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Contact 3-structure QR-warped product submanifold in Sasakian space form

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Abstract: In the present paper we obtain sharp estimates for the squared norm of the second fundamental form in terms of the mapping function for contact 3-structure CR-warped products isometrically immersed in Sasakian space form.

Key words: Warped product, contact QR-warped product, Sasakian space form

1. Introduction

Let $\tilde{M}$ be a hermitian manifold and denoted by J the almost complex structure on $\tilde{M}$. Yano and Ishihara (see [13]) considered a submanifold M whose tangent bundle TM splits into a complex subbundle D and a totally real subbundle $D^{\perp}$. Later, such a submanifold was called a CR-submanifold [4],[3]. Blair and Chen [4] proved that a CR-submanifold of a locally conformal Kaäler manifold is a Cauchy-Riemann manifold in the sense of Greenfield.

Recently, Chen [5] introduced the notion of a CR-warped product submanifold in a Kaäler manifold. He established a sharp relationship between the mapping function f of a warped product CR-submanifold $M_1 \times_f M_2$ of a Kaäler manifold $\tilde{M}$ and the squared norm of the second fundamental form $\|h\|$ [5].

In 1971, Kenmotsu [7] introduced a class of almost contact metric manifolds, called Kenmotsu manifold, which is not Sasakian. Kenmotsu manifolds have been studied by several authors such as Pitiş [12], Özgür [10] and Özgür and De [11].

Let $\bar{M}^{\frac{(n+p)}{4}}$ be a quaternionic Kaäler manifold with real dimension of $n+p$. Let M be an n-dimensional QR-submanifold of QR dimension $(p-3)$ isometrically immersed in a quaternionic Kaäler manifold $\bar{M}^{\frac{(n+p)}{4}}$. Denoting by $\{F_1, F_2, F_3\}$ the quaternionic Kaäler structure of $\bar{M}^{\frac{(n+p)}{4}}$, it follows by definition [8] that there exists a $(p-3)$-dimensional subbundle ν of the normal bundle $TM^{\perp}$

$$F_1\nu_x \subset \nu_x, \quad F_2\nu_x \subset \nu_x, \quad F_3\nu_x \subset \nu_x, \tag{1.1}$$

$$F_1\nu_x^{\perp} \subset T_xM, \quad F_2\nu_x^{\perp} \subset T_xM, \quad F_3\nu_x^{\perp} \subset T_xM, \tag{1.2}$$

for each $x \in M$, where $\nu^{\perp}$ denotes the complementary orthogonal subbundle to ν in $TM^{\perp}$. Thus these are naturally distinguished unit normal vector fields $\{\xi_1, \xi_2, \xi_3\}$ of M such that $\nu_x^{\perp} = \text{span}\{\xi_1, \xi_2, \xi_3\}$ for each

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$x \in M$, and the vector fields U_1, U_2, U_3 defined by

$$U_1 = -F_1\xi_1, \quad U_2 = -F_2\xi_2, \quad U_3 = -F_3\xi_3 \tag{1.3}$$

are tangent to M. On the other hand, each tangent space T_xM is decomposed as

$$T_xM = D_x \oplus D_x^{\perp} \tag{1.4}$$

where D_x is the maximal quaternionic invariant subspace of T_xM defined by

$$D_x = T_xM \cap F_1T_xM \cap F_2T_xM \cap F_3T_xM \tag{1.5}$$

and $D_x^{\perp}$ its orthogonal complement in T_xM. In this case, as shown in [2], $D_x^{\perp} = \mathrm{span}\{U_1, U_2, U_3\}$ and so $D : x \mapsto D_x$ defines an $(n-3)$-dimensional distribution on M. But D cannot be a quaternionic CR-distribution in the sense of [1]. Further, it is clear that

$$F_1T_xM, F_2T_xM, F_3T_xM \subset T_xM \oplus \mathrm{span}\{\xi_1, \xi_2, \xi_3\} \tag{1.6}$$

and, consequently, for any tangent vector X to M, we have following decomposition in tangential and normal components

$$F_iX = \varphi_iX + \eta_i(X)\xi_i \tag{1.7}$$

In the present paper, we study contact 3-structure QR-warped product submanifolds in Sasakian space forms.

We prove estimates of the squared norm of the second fundamental form in terms of the mapping function. Equality cases are investigated.

2. Preliminaries

A $(4m+3)$-dimensional Riemannian manifold $\tilde{M}$ is said to have an *almost contact* 3*-structure* [9] if it admits three contact structure $(\varphi_i, \xi_i, \eta_i)$, $i = 1, 2, 3$, satisfying:

$$\varphi_k = \varphi_i\varphi_j - \eta_j \otimes \xi_i = -\varphi_j\varphi_i + \eta_i \otimes \xi_j$$

$$\xi_k = \varphi_i\xi_j = -\varphi_j\xi_i \quad , \quad \eta_k = \eta_i \circ \varphi_j = -\eta_j \circ \varphi_i.$$

Kuo [9] proved that given an almost contact 3-structure, there exists a Riemannian metric compatible with each of them, and hence we can speak of an *almost contact metric* 3*-structure* $(\varphi_i, \xi_i, \eta_i, \tilde{g})$, $i = 1, 2, 3$.

The almost contact metric structure of $(\varphi, \xi, \eta, \tilde{g})$ on $\tilde{M}$ is called Sasakian structure if

$$\tilde{\nabla}_X\xi = -\varphi(X)$$

$$(\tilde{\nabla}_X\varphi)Y = g(X, Y)\xi - \eta(Y)X.$$

If the three structures $(\varphi_i, \xi_i, \eta_i, \tilde{g})$ are contact metric structures, we say that $\tilde{M}$ has a *contact metric* 3*-structure*. If the three structures are Sasakian, we say that $\tilde{M}$ has a 3*-Sasakian structure*, and $\tilde{M}$ is a 3*-Sasakian manifold*.

We have the following theorem of Kashiwada [6]

Theorem. Every contact metric 3-structure is 3-Sasakian.

A plane section π in $T_p\tilde{M}$ is called a $\varphi-$section if $\varphi_i(\pi) \subseteq \pi$ for some $i = 1, 2, 3$. The sectional curvature of a φ-section is called φ-holomorphic sectional curvature. A Sasakian manifold with constant φ-holomorphic sectional curvature c is called a *Sasakian space form* and is denoted by $\tilde{M}(c)$.

The curvature tensor $\tilde{R}$ of a Sasakian space form is given by

$$\begin{aligned}\tilde{R}(X,Y)Z &= \frac{c+3}{4}\{\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y\} \\ &+ \frac{c-1}{4}\sum_{i=1}^{3}\{[\eta_i(X)Y - \eta_i(Y)X]\eta_i(Z) \\ &\quad +[\tilde{g}(X,Z)\eta_i(Y) - \tilde{g}(Y,Z)\eta_i(X)]\xi_i \\ &\quad -\tilde{g}(Y,\varphi_i Z)\varphi_i X + \tilde{g}(X,\varphi_i Z)\varphi_i Y + 2\tilde{g}(X,\varphi_i Y)\varphi_i Z\}. \end{aligned} \tag{2.1}$$

Let $\tilde{M}$ be a Sasakian manifold and M an n-dimensional submanifold tangent to $\{\xi_i\}$. For any vector field X tangent to M, we put

$$\varphi_i X = P_i X + F_i X \tag{2.2}$$

where P_iX (resp. F_iX) denotes the tangent (resp. normal) component of φ_iX. Then P_i is an endomorphism of the bundle TM and F_i is a normal bundle valued $1-$forms on TM.

The Gauss equation is given by

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= R(X,Y,Z,W) \\ &\quad +g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W))\end{aligned} \tag{2.3}$$

for any vectors X, Y, Z, W tangent to M.

Defining the covariant derivative of h by

$$(\nabla h)(X,Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z)$$

the Codazzi equation is

$$(R(X,Y)Z)^{\perp} = (\nabla h)(X,Y,Z) - (\nabla h)(Y,X,Z). \tag{2.4}$$

Let $p \in M$ and $\{e_1,\ldots,e_n,e_{n+1},\ldots,e_{4m+3}\}$ be an orthonormal basis of the tangent space $T_p\tilde{M}$, such that $e_1,\ldots,e_n$ are tangent to M at p. We denote by H the mean curvature vector, that is

$$H(p) = \frac{1}{n}\sum_{1}^{n} h(e_i,e_i) \tag{2.5}$$

As is known, M is said to be *minimal* if H vanishes identically.

Also, we set

$$h_{ij}^{r}=g(h(e_i,e_j),e_r),\quad i,j\in\{1,\ldots,n\},\quad r\in\{n+1,\ldots,4m+3\} \tag{2.6}$$

as the coefficients of the second fundamental form h with respect to $\{e_1,\ldots,e_n,\ e_{n+1},\ldots,e_{4m+3}\}$, and

$$\| h \|^2=\sum_{i,j=1}^{n} g(h(e_i,e_j),h(e_i,e_j)). \tag{2.7}$$

By analogy with submanifold in a Kaäler manifold, different classes of submanifolds in a Sasakian manifold were considered.

A submanifold M tangent to $\{\xi_i\}$ is called an invariant (resp. anti-invariant) submanifold if $\varphi_i(T_pM)\subset T_pM,\quad \forall p\in M$ (resp. $\varphi_i(T_pM)\subset T_p^{\perp}M,\quad \forall p\in M$).

A submanifold M tangent to $\{\xi_i\}$ is called a contact QR-submanifold if there exists a pair of orthogonal differentiable distributions D and $D^{\perp}$ on M, such that:

(1) $TM=D\oplus D^{\perp}\oplus\{\xi_i\}$, where $\{\xi_i\}$ is the $3-$dimensional distribution spanned by $\{\xi_i\}$

(2) D is invariant by φ_i, i.e. , $\varphi_i(D_p)\subset D_p,\quad \forall p\in M$;

(3) $D^{\perp}$ is anti-invariant by φ_i, i.e. , $\varphi_i(D_p^{\perp})\subset T_p^{\perp}M,\quad \forall p\in M$.

In particular, if $D^{\perp}=\{0\}$ (resp. $D=\{0\}$), M is an invariant (resp. anti-invariant) submanifold.

3. 3-Contact QR-warped product submanifolds

Let (M_1,g_1) and (M_2,g_2) be two Riemannian manifolds and let $f:M_1\to(0,\infty)$ be differentiable function.

The warped product $M=M_1\times_f M_2$ is the product manifold $M_1\times M_2$ endowed with the metric

$$g=g_1+f^2g_2 \tag{3.1}$$

More precisely, if $\pi_1:M_1\times M_2\to M_1$ and $\pi_2:M_1\times M_2\to M_2$ are natural projections, the metric g is defined by

$$g=\pi_1^{*}g_1+(fo\pi_1)^2\pi_2^{*}g_2 \tag{3.2}$$

The function f is called warping function. If $f\equiv 1$, then we have a Riemannian product manifold. If neither f is constant, then we have a non-trivial warped product.

We recall that on a warped product one has

$$\nabla_X Z=X(\ln f)Z \tag{3.3}$$

for any vector field X tangent to M_1 and Z tangent to M_2, where ∇ is the Riemannian connection of the Riemannian metric g.

If X and Z are unit vector fields, it follows that the sectional curvature $K(X\wedge Z)$ of the plan section spanned by X and Z is given by

$$K(X\wedge Z)=g(\nabla_Z^1\nabla_X^1X-\nabla_X^1\nabla_Z^1X,Z)=\frac{1}{f}\{(\nabla_X^1X)f-X^2f\},$$

where ∇^1, ∇^2 are the Riemannian connections of the Riemannian metrics g_1 and g_2 respectively.

A warped product submanifold $M = M_1 \times_f M_2$ of a Sasakian manifold $\tilde{M}$, with M_1 a $(4\alpha + 3)-$dimensional invariant submanifold tangent to $\{\xi_i\}$ and M_2 a β-dimensional anti-invariant submanifold of $\tilde{M}$, is called the *contact QR-warped product submanifold*.

We state the following estimate of the squared norm of the second fundamental form for contact QR-warped product in Sasakian space forms.

Theorem 3.1 *Let $\tilde{M}$ be a $(4m+3)$-dimensional Sasakian manifold and $M = M_1 \times_f M_2$ an n-dimensional contact QR-warped product submanifold, such that M_1 is a $(4\alpha+3)$-dimensional invariant submanifold tangent to $\{\xi_i\}$ and M_2 is a β-dimensional anti-invariant submanifold of $\tilde{M}(c)$. Then:*
(i) The squared norm of the second fundamental form of M satisfies

$$\| h \|^2 \geq 2\beta \| \nabla(\ln f) \|^2 + 6 \tag{3.4}$$

where $\nabla(\ln f)$ is the gradient of $\ln f$.
(ii) If the equality sign of (3.4) holds identically, then M_1 is a totally geodesic submanifold and M_2 is a totally umbilical submanifold of $\tilde{M}$. Moreover, M is a minimal submanifold of $\tilde{M}$.

Proof Let $M = M_1 \times_f M_2$ be a QR-warped product submanifold of a Sasakian manifold $\tilde{M}$, such that M_1 is an invariant submanifold tangent to $\{\xi_i\}$ and M_2 is an anti-invariant submanifold of $\tilde{M}$.

For any unit vector fields X tangent to M_1 and orthogonal to $\{\xi_i\}$ and Z, W tangent to M_2, we have

$$\begin{aligned} g(h(\varphi_i X, Z), \varphi_i Z) &= g(\tilde{\nabla}_Z \varphi_i X, \varphi_i Z) = g(\varphi_i \tilde{\nabla}_Z X, \varphi_i Z) \\ &= g(\tilde{\nabla}_Z X, Z) = g(\nabla_Z X, Z) = X \ln f. \end{aligned} \tag{3.5}$$

Also, we have

$$g(h(\varphi_i X, Z), \varphi_i W) = (X \ln f) g(Z, W) \tag{3.6}$$

and we have

$$\begin{aligned} g(h(X, Z), \varphi_i Z) &= g(\tilde{\nabla}_Z X - \nabla_Z X, \varphi_i Z) \\ &= g(\tilde{\nabla}_Z X, \varphi_i Z) = -g(\varphi_i \tilde{\nabla}_Z X, Z) \\ &= -g(\tilde{\nabla}_Z(\varphi_i X) - (\tilde{\nabla}_Z \varphi_i) X, Z) \\ &= -g(\tilde{\nabla}_Z(\varphi_i X), Z) = -g(\nabla_Z(\varphi_i X), Z) \\ &= -g(\varphi_i X(lnf) Z, Z) = -\varphi_i X(\ln f) \end{aligned} \tag{3.7}$$

for $i = 1, 2, 3$. On the other hand, since the ambient manifold $\tilde{M}$ is a Sasakian manifold, it is easily seen that

$$h(\xi_i, Z) = -\varphi_i Z, \quad i = 1, 2, 3 \tag{3.8}$$

Obviously, (3.8) implies $\xi_i \ln f = 0$.

Let $\{e_i, \varphi_k e_i, f_j, \xi_k \mid i = 1, \ldots, \alpha,\ j = 1, \ldots, \beta,\ k = 1, 2, 3\}$ be a local orthonormal frame on M such that $e_i, \xi_k, i = 1, \ldots, \alpha, k = 1, 2, 3$ are tangent to M_1 and $f_1, ..., f_\beta$ are tangent to M_2. Therefore, by (3.6), (3.7) and (3.8), we have

$$\begin{aligned} \parallel \nabla \ln f \parallel^2 &= \sum_{i=1}^{\alpha}(e_i \ln f)^2 + \sum_{i=1}^{\alpha}(\varphi_1 e_i \ln f)^2 + \sum_{i=1}^{\alpha}(\varphi_2 e_i \ln f)^2 + \sum_{i=1}^{\alpha}(\varphi_3 e_i \ln f)^2 \\ &= \frac{1}{3\beta}\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}\{\parallel h(\varphi_1 e_i, f_j) \parallel^2 + \parallel h(\varphi_2 e_i, f_j) \parallel^2 + \parallel h(\varphi_3 e_i, f_j) \parallel^2\} \\ &\quad + \frac{3}{\beta}\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} \parallel h(e_i, f_j) \parallel^2 \\ &= \frac{1}{3\beta} \parallel h(D, D^\perp) \parallel^2 + \frac{8}{3\beta} \parallel h(e_i, f_j) \parallel^2 \\ &= \frac{1}{3\beta} \parallel h(D, D^\perp) \parallel^2 + \frac{8}{9} \parallel \nabla \ln f \parallel^2 - \frac{8}{9}\sum_{i=1}^{\alpha}(e_i \ln f)^2. \end{aligned}$$

Therefore,

$$\frac{1}{9} \parallel \nabla \ln f \parallel^2 = \frac{1}{3\beta} \parallel h(D, D^\perp) \parallel^2 - \frac{8}{9}\sum_{i=1}^{\alpha}(e_i \ln f)^2. \tag{3.9}$$

From the property of $\varphi-$invariant D, and permutations of $\varphi_1 e_i, \varphi_2 e_i, \varphi_3 e_i$, we obtain the following three analogous relations:

$$\frac{1}{9} \parallel \nabla \ln f \parallel^2 = \frac{1}{3\beta} \parallel h(D, D^\perp) \parallel^2 - \frac{8}{9}\sum_{i=1}^{\alpha}(\varphi_1 e_i \ln f)^2 \tag{3.10}$$

$$\frac{1}{9} \parallel \nabla \ln f \parallel^2 = \frac{1}{3\beta} \parallel h(D, D^\perp) \parallel^2 - \frac{8}{9}\sum_{i=1}^{\alpha}(\varphi_2 e_i \ln f)^2 \tag{3.11}$$

$$\frac{1}{9} \parallel \nabla \ln f \parallel^2 = \frac{1}{3\beta} \parallel h(D, D^\perp) \parallel^2 - \frac{8}{9}\sum_{i=1}^{\alpha}(\varphi_3 e_i \ln f)^2 \tag{3.12}$$

Summing the above relations, we have

$$\parallel \nabla \ln f \parallel^2 = \frac{1}{\beta} \parallel h(D, D^\perp) \parallel^2 .$$

On the other hand, since $h(\xi_i, \xi_i) = 0$ for $i = 1, 2, 3$, then

$$\begin{aligned} \|h\|^2 &= \|h(D, D)\|^2 + 2\|h(D, D^\perp)\|^2 + \|h(D^\perp, D^\perp)\|^2 \\ &\quad + \sum_{i \neq j} \|h(\xi_i, \xi_j)\|^2 \geq 2\|h(D, D^\perp)\|^2 + 6, \end{aligned}$$

therefore the inequality (3.4) is immediately obtained.

Denote by h'' the second fundamental form of M_2 in M. Then, we get

$$g(h''(Z,W),X) = g(\nabla_Z W, X) = -(X \ln f) g(Z,W)$$

or equivalently

$$h''(Z,W) = -g(Z,W)\nabla(\ln f) \tag{3.13}$$

If the equality sign of (3.4) identically holds, then we obtain

$$h(D,D) = 0, \quad h(D^{\perp}, D^{\perp}) = 0, \quad h(D, D^{\perp}) \subset \varphi_i D^{\perp}. \tag{3.14}$$

The first condition (3.14) implies that M_1 is totally geodesic in M. On the other hand, one has

$$\tilde{g}(h(X, \varphi_i Y), \varphi_i Z) = \tilde{g}(\tilde{\nabla}_X \varphi_i Y, \varphi_i Z) = \tilde{g}(\nabla_X Y, Z) = 0, \tag{3.15}$$

where X, Y are tangent to M_1 and Z is tangent to M_2. Thus M_1 is totally geodesic in $\tilde{M}$.

The second condition in (3.14) and (3.13) imply that M_2 is totally umbilical submanifold in $\tilde{M}$.

Moreover, by (3.14), it follows that M is a minimal submanifold of $\tilde{M}$. □

In particular, if the ambient space is a Sasakian space form, one has the following corollary.

Corollary 3.2 *Let $\tilde{M}(c)$ be a $(4m+3)-$dimensional Sasakian space form of constant $\varphi-$holomorphic sectional curvature c and $M = M_1 \times_f M_2$ an $n-$dimensional non-trivial contact QR-warped product submanifold, satisfying*

$$\| h \|^2 = 2\beta \| \nabla(\ln f) \|^2 + 6. \tag{3.16}$$

Then, we have

(a) M_1 is a totally geodesic invariant submanifold of $\tilde{M}(c)$. Hence M_1 is a Sasakian space form of constant $\varphi-$holomorphic sectional curvature c.

(b) M_2 is a totally umbilical anti-invariant submanifold of $\tilde{M}(c)$. Hence M_1 is a real space form of sectional curvature $\varepsilon \geq (c+3)/4$.

Proof Statement (a) follows from Theorem 3.1.

Also, we know that M_2 is a totally umbilical submanifold of $\tilde{M}(c)$. The Gauss equation implies that M_2 is a real space form of sectional curvature $\varepsilon \geq (c+3)/4$.

Moreover, by (3.3),we see that $\varepsilon = (c+3)/4$ if and only if the warping function f is constant. □

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