xi^V = -\frac{1}{2}\phi X^V$. □

Example 3.1 *Let $V(M) = \{V, \pi, M\}$ be a vector bundle with the total space $V = R^6$ is a 6-dimensional C^∞-manifold and the base space $M = R^3$ is a 3-dimensional C^∞-manifold. Let $x^i, 1 \le i \le 3$ and $y^a, 1 \le a \le 3$ be the coordinates of $u = (x, y) \in V$ that is $u = \left(x^1, x^2, x^3, y^1, y^2, y^3\right) \in V$.*

The local base of N_u is $\left(\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2}, \frac{\delta}{\delta x^3}\right)$ and that of V^v_u is $\left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3}\right)$.

Let$X = X^i\frac{\delta}{\delta x^i} + \tilde{X}^a\frac{\partial}{\partial y^a}\forall X \in T_u(V)$. Then $X^H = X^1\frac{\delta}{\delta x^1} + X^2\frac{\delta}{\delta x^2} + X^3\frac{\delta}{\delta x^3}$, $X^V = \tilde{X}^1\frac{\partial}{\partial y^1} + \tilde{X}^2\frac{\partial}{\partial y^2} + \tilde{X}^3\frac{\partial}{\partial y^3}$ where $X^H \in N_u$ and $X^V \in V^v_u$. Similarly Y can be written as

$$Y^H = Y^1\frac{\delta}{\delta x^1} + Y^2\frac{\delta}{\delta x^2} + Y^3\frac{\delta}{\delta x^3}, Y^V = \tilde{Y}^1\frac{\partial}{\partial y^1} + \tilde{Y}^2\frac{\partial}{\partial y^2} + \tilde{Y}^3\frac{\partial}{\partial y^3}.$$

Let η be a 1-form, $\eta = \eta_i dx^i + \tilde{\eta}_a\delta y^a$ then $\eta^H = \eta_1 dx^1 + \eta_2 dx^2 + \eta_3 dx^3$ and $\eta^V = \tilde{\eta}_1\delta y^1 + \tilde{\eta}_2\delta y^2 + \tilde{\eta}_3\delta y^3$ where $\eta = \eta^H + \eta^V$ and $\eta^H\left(X^V\right) = 0$ and $\eta^V\left(X^H\right) = 0$. We put $\eta^H = \frac{1}{2}\left(dx^3 - x^2dx^1\right)$ and $\eta^V = \frac{1}{2}\left(\delta y^3 - y^2\delta y^1\right)$. Then the structure vector field ξ is given by $\xi = 2\left(\frac{\delta}{\delta x^3} + \frac{\partial}{\partial y^3}\right)$ and ξ is decomposed as $\xi^H = 2\frac{\delta}{\delta x^3}$ and $\xi^V = 2\frac{\partial}{\partial y^3}$. The tensor field ϕ^H of type $(1,1)$ and ϕ^V of type $(1,1)$ by a matrix form is given by

$$\phi^H = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & x^2 & 0 \end{bmatrix}, \phi^V = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & y^2 & 0 \end{bmatrix}.$$

The Riemann metric tensor field $G = G^H + G^V$ is given by

$$G^H = \frac{1}{4}\left(dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + \eta^H \otimes \eta^H\right) = \frac{1}{4}\left(\left(1 + \left(x^2\right)^2\right)\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 - 2x^2\left(dx^1\right)\left(dx^3\right)\right)$$

$$G^V = \frac{1}{4}\left(\delta y^1 \otimes \delta y^1 + \delta y^2 \otimes \delta y^2 + \eta^V \otimes \eta^V\right) = \frac{1}{4}\left(\left(1 + \left(y^2\right)^2\right)\left(\delta y^1\right)^2 + \left(\delta y^2\right)^2 + \left(\delta y^3\right)^2 - 2y^2\left(\delta y^1\right)\left(\delta y^3\right)\right).$$

Thus we give a metric tensor field G by a matrix form

$$G^H = \frac{1}{4}\begin{bmatrix} 1 + \left(x^2\right)^2 & 0 & -x^2 \\ 0 & 1 & 0 \\ -x^2 & 0 & 1 \end{bmatrix}, G^V = \frac{1}{4}\begin{bmatrix} 1 + \left(y^2\right)^2 & 0 & -y^2 \\ 0 & 1 & 0 \\ -y^2 & 0 & 1 \end{bmatrix}.$$

We analyze that $\eta^H\left(\xi^H\right) = 1$, $\eta^V\left(\xi^V\right) = 1$, $\phi^H\left(\xi^H\right) = 0$, $\eta^H\left(\xi^V\right) = 0$, $\phi^V\left(\xi^V\right) = 0$, $\eta^V\left(\xi^H\right) = 0$, $\left(\phi^H\right)^2 X^H = -X^H + \eta^H\left(X^H\right)\xi^H$ and $\left(\phi^V\right)^2 X^V = -X^V + \eta^V\left(X^V\right)\xi^V$, hence (ϕ, ξ, η) is an almost contact Finsler structure on R^6.

On the other hand, we formulize that

$$\eta^H\left(X^H\right)=G^H\left(X^H,\xi^H\right),\eta^V\left(X^V\right)=G^V\left(X^V,\xi^V\right)$$

$$G^H\left(\phi X^H,\phi Y^H\right)=G^H\left(X^H,Y^H\right)-\eta^H\left(X^H\right)\eta^H\left(Y^H\right),\ G^V\left(\phi X^V,\phi Y^V\right)=G^V\left(X^V,Y^V\right)-\eta^V\left(X^V\right)\eta^V\left(Y^V\right)$$

$$\eta^H\left(X^H\right)=\frac{1}{2}\left(dx^3-x^2dx^1\right)\left(X^1\frac{\delta}{\delta x^1}+X^2\frac{\delta}{\delta x^2}+X^3\frac{\delta}{\delta x^3}\right)=\frac{1}{2}\left(X^3-X^1x^2\right) \tag{3.35}$$

$$G^H\left(X^H,\xi^H\right)=\tfrac{1}{4}\left[\begin{array}{ccc}X^1 & X^2 & X^3\end{array}\right]\left[\begin{array}{ccc}1+\left(x^2\right)^2 & 0 & -x^2\\ 0 & 1 & 0\\ -x^2 & 0 & 1\end{array}\right]\left[\begin{array}{c}0\\0\\2\end{array}\right]=\tfrac{1}{4}\left[\begin{array}{ccc}X^1 & X^2 & X^3\end{array}\right]\left[\begin{array}{c}-2x^2\\0\\2\end{array}\right]$$
$$=\tfrac{1}{4}\left(-2X^1x^2+2X^3\right)=\tfrac{1}{2}\left(X^3-X^1x^2\right). \tag{3.36}$$

From (3.35) *and* (3.36) *we get* $\eta^H\left(X^H\right)=G^H\left(X^H,\xi^H\right)$*. Similarly, we have*

$$\eta^V\left(X^V\right)=\frac{1}{2}\left(\delta y^3-y^2\delta y^1\right)\left(\tilde{X}^1\frac{\partial}{\partial y^1}+\tilde{X}^2\frac{\partial}{\partial y^2}+\tilde{X}^3\frac{\partial}{\partial y^3}\right)=\frac{1}{2}\left(\tilde{X}^3-\tilde{X}^1y^2\right)\eta^V=G^V\left(X^V,\xi^V\right),$$

$$\phi^H\left(X^H\right)=\left(X^2,-X^1,X^2x^2\right),\phi^H\left(Y^H\right)=\left(Y^2,-Y^1,Y^2x^2\right),$$

$$\phi^V\left(X^V\right)=\left(\tilde{X}^2,-\tilde{X}^1,\tilde{X}^2y^2\right),\phi^V\left(Y^V\right)=\left(\tilde{Y}^2,-\tilde{Y}^1,\tilde{Y}^2y^2\right),$$

$$G^H\left(\phi^HX^H,\phi^HY^H\right)=\frac{1}{4}\left(X^1Y^1+X^2Y^2\right),G^V\left(\phi^VX^V,\phi^VY^V\right)=\frac{1}{4}\left(\tilde{X}^1\tilde{Y}^1,+\tilde{X}^2\tilde{Y}^2\right),$$

$$G^H\left(X^H,Y^H\right)=\frac{1}{4}\left\{\left(Y^1\left(1+\left(x^2\right)^2\right)-Y^3x^2\right)X^1+X^2Y^2+X^3\left(Y^3-Y^1x^2\right)\right\},$$

$$G^V\left(X^V,Y^V\right)=\frac{1}{4}\left\{\left(\tilde{Y}^1\left(1+\left(y^2\right)^2\right)-\tilde{Y}^3y^2\right)\tilde{X}^1+\tilde{X}^2\tilde{Y}^2+\tilde{X}^3\left(\tilde{Y}^3-\tilde{Y}^1y^2\right)\right\},$$

$$\eta^H\left(X^H\right)\eta^H\left(Y^H\right)=\frac{1}{4}\left[X^3Y^3+X^1Y^1\left(x^2\right)^2-X^1Y^3x^2-X^3Y^1x^2\right],$$

$$\eta^V\left(X^V\right)\eta^V\left(Y^V\right)=\frac{1}{4}\left[\tilde{X}^3\tilde{Y}^3+\tilde{X}^1\tilde{Y}^1\left(y^2\right)^2-\tilde{X}^1\tilde{Y}^3y^2-\tilde{X}^3\tilde{Y}^1y^2\right].$$

Thus, we get $G^H\left(\phi X^H,\phi Y^H\right)=G^H\left(X^H,Y^H\right)-\eta^H\left(X^H\right)\eta^H\left(Y^H\right),\ G^V\left(\phi X^V,\phi Y^V\right)=G^V\left(X^V,Y^V\right)-\eta^V\left(X^V\right)\eta^V\left(Y^V\right)$ *and hence* (ϕ, ξ, η, G) *is an almost contact Finsler metric structure.*

$$G^H\left(X^H,\phi Y^H\right)=\frac{1}{4}\left[\begin{array}{ccc}X^1 & X^2 & X^3\end{array}\right]\left[\begin{array}{ccc}1+\left(x^2\right)^2 & 0 & -x^2\\ 0 & 1 & 0\\ -x^2 & 0 & 1\end{array}\right]\left[\begin{array}{c}Y^2\\-Y^1\\Y^2x^2\end{array}\right]=\frac{1}{4}\left(X^1Y^2-X^2Y^1\right).$$

Also, we know that $d\eta^H=\frac{1}{2}\left(dx^1\wedge dx^2\right)$*. By using this equality, we obtain* $d\eta^H\left(X^H,Y^H\right)=G^H\left(X^H,\phi Y^H\right)$*. Similarly we get*

$$G^V\left(X^V,\phi Y^V\right)=\frac{1}{4}\left[\begin{array}{ccc}\tilde{X}^1 & \tilde{X}^2 & \tilde{X}^3\end{array}\right]\left[\begin{array}{ccc}1+\left(y^2\right)^2 & 0 & -y^2\\ 0 & 1 & 0\\ -y^2 & 0 & 1\end{array}\right]\left[\begin{array}{c}\tilde{Y}^2\\-\tilde{Y}^1\\\tilde{Y}^2y^2\end{array}\right]=\frac{1}{4}\left(\tilde{X}^1\tilde{Y}^2-\tilde{X}^2\tilde{Y}^1\right).$$

By using $d\eta^V = \frac{1}{2}\left(\delta y^1 \wedge \delta y^2\right)$*, we derived* $d\eta^V\left(X^V, Y^V\right) = G^V\left(X^V, \phi Y^V\right)$*. As a result we come up with the following equation:*

$$d\eta^H\left(X^H, Y^H\right) = G^H\left(X^H, \phi Y^H\right) = \Omega\left(X^H, Y^H\right), d\eta^V\left(X^V, Y^V\right) = G^V\left(X^V, \phi Y^V\right) = \Omega\left(X^V, Y^V\right). \quad (3.37)$$

Then the almost contact metrical Finsler structure (ϕ, ξ, η, G) *is called almost Sasakian Finsler structure.*

Because of $\eta \wedge (d\eta) \neq 0$, (ϕ, ξ, η, G) *is a contact metrical Finsler structure. The vector fields* $X_1 = 2\left(\frac{\delta}{\delta x^2} + \frac{\partial}{\partial y^2}\right)$, $X_2 = 2\left(\frac{\delta}{\delta x^1} + x^2\frac{\delta}{\delta x^3} + \frac{\partial}{\partial y^1} + y^2\frac{\partial}{\partial y^3}\right)$ *and* $\xi = 2\left(\frac{\delta}{\delta x^3} + \frac{\partial}{\partial y^3}\right)$ *form a* ϕ*-basis for the contact metrical Finsler structure, where these are decomposed as*

$$X_1^H = 2\left(\frac{\delta}{\delta x^2}\right), X_1^V = 2\left(\frac{\partial}{\partial y^2}\right), X_2^H = 2\left(\frac{\delta}{\delta x^1} + x^2\frac{\delta}{\delta x^3}\right), X_2^V = 2\left(\frac{\partial}{\partial y^1} + y^2\frac{\partial}{\partial y^3}\right),$$

$$\xi^H = 2\left(\frac{\delta}{\delta x^3}\right), \xi^V = 2\left(\frac{\partial}{\partial y^3}\right).$$

On the other hand, we can see that$N_\phi + d\eta \otimes \xi = 0$*, that is* $N_\phi^H + d\eta^H \otimes \xi^H = 0$ *and* $N_\phi^V + d\eta^V \otimes \xi^V = 0$*. Hence the contact metrical Finsler structure is normal.*

4. The curvature of a Finsler connection

The curvature of a Finsler connection ∇ is given by:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \forall X,Y,Z \in T_u(V). \quad (4.1)$$

As ∇ preserves by parallelism the horizontal and the vertical distributions, from (4.1) we have that the operator $R(X,Y)$ carries horizontal vector fields into horizontal vector fields and vertical vector fields into verticals. Consequently,

$$R(X,Y)Z = R^H(X,Y)Z^H + R^V(X,Y)Z^V, \forall X,Y,Z \in T_u(V). \quad (4.2)$$

Noting that the operator $R(X,Y)$ is skew-symmetric with respect to X and Y, a theorem follows [1]:

Theorem 4.1 *The curvature of a Finsler connection* ∇ *on the tangent space* $T_u(V)$ *is completely determined by the following six Finsler tensor fields:*

$$\begin{aligned}
R\left(X^H, Y^H\right)Z^H &= \nabla_X^H\nabla_Y^H Z^H - \nabla_Y^H\nabla_X^H Z^H - \nabla_{[X^H,Y^H]}Z^H,\\
R\left(X^H, Y^H\right)Z^V &= \nabla_X^H\nabla_Y^H Z^V - \nabla_Y^H\nabla_X^H Z^V - \nabla_{[X^H,Y^H]}Z^V,\\
R\left(X^V, Y^H\right)Z^H &= \nabla_X^V\nabla_Y^H Z^H - \nabla_Y^H\nabla_X^V Z^H - \nabla_{[X^V,Y^H]}Z^H,\\
R\left(X^V, Y^H\right)Z^V &= \nabla_X^V\nabla_Y^H Z^V - \nabla_Y^H\nabla_X^V Z^V - \nabla_{[X^V,Y^H]}Z^V,\\
R\left(X^V, Y^V\right)Z^H &= \nabla_X^V\nabla_Y^V Z^H - \nabla_Y^V\nabla_X^V Z^H - \nabla_{[X^V,Y^V]}Z^H,\\
R\left(X^V, Y^V\right)Z^V &= \nabla_X^V\nabla_Y^V Z^V - \nabla_Y^V\nabla_X^V Z^V - \nabla_{[X^V,Y^V]}Z^V.
\end{aligned} \quad (4.3)$$

Then the curvature tensor of a Finsler connection ∇ *has only three different components with respect to the Berwald basis. These are given by:*

$$R\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^h} = R_{h\ jk}^{i}\frac{\delta}{\delta x^i}, R\left(\frac{\partial}{\partial y^k}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^h} = P_{h\ jk}^{i}\frac{\delta}{\delta x^i}, R\left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial x^j}\right)\frac{\delta}{\delta x^h} = S_{h\ jk}^{i}\frac{\delta}{\delta x^i}. \quad (4.4)$$

These three components are the first, third and fifth Finsler tensors from (4.3). *The other three Finsler tensors from* (4.3) *have the same local components* $R^i_{h\,jk}$*,* $P^i_{h\,jk}$*, and* $S^i_{h\,jk}$*.*

$$\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^h}=R^i_{h\,jk}\frac{\partial}{\partial y^i},R\left(\frac{\partial}{\partial y^k},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^h}=P^i_{h\,jk}\frac{\partial}{\partial y^i},R\left(\frac{\partial}{\partial y^k},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial y^h}=S^i_{h\,jk}\frac{\partial}{\partial y^i}. \tag{4.5}$$

(4.5)

So, a Finsler connection $\nabla\Gamma=(N^i_j,\ F^i_{jk},\ C^i_{jk})$ *has only three local components* $R^i_{h\,jk}$*,* $P^i_{h\,jk}$*,* $S^i_{h\,jk}$ *[1].*

For a Finsler connection ∇*, consider the torsion T, defined as usual*

$$T(X,Y)=\nabla_XY-\nabla_YX-[X,Y],\forall X,Y\in T_u(V). \tag{4.6}$$

Breaking T down into horizontal and vertical parts gives the torsion of a Finsler connection, ∇ *on* $T_u(V)$ *is completely determined by the following Finsler tensor fields [1]:*

$$\begin{aligned}&T^H\left(X^H,Y^H\right)=\nabla^H_XY^H-\nabla^H_YX^H-\left[X^H,Y^H\right]^H,T^V\left(X^H,Y^H\right)=-\left[X^H,Y^H\right]^V,\\&T^H\left(X^H,Y^V\right)=-\nabla^V_YX^H-\left[X^H,Y^V\right]^H,T^V\left(X^H,Y^V\right)=\nabla^H_XY^V-\left[X^H,Y^V\right]^V,\\&T^V\left(X^V,Y^V\right)=\nabla^V_XY^V-\nabla^V_YX^V-\left[X^V,Y^V\right]^V.\end{aligned} \tag{4.7}$$

Let ∇ *be the torsion free Finsler connection, then we get*

$$\begin{aligned}&\left[X^H,Y^H\right]^H=\nabla^H_XY^H-\nabla^H_YX^H,\ \left[X^H,Y^H\right]^V=0,\ \left[X^H,Y^V\right]^H=-\nabla^V_YX^H,\\&\left[X^H,Y^V\right]^V=\nabla^H_XY^V,\ \left[X^V,Y^V\right]^V=\nabla^V_XY^V-\nabla^V_YX^V.\end{aligned} \tag{4.8}$$

Theorem 4.2 *In order for a (n+m)-dimensional Finsler manifold V to be K-contact, it is necessary and sufficient that the following two conditions are satisfied:*

1. V admits a unit Killing vector field ξ*;*

2. The flag curvature for plane sections containing ξ *are equal to* $\frac{1}{4}$ *at every point of V.*

Proof Let V be a K-contact manifold. From (4.3) and (3.30), we have

$$\begin{aligned}G^H\left(R\left(X^H,\xi^H\right)\xi^H,X^H\right)&=G^H\left(\nabla^H_X\nabla^H_\xi\xi^H-\nabla^H_\xi\nabla^H_X\xi^H-\nabla^H_{[X^H,\xi^H]}\xi^H,X^H\right)\\&=\tfrac{1}{4}G^H\left(X^H,X^H\right)=\tfrac{1}{4},G^V\left(R\left(X^V,\xi^V\right)\xi^V,X^V\right)\\&=G^V\left(\nabla^V_X\nabla^V_\xi\xi^V-\nabla^V_\xi\nabla^V_X\xi^V-\nabla^V_{[X,\xi]}\xi^V,X^V\right)=\tfrac{1}{4}G^V\left(X^V,X^V\right)=\tfrac{1}{4},\end{aligned}$$

where X^H is a unit vector field orthogonal to ξ^H and X^V is a unit vector field orthogonal to ξ^V. Hence

$$\begin{aligned}G(R(X,\xi)\xi,X)&=G^H\left(R^H(X,\xi)\xi^H,X^H\right)+G^V\left(R^V(X,\xi)\xi^V,X^V\right)\\&=\tfrac{1}{4}\left(G^H\left(X^H,X^H\right)+G^V\left(X^V,X^V\right)\right)=\tfrac{1}{4}G(X,X)=\tfrac{1}{4}.\end{aligned}$$

Thus we obtain $K(X,\xi)=\frac{G(R(X,\xi)\xi,X)}{G(X,X)}=\frac{1}{4}$.

Conversely, we suppose that M satisfies the conditions (1.1) and (1.2). Since ξ is a Killing vector field, we have

$$d\eta\left(X^H,Y^H\right)=\left(G^H\left(\nabla^H_X\xi^H,Y^H\right)-G^H\left(\nabla^H_Y\xi^H,X^H\right)\right)=-2G\left(\nabla^H_Y\xi^H,X^H\right)=G\left(X^H,\phi Y^H\right),$$

$$d\eta\left(X^V, Y^V\right) = G\left(X^V, \phi Y^V\right).$$

Consequently, $(\phi,\ \eta,\ \xi,\ G)$ is a K-contact Finsler structure on V.

Let $(\phi,\ \eta,\ \xi,\ G)$ be a contact metrical Finsler structure on V. If the metric structure of V is normal, then V is mentioned to have a Sasakian Finsler structure and V is called a *Sasakian Finsler manifold.* □

Theorem 4.3 *An almost contact metrical Finsler structure* $(\phi,\ \eta,\ \xi,\ G)$ *on* V *is a Sasakian Finsler structure if and only if*

$$\left(\nabla^H_X\phi\right)Y^H = \frac{1}{2}\left[G^H\left(X^H, Y^H\right)\xi^H - \eta^H\left(Y^H\right)X^H\right], \tag{4.9}$$

$$\left(\nabla^V_X\phi\right)Y^V = \frac{1}{2}\left[G^V\left(X^V, Y^V\right)\xi^V - \eta^V\left(Y^V\right)X^V\right]. \tag{4.10}$$

Proof If the structure is normal, we have $\Omega = d\eta$ and $N^{(1)} = N^{(2)} = 0$. Thus, by using (3.14), (3.18) and (3.19), we get $2G^H\left(\left(\nabla^H_X\phi\right)Y^H, \xi^H\right) = -d\Omega\left(X^H, Y^H, \xi^H\right) + d\eta\left(\phi Y^H, X^H\right) = G^H\left(Y^H, X^H\right) - \eta^H\left(X^H\right)\eta^H\left(Y^H\right)$. Thus we have $\left(\nabla^H_X\phi\right)Y^H = \frac{1}{2}\left[G^H\left(X^H, Y^H\right)\xi^H - \eta^H\left(Y^H\right)X^H\right]$.

Similarly, from Lemma 3.3, we have

$$2G^V\left(\left(\nabla^V_X\phi\right)Y^V, \xi^V\right) = G^V\left(Y^V, X^V\right) - \eta^V\left(X^V\right)\eta^V\left(Y^V\right).$$

Thus we get $\left(\nabla^V_X\phi\right)Y^V = \frac{1}{2}\left[G^V\left(X^V, Y^V\right)\xi^V - \eta^V\left(Y^V\right)X^V\right]$.

Conversely, we suppose that the structure satisfies (4.9) and (4.10). Putting $Y^H = \xi^H$ in (4.9) we have $-\phi\nabla^H_X\xi^H = \frac{1}{2}\left(\eta^H\left(X^H\right)\xi^H - X^H\right)$, and putting $Y^V = \xi^V$ in (4.10), we can get $-\phi\nabla^V_X\xi^V = \frac{1}{2}\left(\eta^V\left(X^V\right)\xi^V - X^V\right)$, and hence, applying ϕ to this, we obtain $\nabla^H_X\xi^H = -\frac{1}{2}\phi X^H$ and $\nabla^V_Y\xi^V = -\frac{1}{2}\phi Y^V$. Since ξ is skew-symmetric, we prove that ξ^H and ξ^V is a Killing vector field. Moreover, we obtain

$$d\eta\left(X^H, Y^H\right) = \frac{1}{2}\left(\left(\nabla^H_X\eta\right)Y^H - \left(\nabla^H_Y\eta\right)X^H\right) = G\left(X^H, \phi Y^H\right) = \Omega\left(X^H, Y^H\right),$$

$$d\eta\left(X^V, Y^V\right) = \frac{1}{2}\left(\left(\nabla^V_X\eta\right)Y^V - \left(\nabla^V_Y\eta\right)X^V\right) = G\left(X^V, \phi Y^V\right) = \Omega\left(X^V, Y^V\right).$$

Thus the structure is a contact metric Sasakian structure.

If $(\phi,\ \eta,\ \xi,\ G)$ is a Sasakian Finsler structure on V, from (4.9) and (4.10) we obtain

$$R\left(X^H, Y^H\right)\xi^H = \frac{1}{4}\left(\eta^H\left(Y^H\right)X^H - \eta^H\left(X^H\right)Y^H\right), \tag{4.11}$$

$$R\left(X^V, Y^V\right)\xi^V = \frac{1}{4}\left(\eta^V\left(Y^V\right)X^V - \eta^V\left(X^V\right)Y^V\right). \tag{4.12}$$

That is, we have

$$\begin{aligned} R(X,Y)\xi &= R^H(X,Y)\xi^H + R^V(X,Y)\xi^V = R\left(X^H, Y^H\right)\xi^H + R\left(X^V, Y^V\right)\xi^V \\ &= \tfrac{1}{4}\left[\eta^H\left(Y^H\right)X^H + \eta^V\left(Y^V\right)X^V - \eta^H\left(X^H\right)Y^H - \eta^V\left(X^V\right)Y^V\right]. \end{aligned} \tag{4.13}$$

□

Theorem 4.4 *Let V be a (n+m)-dimensional Finsler manifold admitting a unit Killing vector field ξ. Then V is a Sasakian Finsler manifold if and only if*

$$R(X,\xi)Y = \frac{1}{4}\left[-G^H(X,Y)\xi^H - G^V(X,Y)\xi^V + \eta^H\left(Y^H\right)X^H + \eta^V\left(Y^V\right)X^V\right]. \qquad (4.14)$$

Proof

$$R^H(X,\xi)Y^H = \nabla^H_X\nabla^H_\xi Y^H - \nabla^H_\xi\nabla^H_X Y^H - \nabla^H_{[X,\xi]}Y^H = -\tfrac{1}{2}\left(\nabla^H_X\phi\right)Y^H$$
$$= \tfrac{1}{4}\left[-G\left(X^H,Y^H\right)\xi^H + \eta^H\left(Y^H\right)X^H\right],$$

$$R^V(X,\xi)Y^V = -\frac{1}{2}\left(\nabla^V_X\phi\right)Y^V = -\frac{1}{4}\left[G\left(X^V,Y^V\right)\xi^V - \eta^V\left(Y^V\right)X^V\right].$$

From these equations mentioned above, we have the equation.

Let $(.\phi,\ \eta,\ \xi,\ G)$ be a Sasakian Finsler structure on *V*. From (4.9) and (4.10), we realize that

$$R\left(X^H,Y^H\right)\phi Z^H = \phi R\left(X^H,Y^H\right)Z^H + \tfrac{1}{4}\left\{G\left(\phi X^H,Z^H\right)Y^H - G\left(Y^H,Z^H\right)\phi X^H + G\left(X^H,Z^H\right)\phi Y^H - G\left(\phi Y^H,Z^H\right)X^H\right\}, \qquad (4.15)$$

$$R\left(X^V,Y^V\right)\phi Z^V = \phi R\left(X^V,Y^V\right)Z^V + \tfrac{1}{4}\left\{G\left(\phi X^V,Z^V\right)Y^V - G\left(Y^V,Z^V\right)\phi X^V + G\left(X^V,Z^V\right)\phi Y^V - G\left(\phi Y^V,Z^V\right)X^V\right\}, \qquad (4.16)$$

$$R\left(X^H,Y^H\right)\phi Z^V = \phi R\left(X^H,Y^H\right)Z^V, \qquad (4.17)$$

$$R\left(X^H,Y^V\right)\phi Z^V = \phi R\left(X^H,Y^V\right)Z^V - \frac{1}{4}\left\{G\left(Y^V,Z^V\right)\phi X^H - G\left(\phi Y^V,Z^V\right)X^H\right\}, \qquad (4.18)$$

$$R\left(X^V,Y^H\right)\phi Z^V = \phi R\left(X^V,Y^H\right)Z^V + \frac{1}{4}\left\{G\left(\phi X^V,Z^V\right)Y^H + G\left(X^V,Z^V\right)\phi Y^H\right\}, \qquad (4.19)$$

$$R\left(X^V,Y^V\right)\phi Z^H = \phi R\left(X^V,Y^V\right)Z^H, \qquad (4.20)$$

$$R\left(X^V,Y^H\right)\phi Z^H = \phi R\left(X^V,Y^H\right)Z^H - \frac{1}{4}\left\{G\left(Y^H,Z^H\right)\phi X^V - G\left(\phi Y^H,Z^H\right)X^V\right\}, \qquad (4.21)$$

$$R\left(X^H,Y^V\right)\phi Z^H = \phi R\left(X^H,Y^V\right)Z^H + \frac{1}{4}\left\{G\left(\phi X^H,Z^H\right)Y^V + G\left(X^H,Z^H\right)\phi Y^V\right\}, \qquad (4.22)$$

$$R(X,Y)\phi Z = R(X,Y)\phi Z^H + R(X,Y)\phi Z^V. \qquad (4.23)$$

From (4.15), (4.16), (4.17), (4.18), (4.19), we also have the following equations:

$$R\left(X^H,Y^H\right)Z^H = -\phi R\left(X^H,Y^H\right)\phi Z^H + \tfrac{1}{4}\left\{G\left(Y^H,Z^H\right)X^H - G\left(X^H,Z^H\right)Y^H - G\left(\phi Y^H,Z^H\right)\phi X^H + G\left(\phi X^H,Z^H\right)\phi Y^H\right\}, \qquad (4.24)$$

$$R\left(X^V,Y^V\right)Z^V = -\phi R\left(X^V,Y^V\right)\phi Z^V + \tfrac{1}{4}\left\{G\left(Y^V,Z^V\right)X^V - G\left(X^V,Z^V\right)Y^V - G\left(\phi Y^V,Z^V\right)\phi X^V + G\left(\phi X^V,Z^V\right)\phi Y^V\right\}, \qquad (4.25)$$

$$R\left(X^H,Y^H\right)Z^V = -\phi R\left(X^H,Y^H\right)\phi Z^V, \qquad (4.26)$$

$$R\left(X^H,Y^V\right)Z^V = -\phi R\left(X^H,Y^V\right)\phi Z^V + \frac{1}{4}\left\{G\left(Y^V,Z^V\right)X^H - G\left(\phi Y^V,Z^V\right)\phi X^H\right\}, \qquad (4.27)$$

$$R\left(X^V, Y^H\right) Z^V = -\phi R\left(X^V, Y^H\right) \phi Z^V - \frac{1}{4}\left\{G\left(X^V, Z^V\right) Y^H + G\left(\phi X^V, Z^V\right) \phi Y^H\right\}, \tag{4.28}$$

$$R\left(X^V, Y^V\right) Z^H = -\phi R\left(X^V, Y^V\right) \phi Z^H, \tag{4.29}$$

$$R\left(X^V, Y^H\right) Z^H = -\phi R\left(X^V, Y^H\right) \phi Z^H + \frac{1}{4}\left\{G\left(Y^H, Z^H\right) X^H - G\left(\phi Y^H, Z^H\right) \phi X^H\right\}, \tag{4.30}$$

$$R\left(X^H, Y^V\right) Z^H = -\phi R\left(X^H, Y^V\right) \phi Z^H - \frac{1}{4}\left\{G\left(X^H, Z^H\right) Y^V + G\left(\phi X^H, Z^H\right) \phi Y^V\right\}, \tag{4.31}$$

$$\begin{aligned} &G\left(R\left(\phi X^H, \phi Y^H\right) \phi Z^H, \phi W^H\right) = G\left(R\left(X^H, Y^H\right) Z^H, W^H\right) \\ &+\tfrac{1}{4}\left\{-\eta^H\left(Y^H\right)\eta^H\left(Z^H\right) G\left(X^H, W^H\right) - \eta^H\left(X^H\right)\eta^H\left(W^H\right) G\left(Y^H, Z^H\right)\right. \\ &\left.+\eta^H\left(Y^H\right)\eta^H\left(W^H\right) G\left(X^H, Z^H\right) + \eta^H\left(X^H\right)\eta^H\left(Z^H\right) G\left(Y^H, W^H\right)\right\}, \end{aligned} \tag{4.32}$$

$$\begin{aligned} &G\left(R\left(\phi X^V, \phi Y^V\right) \phi Z^V, \phi W^V\right) = G\left(R\left(X^V, Y^V\right) Z^V, W^V\right) \\ &+\tfrac{1}{4}\left\{-\eta^V\left(Y^V\right)\eta^V\left(Z^V\right) G\left(X^V, W^V\right) - \eta^V\left(X^V\right)\eta^V\left(W^V\right) G\left(Y^V, Z^V\right)\right. \\ &\left.+\eta^V\left(Y^V\right)\eta^V\left(W^V\right) G\left(X^V, Z^V\right) + \eta^V\left(X^V\right)\eta^V\left(Z^V\right) G\left(Y^V, W^V\right)\right\}. \end{aligned} \tag{4.33}$$

A plane section in N_u is called a *horizontal ϕ-section* if there exists a unit vector X^H in N_u orthogonal to ξ^H such that $\left\{X^H, \phi X^H\right\}$ and a plane section in V_u^v is called a *vertical ϕ-section* if there exists a unit vector X^V in V_u^v orthogonal to ξ^V such that $\left\{X^V, \phi X^V\right\}$. Then the horizontal flag curvature

$$K\left(X^H, \phi X^H\right) = G^H\left(R\left(X^H, \phi X^H\right) \phi X^H, X^H\right) \tag{4.34}$$

is called a *horizontal ϕ-sectional curvature*, which will be denoted by $K^H\left(X^H\right)$. Vertical flag curvature

$$K\left(X^V, \phi X^V\right) = G^V\left(R\left(X^V, \phi X^V\right) \phi X^V, X^V\right) \tag{4.35}$$

is called a *vertical ϕ-sectional curvature*, which will be denoted by $K^V\left(X^V\right)$. On a Sasakian Finsler manifold the ϕ-sectional curvature is $K(X) = K^H\left(X^H\right) + K^V\left(X^V\right)$. □

Proposition 4.1 *Let (ϕ, η, ξ, G) be a K-contact Finsler structure on V. If V is locally symmetric, then V is a Sasakian Finsler manifold with constant curvature* $\frac{1}{4}$.

Proof For $X^H, Y^H, Z^H, \xi^H \in N_u$ from (4.9), (4.10), (4.11) and (4.12), we get

$$\left(\nabla_Z^H R\right)\left(X^H, Y^H, \xi^H\right) = \frac{1}{4}\left\{G\left(Z^H, X^H\right) Y^H - G\left(Z^H, Y^H\right) X^H\right\} - R\left(X^H, Y^H\right) Z^H. \tag{4.36}$$

Since V is locally symmetric, that is, $\nabla_Z^H R = 0$, from (4.36) we obtain

$$R\left(X^H, Y^H\right) Z^H = \frac{1}{4}\left\{G\left(Z^H, Y^H\right) X^H - G\left(Z^H, X^H\right) Y^H\right\}.$$

Thus for any orthonormal pair $\left\{X^H, Y^H\right\}$, we get

$$K\left(X^H, Y^H\right) = G\left(R\left(X^H, Y^H\right) Y^H, X^H\right) = \frac{1}{4}.$$

Similarly for $X^V, Y^V, Z^V, \xi^V \in V_u^v$, we get

$$R\left(X^V, Y^V\right) Z^V = \frac{1}{4}\{G\left(Z^V, Y^V\right) X^V - G\left(Z^V, X^V\right) Y^V\},$$

and for any orthonormal pair $\left\{X^V, Y^V\right\}$, we obtain $K\left(X^V, Y^V\right) = G\left(R\left(X^V, Y^V\right) Y^V, X^V\right) = \frac{1}{4}$.

For any orthonormal pair $\{X, Y\}$, we get $K(X, Y) = \frac{G^H\left(R\left(X^H, Y^H\right)Y^H, X^H\right) + G^V\left(R\left(X^V, Y^V\right)Y^V, X^V\right)}{G^H\left(X^H, X^H\right)G^H\left(Y^H, Y^H\right) + G^V\left(X^V, X^V\right)G^V\left(Y^V, Y^V\right)} = \frac{1}{4}$ which shows us that the sectional curvature of V is $\frac{1}{4}$. The *horizontal Ricci tensor* S^H of a *(4n+2)*-dimensional Sasakian Finsler manifold V is given by

$$\begin{aligned} S^H\left(X^H, Y^H\right) &= \textstyle\sum_{i=1}^{2n} G\left(R\left(X^H, E_i^H\right) E_i^H, Y^H\right) + G\left(R\left(X^H, \xi^H\right) \xi^H, Y^H\right) \\ &= \textstyle\sum_{i=1}^{2n} G\left(R\left(E_i^H, X^H\right) Y^H, E_i^H\right) + G\left(R\left(\xi^H, X^H\right) Y^H, \xi^H\right), \end{aligned}$$

where $\left\{E_1^H, E_2^H, ..., E_{2n}^H, \xi^H\right\}$ is a local orthonormal frame of N_u.

The *vertical Ricci tensor* of a *(4n+2)*-dimensional Sasakian Finsler manifold V is given by

$$\begin{aligned} S^V\left(X^V, Y^V\right) &= \textstyle\sum_{i=1}^{2n} G\left(R\left(X^V, E_i^V\right) E_i^V, Y^V\right) + G\left(R\left(X^V, \xi^V\right) \xi^V, Y^V\right) \\ &= \textstyle\sum_{i=1}^{2n} G\left(R\left(E_i^V, X^V\right) Y^V, E_i^V\right) + G\left(R\left(\xi^V, X^V\right) Y^V, \xi^V\right) \end{aligned}$$

where $\left\{E_1^V, E_2^V, ..., E_{2n}^V, \xi^V\right\}$ is a local orthonormal frame of V_u^v. Thus the Ricci tensor S of a *(4n+2)*-dimensional Sasakian Finsler manifold V is given by

$$\begin{aligned} S(X, Y) &= S^H(X, Y) + S^V(X, Y) = S\left(X^H, Y^H\right) + S\left(X^V, Y^V\right) \\ &= \textstyle\sum_{i=1}^{2n} G\left(R\left(X^H, E_i^H\right) E_i^H, Y^H\right) + G\left(R\left(X^H, \xi^H\right) \xi^H, Y^H\right) \\ &+ \textstyle\sum_{i=1}^{2n} G\left(R\left(X^V, E_i^V\right) E_i^V, Y^V\right) + G\left(R\left(X^V, \xi^V\right) \xi^V, Y^V\right). \end{aligned} \tag{4.37}$$

□

Proposition 4.2 *A contact metric structure (ϕ, η, ξ, G) on a Finsler manifold of dimension (4n+2) is K-contact if and only if* $S\left(\xi^H, \xi^H\right) = \frac{n}{2}, S\left(\xi^V, \xi^V\right) = \frac{n}{2}$.

Proof From (4.37) and (4.14), we have

$$S\left(\xi^H, \xi^H\right) = \sum_{i=1}^{2n} G\left(R\left(E_i^H, \xi^H\right) \xi^H, E_i^H\right) = \frac{1}{4}\sum_{i=1}^{2n} G\left(E_i^H, E_i^H\right) - \frac{1}{4}\sum_{i=1}^{2n} \eta^H\left(E_i^H\right) \eta^H\left(E_i^H\right).$$

Since E_i^H and ξ^H orthogonal, we can take $\eta^H\left(E_i^H\right) = 0$, thus we have $S\left(\xi^H, \xi^H\right) = \frac{n}{2}$.

$$S\left(\xi^V, \xi^V\right) = \sum_{i=1}^{2n} G\left(R\left(E_i^V, \xi^V\right) \xi^V, E_i^V\right) = \frac{1}{4}\sum_{i=1}^{2n} G\left(E_i^V, E_i^V\right) - \frac{1}{4}\sum_{i=1}^{2n} \eta^V\left(E_i^V\right) \eta^V\left(E_i^V\right).$$

Since E_i^V and ξ^V orthogonal, we can take $\eta^V\left(E_i^V\right) = 0$, thus we have $S\left(\xi^V, \xi^V\right) = \frac{n}{2}$. □

Lemma 4.1 *The Ricci tensor S of a (4n+2)-dimensional Sasakian Finsler manifold satisfies the following equations:*

$$S(X,\xi) = S\left(X^H,\xi^H\right) + S\left(X^V,\xi^V\right) = \frac{n}{2}\eta^H\left(X^H\right) + \frac{n}{2}\eta^V\left(X^V\right) = \frac{n}{2}\left(\eta^H\left(X^H\right) + \eta^V\left(X^V\right)\right) = \frac{n}{2}\eta(X),$$

$$S(\phi X,\phi Y) = S\left(\phi X^H,\phi Y^H\right) + S\left(\phi X^V,\phi Y^V\right) - \frac{n}{2}\eta^H\left(X^H\right)\eta^H\left(Y^H\right) - \frac{n}{2}\eta^V\left(X^V\right)\eta^V\left(Y^V\right).$$

5. Conclusion

For the Sasakian Finsler structure (ϕ, η, ξ, G) on V, the following relations hold:

$$\phi.\phi = -\mathrm{I}_n + \eta^H\otimes\xi^H + \eta^V\otimes\xi^V,\ \phi\xi^H = 0,\ \phi\xi^V = 0,\ \eta^H\left(\xi^H\right) + \eta^V\left(\xi^V\right) = 1,$$

$$\eta^H\left(\phi X^H\right) = 0, \eta^V\left(\phi X^H\right) = 0, \eta^H\left(\phi X^V\right) = 0, \eta^V\left(\phi X^V\right) = 0,$$

$$G^H(\phi X,\phi Y) = G^H(X,Y) - \eta^H\left(X^H\right)\eta^H\left(Y^H\right), G^V(\phi X,\phi Y) = G^V(X,Y) - \eta^V\left(X^V\right)\eta^V\left(Y^V\right),$$

$$G^H(X,\xi) = \eta^H\left(X^H\right), G^V(X,\xi) = \eta^V\left(X^V\right), N_\phi + d\eta^H\otimes\xi^H + d\eta^V\otimes\xi^V = 0,$$

$$\Omega\left(X^H,Y^H\right) = G^H(X,\phi Y) = d\eta\left(X^H,Y^H\right), \Omega\left(X^V,Y^V\right) = G^V(X,\phi Y) = d\eta\left(X^V,Y^V\right),$$

$$\nabla^H_\xi\phi = 0, \nabla^V_\xi\phi = 0, \nabla^H_X\xi^H = -\frac{1}{2}\phi X^H, \nabla^V_X\xi^V = -\frac{1}{2}\phi X^V,$$

$$\begin{aligned}
\left(\nabla^H_X\phi\right)Y^H &= \tfrac{1}{2}\left[G^H\left(X^H,Y^H\right)\xi^H - \eta^H\left(Y^H\right)X^H\right], \left(\nabla^V_X\phi\right)Y^V \\
&= \tfrac{1}{2}\left[G^V\left(X^V,Y^V\right)\xi^V - \eta^V\left(Y^V\right)X^V\right], R\left(X^H,Y^H\right)Z^H \\
&= \tfrac{1}{4}\left\{G\left(Z^H,Y^H\right)X^H - G\left(Z^H,X^H\right)Y^H\right\}(V \text{ is locally symmetric}), R^H\left(X^H,Y^H\right)\xi^H \\
&= \tfrac{1}{4}\left(\eta^H\left(Y^H\right)X^H - \eta^H\left(X^H\right)Y^H\right), R^V\left(X^V,Y^V\right)\xi^V \\
&= \tfrac{1}{4}\left(\eta^V\left(Y^V\right)X^V - \eta^V\left(X^V\right)Y^V\right), R^H\left(X^H,\xi^H\right)Y^H \\
&= \tfrac{1}{4}\left[-G\left(X^H,Y^H\right)\xi^H + \eta^H\left(Y^H\right)X^H\right], S\left(\xi^H,\xi^H\right) = \tfrac{n}{2},
\end{aligned}$$

$$R^V\left(X^V,\xi^V\right)Y^V = \frac{1}{4}\left[-G\left(X^V,Y^V\right)\xi^V + \eta^V\left(Y^V\right)X^V\right], S\left(\xi^V,\xi^V\right) = \frac{n}{2},$$

$$K\left(X^V,Y^V\right) = G\left(R\left(X^V,Y^V\right)Y^V,X^V\right) = \frac{1}{4}, K\left(X^H,Y^H\right) = G\left(R\left(X^H,Y^H\right)Y^H,X^H\right) = \frac{1}{4},$$

$$S(X,\xi) = S\left(X^H,\xi^H\right) + S\left(X^V,\xi^V\right) = \frac{n}{2}\left(\eta^H\left(X^H\right) + \eta^V\left(X^V\right)\right) = \frac{n}{2}\eta(X).$$

$\forall X^V, Y^V,\ \xi^V \in V^v_u$ and $\forall X^H, Y^H, \xi^H \in N_u$, where a linear connection ∇ on V denotes Finsler connection, ϕ is the tensor field of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ on V, η is a 1-form, S is the Ricci tensor, R is the Riemann curvature tensor, G is the Finsler metric structure on V, K is the flag curvature on V. Next, let us set the equation below

$2d\tilde{\eta}(X,Y) = X(\tilde{\eta}(Y)) - Y(\tilde{\eta}(X)) - \tilde{\eta}[X,Y], \forall X = X^H \in N_u, Y = Y^H \in N_u, \forall \tilde{\eta} = \eta^H \in N_u^*$. If we get $\tilde{\phi} = \phi^H$, $\tilde{\eta} = \eta^H$, $\tilde{\xi} = \xi^H$, $\tilde{g} = G^H$, the standard Sasakian structure of the base space M^{2n+1} is $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$. Then we have the following equations:

$$N_{\tilde{\phi}} + 2d\tilde{\eta} \otimes \tilde{\xi} = 0, \nabla_X \tilde{\xi} = -\phi X, L_{\tilde{\xi}} \tilde{g} = 0, \nabla_{\tilde{\xi}} \tilde{\phi} = 0,$$

$$\left(\nabla_X \tilde{\phi}\right) Y = \tilde{g}(X,Y)\tilde{\xi} - \tilde{\eta}(Y) X, R\left(X, \tilde{\xi}\right) Y = \tilde{\eta}(Y) X - \tilde{g}(X,Y)\tilde{\xi},$$

$$\tilde{g}\left(R\left(X, \tilde{\xi}\right)\tilde{\xi}, X\right) = 1, S\left(\tilde{\xi}, \tilde{\xi}\right) = 2n, R(X,Y)\tilde{\xi} = \tilde{\eta}(Y) X - \tilde{\eta}(X) Y,$$

$K(X,Y) = \tilde{g}(R(X,Y)Y,X) = 1$ (sectional curvature for orthonormal pair {X,Y}).

The structure $\left(\phi^H, \eta^H, \xi^H, G^H\right)$ on N_u is Sasakian Finsler if and only if the base manifold M^{2n+1} with the structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ has positive constant curvature 1 in which case M^{2n+1} is Sasakian manifold and N_u is Sasakian Finsler manifold.

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