

On Nash's 4-sphere and Property 2R

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Received: 01.04.2011 • Accepted: 16.01.2012 • Published Online: 19.03.2013 • Printed: 22.04.2013

Abstract: D. Nash defined a family of homotopy 4-spheres in [11]. Proving that his manifolds $\mathcal{S}_{m,n,m',n'}$ are all real S^4 , we show that they have handle decomposition with no 1-handles, two 2-handles and two 3-handles. The handle structures give new potential counterexamples to the Property 2R conjecture.

Key words: Homotopy 4-spheres, Property 2R, handle calculus

1. Introduction

The smooth Poincaré conjecture in 4-dimension is still open. Though many people [3, 4] have proposed potential counterexamples, it was [1, 8, 9, 10] that proved some of them are standard S^4 . D. Nash [11] also proposed potential counterexamples to the conjecture. Most recently S. Akbulut [2] proved that these manifolds are all standard. In this article I will give an alternative proof and furthermore remark on some handle decompositions which appear in the proof.

Nash's manifolds are constructed by use of logarithmic transformations along four tori in some 4-manifold. In this section we will give a brief review of logarithmic transformation. For the remark of the handle decomposition as mentioned above, we review notions Property nR and generalized Property R.

1.1. Logarithmic transformation

We review the notation of the logarithmic transformation. Let $T \subset X^4$ be a torus embedding with the trivial normal bundle $\nu(T) = D^2 \times T$ in 4-manifold X . Removing the neighborhood, we reglue it with the map $\varphi: \partial D^2 \times T^2 \rightarrow \partial \nu(T)$ satisfying

$$\varphi(\partial D^2 \times \{\text{pt}\}) = p\mu + q\gamma,$$

where μ is the meridian of T and $[\gamma]$ is a primitive element in $H_1(T)$, so that we obtain the following manifold.

Definition 1 *The surgery whose gluing map is φ as above*

$$X - \nu(T) \cup_{\varphi} (D^2 \times T)$$

is called the (p/q) -logarithmic transformation along T with direction γ . If γ is a fixed curve, then we simply call (p/q) -surgery.

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2010 AMS Mathematics Subject Classification: 57R65, 57M25.

1.2. Generalized Property R conjecture

Property R conjecture was proved by Gabai [7]. M. Scharlemann and A. Thompson in [12] generalized Property R as follows.

Definition 2 ([12]) *We say that a knot K has Property nR if K satisfies the following property. If any n -component link L containing K as a component yields $\#^n S^1 \times S^2$ by an integral Dehn surgery, then after some handle slides the framed link can be reduced to the n -component unlink.*

The $n = 1$ case is equivalent to original Property R.

Conjecture 1 (Generalized Property R Conjecture) *All knots have Property nR for any $n \geq 1$.*

The generalized Property R conjecture is still open. The homotopy 4-spheres by D. Nash in [11] are standard, however we show that diagrams coming from handle decompositions might be counterexamples to the generalized Property R conjecture.

We can find Figure 1 along the way to prove that Nash’s manifolds are standard (Theorem 1).

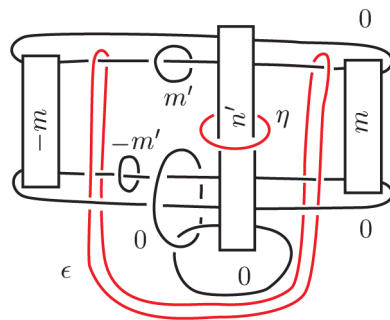


Figure 1. Examples which might not have Property 2R.

The framed link with black color is a presentation of S^3 . The 0-surgery along the 2-component link (with red color) gives rise to $\#^2 S^2 \times S^1$, because the framed link gives a diagram of a Nash homotopy 4-sphere, which is indeed real S^4 (Corollary 2).

Question 1 *Are η and ϵ in Figure 1 examples not having Property 2R for any non-zero integers m, n', m' ?*

2. Nash’s manifolds

D. Nash in [11] defined a new family of homotopy 4-spheres as follows. Let A be a 4-manifold with the handle diagram Figure 2. Since A , as in [5], is constructed by attaching two 2-handles to $D^2 \times T_0^2 \subset (D^2 \times S^1) \times S^1$ along the Bing double of $D^2 \times S^1$, A also includes Bing tori B_T as in Figure 3, where T_0^2 is the punctured torus. As a fundamental fact the (0)-surgery along B_T yields $T_0^2 \times T_0^2$. In addition, two core tori reglued by use of the surgery are $T_1 = S_\alpha^1 \times S_\gamma^1$ and $T_2 = S_\beta^1 \times S_\gamma^1$, where S_α^1, S_β^1 and S_γ^1 are generating circles in $T_0^2 \subset T^2$. In other words, (0)-surgery along T_1^2 and T_2^2 yields A back.

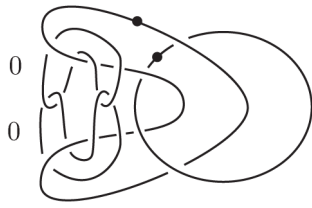


Figure 2. A.



Figure 3. Bing tori.

Now we take two copies of $T_0^2 \times T_0^2$ to glue the boundaries $S^1 \times T_0^2 \cup T_0^2 \times S^1$ to each other by gluing map $\phi : S^1 \times T_0^2 \cup T_0^2 \times S^1 \rightarrow S^1 \times T_0^2 \cup T_0^2 \times S^1$, such that the two components are exchanged between each other. We call the resulting manifold X . Such a construction is generalized to ℓ -component case according to Fintushel and Stern's work [5], and it is called a *pinwheel construction*.

Here we define the $(m/1)$ -surgery of $T_0^2 \times T_0^2$ along T_1 with direction S_α^1 and simultaneously the $(n/1)$ -surgery along T_2 with direction S_β^1 as $X_{m,n}$. We denote by the same gluing map ϕ , $X_{m,n} \cup_\phi -X_{m',n'}$ as $\mathcal{S}_{m,n,m',n'}$. This minus notation denotes reversing the orientation. Namely, $\mathcal{S}_{m,n,m',n'}$ is obtained from four logarithmic transformations of X . From the construction immediately we have the following.

Lemma 1 *For any integers m, n, m', n' the following diffeomorphism holds:*

$$\mathcal{S}_{m,n,m',n'} \cong \mathcal{S}_{m',n',m,n}.$$

Nash obtained the result (Theorem 3.2 in [11]) in which the manifolds $\mathcal{S}_{m,n,m',n'}$ are all homotopy 4-spheres. Namely, $\mathcal{S}_{m,n,m',n'}$ are candidates of counterexamples to the 4-dimensional smooth Poincaré conjecture. Are these manifolds diffeomorphic to standard S^4 ? Here we give an affirmative answer.

Theorem 1 (Nash's manifolds are standard.) *The manifolds $\mathcal{S}_{m,n,m',n'}$ are all diffeomorphic to the standard 4-sphere.*

S. Akbulut independently proved the same result in [2].

3. Handle decomposition of $\mathcal{S}_{m,n,m',n'}$

3.1. The diagram of $X_{m,n}$

Lemma 2 *A handle decomposition of $X_{m,n}$ is Figure 4.*

Proof First the handle picture of T^4 is the left of Figure 6. Recall that T^3 is obtained from 0-surgery along the Borromean ring. Since $T_0^2 \times T_0^2$ is obtained by removing $D_0^2 \times T^2 \subset T^2 \times D^2$ and $T_0^2 \times D_0^2 \subset T_0^2 \times T^2$, the diagram is the right of Figure 6. Since $(m/1)$ and $(n/1)$ -logarithmic transformations correspond to the $(0, -1/m, -1/n)$ surgery over the Borromean ring, we get Figure 4 as a diagram of $X_{m,n}$. \square

The 3 arrows on the right in Figure 6 denote $S_\alpha^1, S_\beta^1, S_\gamma^1$ generating circles above.

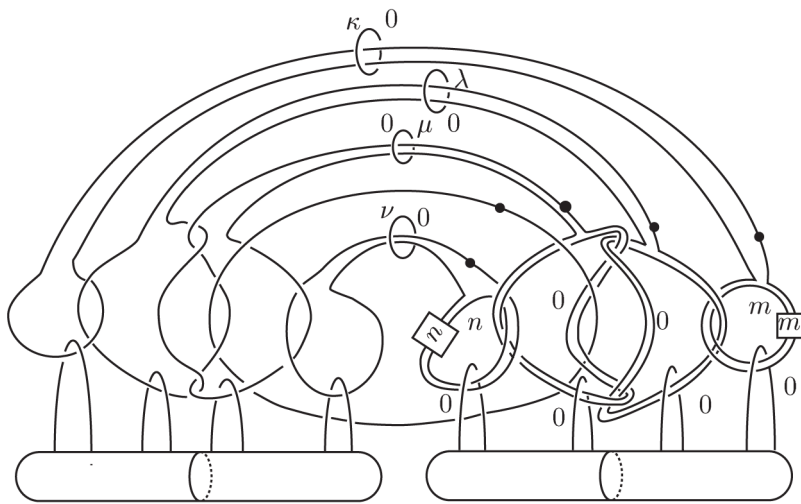


Figure 4. $X_{m,n}$

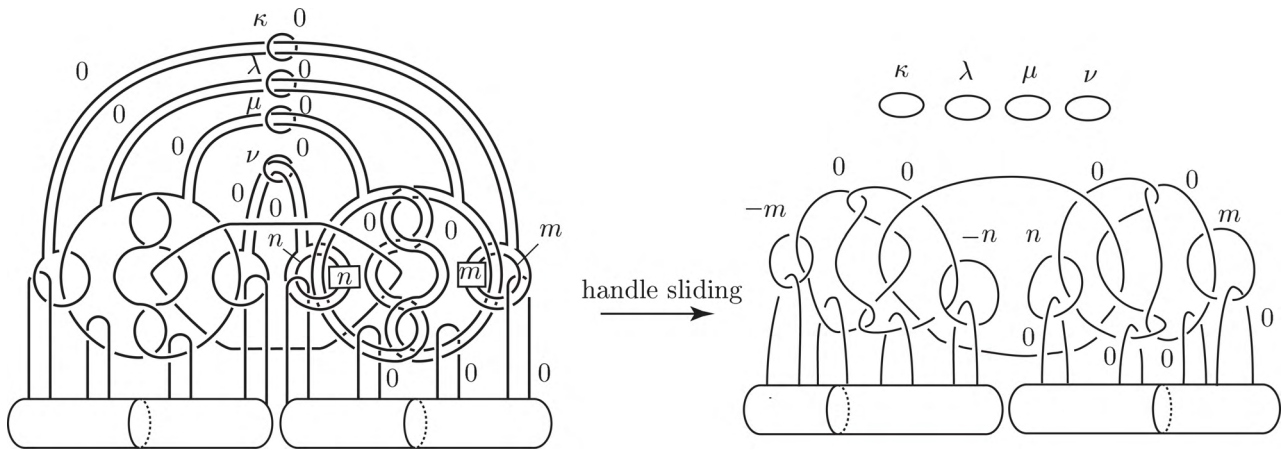


Figure 5. κ, λ, μ and ν are unlink.

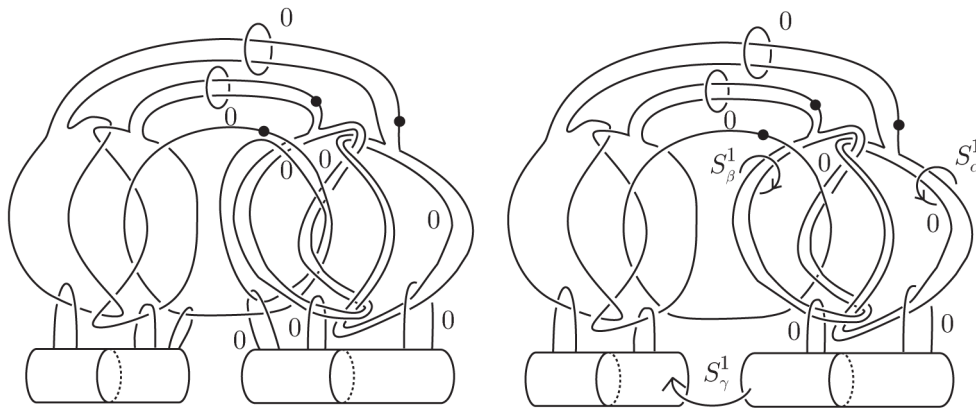


Figure 6. T^4 and $T^2_0 \times T^2_0$.

3.2. Upside-down of $X_{m,n}$

Next we perform the upside down of the manifolds $X_{m,n}$. The right four 2-handles in Figure 4 which are along two components of Borromean ring, and the two meridians as each runs through the adjacent 1-handles once, canceled each other. In addition, the top four 2-handles are isotopic to trivial unlinks on the boundary of the 2-handlebody of Figure 4, except κ, λ, μ and ν . For, the left in Figure 5 represents the boundary of the 2-handlebody except κ, λ, μ and ν . Sliding the two 0-framed links and m and n -framed links over the smaller links which sit inside them, we get the left picture in Figure 5. Thus these attaching circles (κ, λ, μ and ν) are canceled out with four 3-handles.

Therefore dual 2-handles for 2-handlebody of $X_{m,n}$ are the meridians for the bottom four 2-handles in Figure 4. The dual 2-handles are, hence, as four red lines in Figure 7. Then by handle sliding we get the diagram in Figure 8.

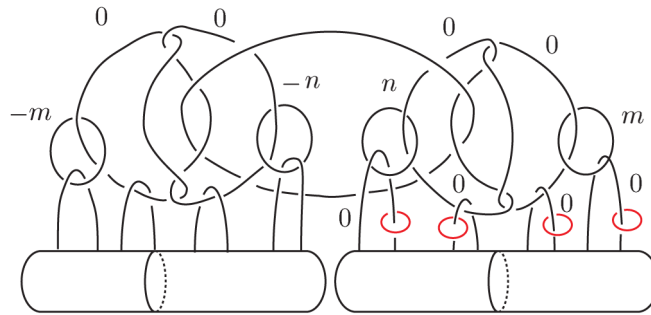


Figure 7. The dual handles.

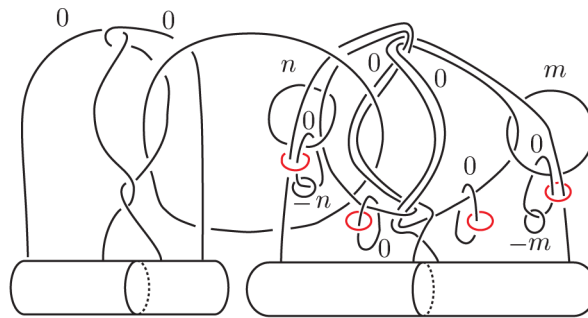


Figure 8. The dual handles.

In addition, several handle slides give Figures 9 and 10. Here, abbreviating the two handles as a box colored as in Figure 11, we get Figure 12. Using the notation and isotopy we get Figure 13, and keep track of the red two handles by the symmetry that exchanges the pair of links (a, b) to (c, d) ; hence, we get Figure 14. Keeping track of the diagram by the converse motion (Figures 13, 12, 10, 9, 8, 7 and 4) from the diagram in the form, we get Figure 15 and Figures 16, 7-level. Two 0-framed components (two unknots) in the diagram in Figure 16 correspond to 3-handles of $X_{m,n}$. Replacing the two framed links with dotted 1-handles, we get a handle diagram of $\mathcal{S}_{m,n,m',n'}$ (Figure 17).

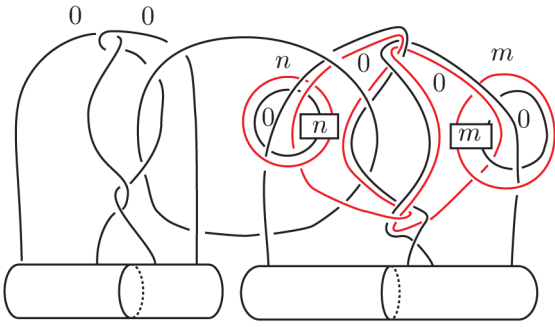


Figure 9. The dual handles.

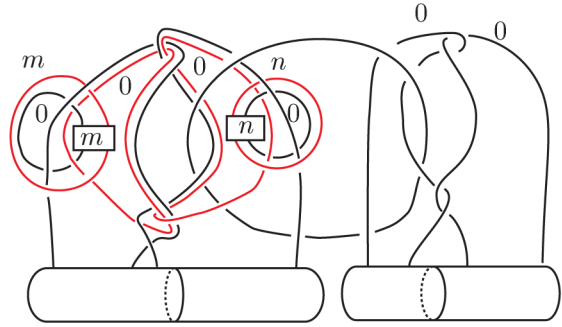


Figure 10. The dual handles.

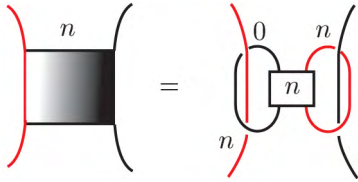


Figure 11. An abbreviation.

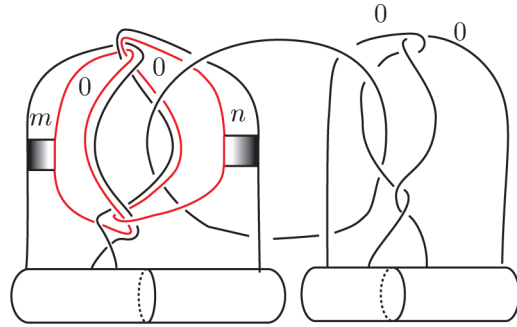


Figure 12. The dual handles on $\partial X_{m,n}$.

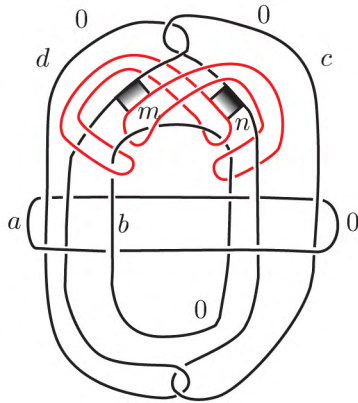


Figure 13. The dual handles on $\partial X_{m,n}$.

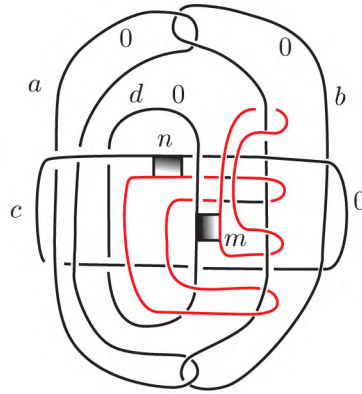


Figure 14. The symmetry on $\partial X_{m,n}$ by ϕ .

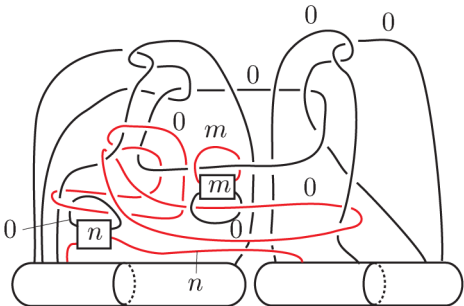


Figure 15. The four 2-handles attached by ϕ .

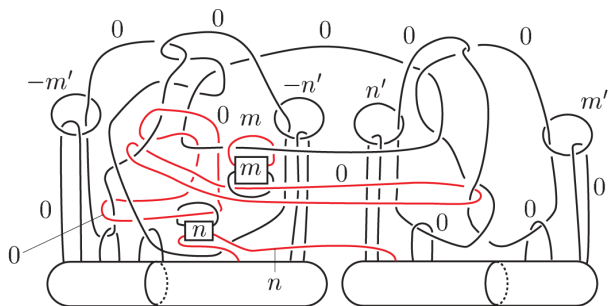


Figure 16. The four 2-handles attached by ϕ .

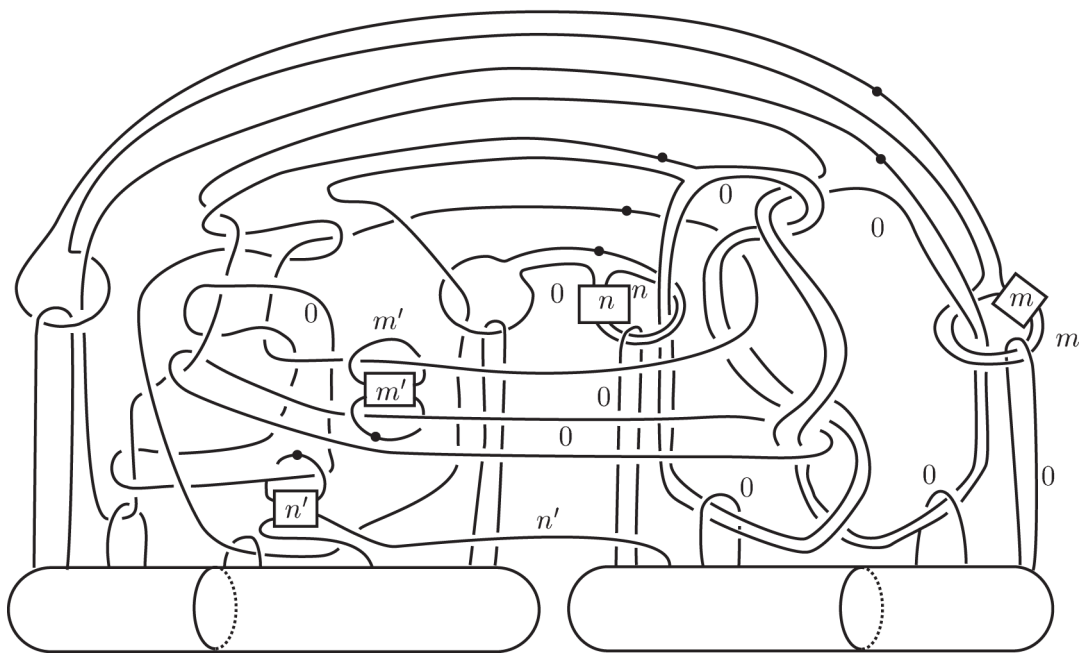


Figure 17. $\mathcal{S}_{m,n,m',n'}$.

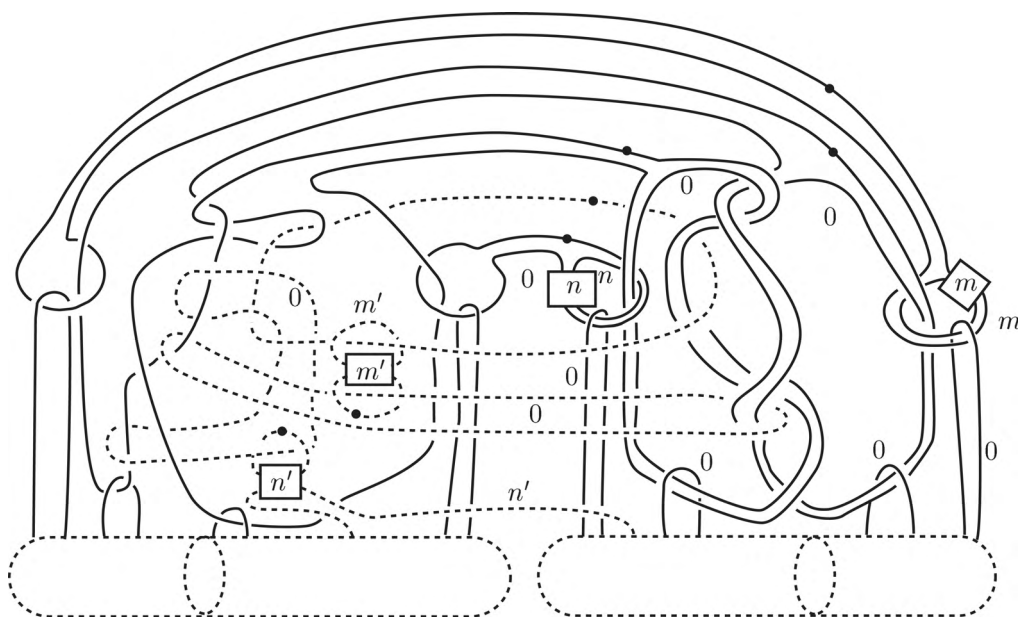


Figure 18. The four canceling pairs of $\mathcal{S}_{m,n,m',n'}$.

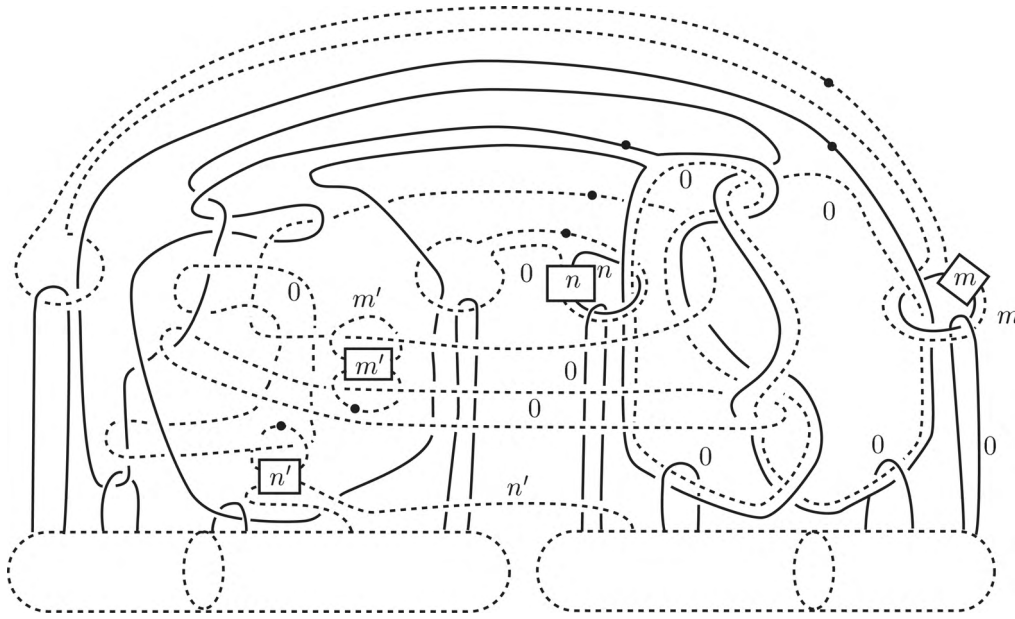


Figure 19. The six canceling pairs of $\mathcal{S}_{m,n,m',n'}$.

4. Handle calculus of $\mathcal{S}_{m,n,m',n'}$

Proposition 1 *Each of the manifolds $\mathcal{S}_{m,n,m',n'}$ has a handle decomposition without 1-handles. In addition the handle decomposition has 4 2-handles.*

Proof To prove this lemma, we will find eight 1,2-canceling pairs in Figure 17. Any canceling pair is drawn by dotted line. Here the only 1-handle goes on drawing as the ball description. First we take 4 pairs as below in Figure 18. We take 2 more pairs as in Figure 19 and successively 2 more pairs as in Figure 20.

We get a handle decomposition (Figure 21):

$$\mathcal{S}_{m,n,m',n'} = D^4 \cup^4 2\text{-handles} \cup^4 3\text{-handles} \cup 4\text{-handle.} \tag{1}$$

□

In (1) we denote the framed link for the attaching 2-handles by $\mathcal{F}_{m,n,m',n'}$. The four attaching circles e, f, g and h in Figure 21 are $\mathcal{F}_{m,n,m',n'}$.

Next we show the following.

Lemma 3 *$\mathcal{F}_{m,n,m',n'}$ are, after several handle slides, isotopic to $\mathcal{F}_{m,0,m',n'}$. Furthermore, two g and h of them are separated as a 2-component unlink after handle slides.*

Proof We denote canceling 2-handles by black lines and 2-handles $\mathcal{F}_{m,n,m',n'}$ by red lines. Two handle slides give Figure 22. Sliding a handle as indicated in this figure, we get Figure 23, and by isotopy we get Figure 24. Turning the diagram in the direction of the arrow in Figure 24 n' times, we obtain Figure 25. Removing bottom canceling 1,2-handle pair, we get Figure 26. Sliding h to g , and canceling two $\pm n$ -framed 2-handle with two 0-framed 2-handles, we get Figure 27. The curve g in Figure 27 is untied by use of several handle slides to get a separated 2-handle as in Figure 28. At this time the n and $-n$ boxes are untied by rotating (Figure 29). Sliding and canceling handles, we get Figures 30 and 31. In the form we can untie f by a handle slide as in Figure 32. Iterating this process, we get Figure 33. □

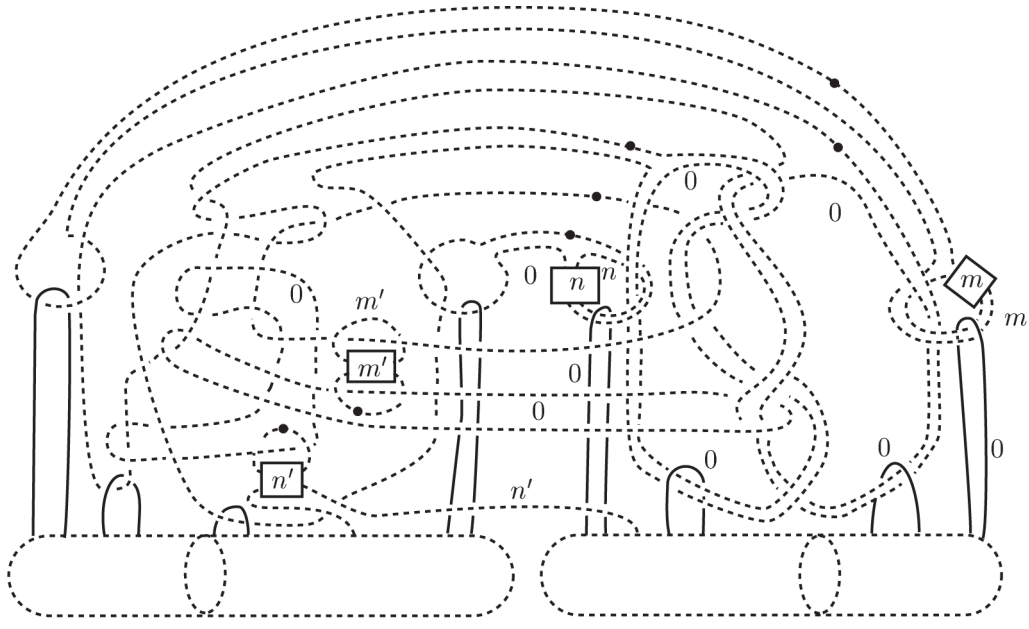


Figure 20. The eight canceling pairs of $\mathcal{S}_{m,n,m',n'}$.

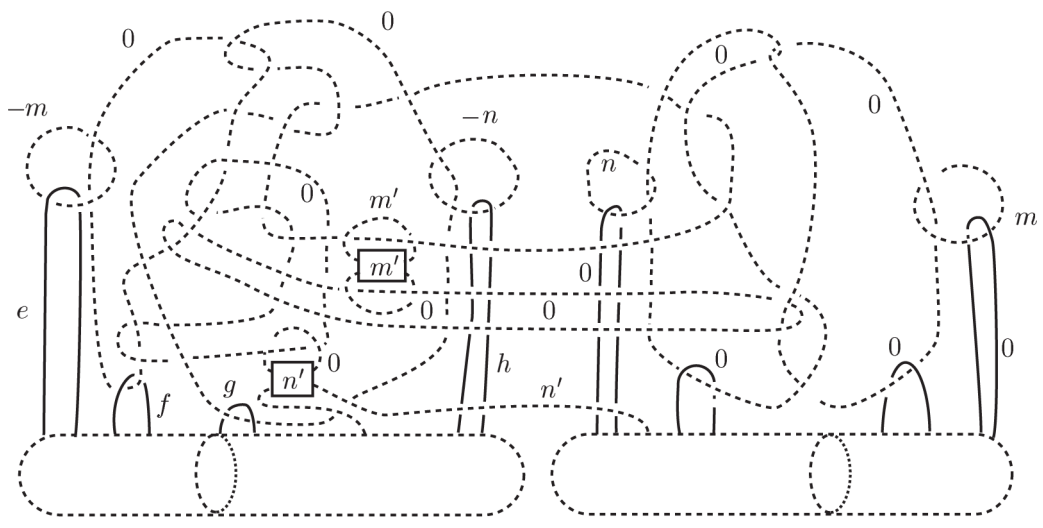


Figure 21. The handle decomposition $\mathcal{F}_{m,n,m',n'}$.

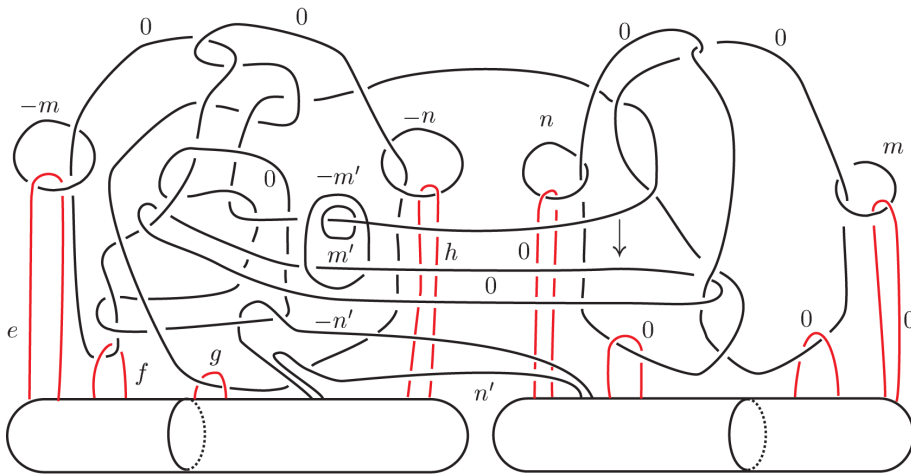


Figure 22. A handle slide as indicated by the arrow.

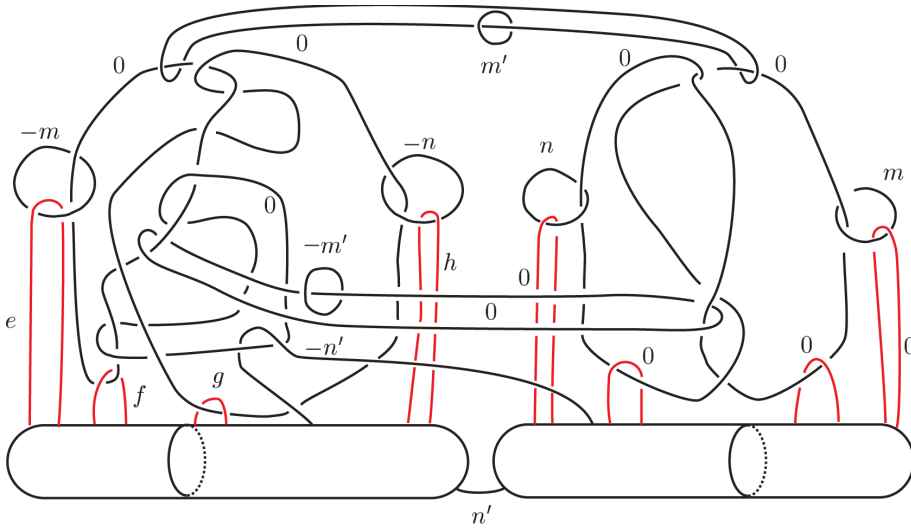


Figure 23. 2-handles in ∂D^4 .

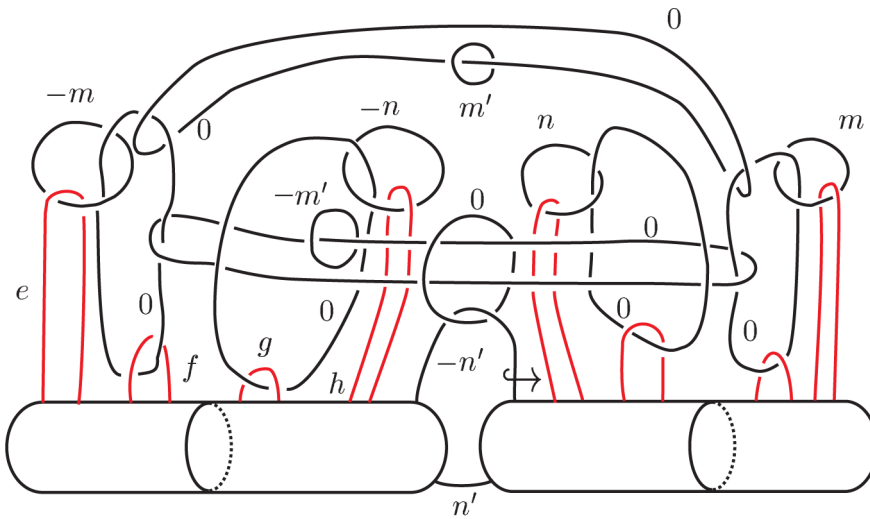


Figure 24. 2-handles in ∂D^4 .

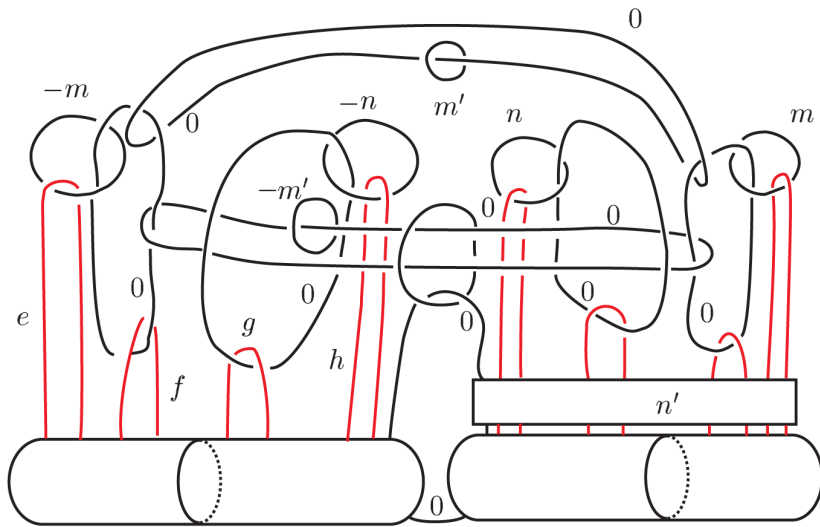


Figure 25. 2-handles in ∂D^4 .

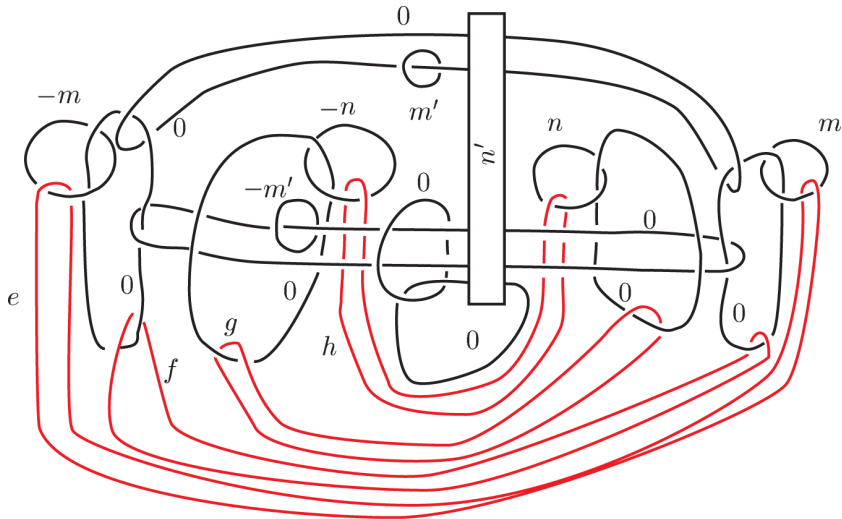


Figure 26. 2-handles in ∂D^4 .

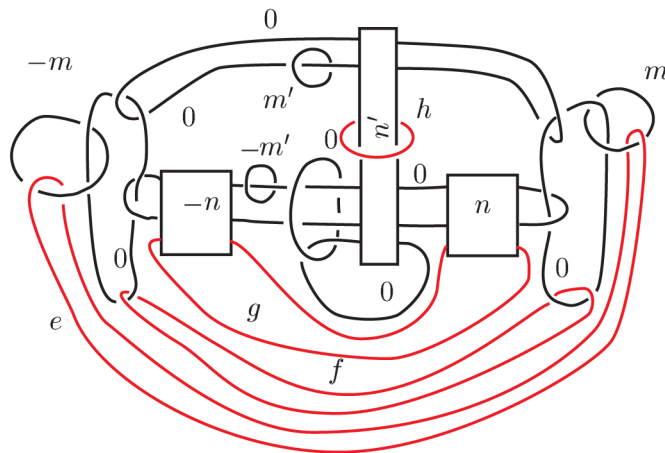


Figure 27. 2-handles in ∂D^4 .

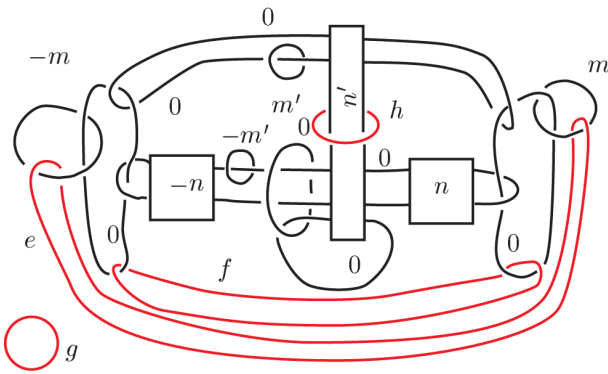


Figure 28. 2-handles in ∂D^4 .

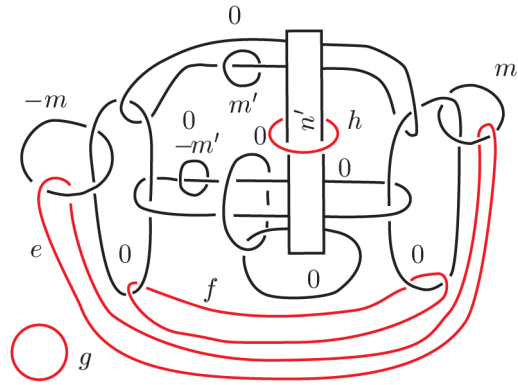


Figure 29. 2-handles in ∂D^4 .

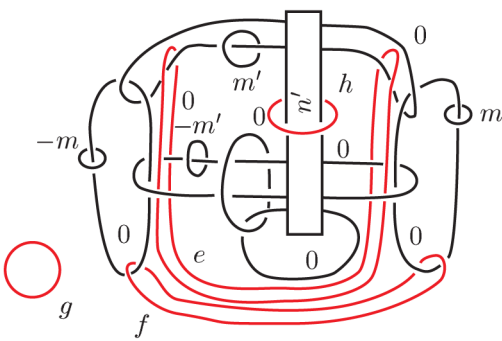


Figure 30. 2-handles in ∂D^4 .

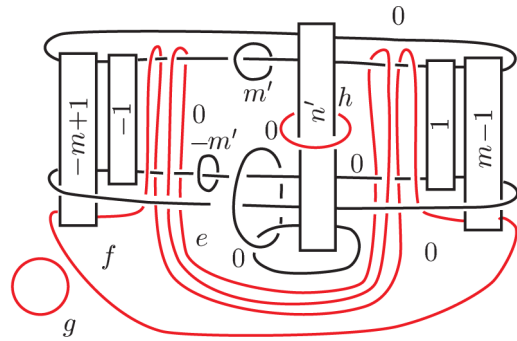


Figure 31. 2-handles in ∂D^4 .

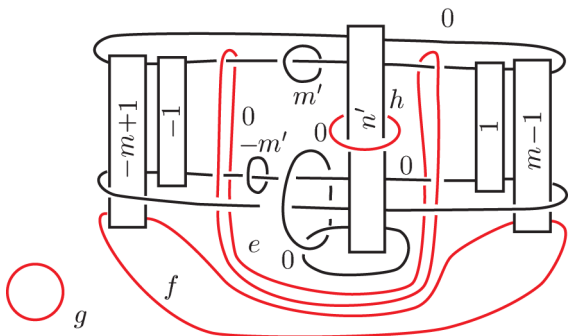


Figure 32. 2-handles in ∂D^4 .

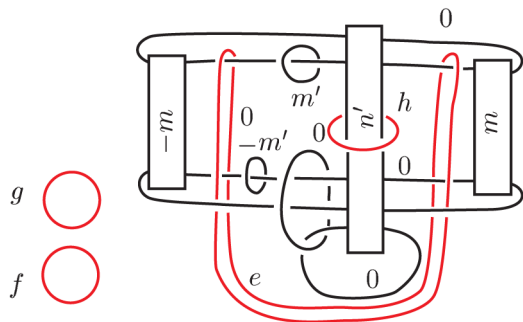


Figure 33. 2-handles in ∂D^4 .

4.1. Nash's manifolds as a torus surgery

In the subsection we show that each Nash's manifold is constructed by a logarithmic transformation along a single torus.

Proposition 2 For any integers m, n, m', n' we have

$$\mathcal{S}_{m,n,0,n'} \cong \mathcal{S}_{m,n,m',0} \cong S^4.$$

Proof Putting $m' = 0$, we have Figure 34. Canceling two pairs of 1 and 2-handle, we get Figure 35. By isotopy the picture becomes Figure 36. The resulting manifold is the surgering of $S^3 \times S^1$ along $\{\text{pt}\} \times S^1$ framing n' .

Namely, the manifold has the same diagram as Figure 37. This is diffeomorphic to S^4 . The manifold $\mathcal{S}_{m,n,m',0}$ is also diffeomorphic to S^4 in a similar way. \square

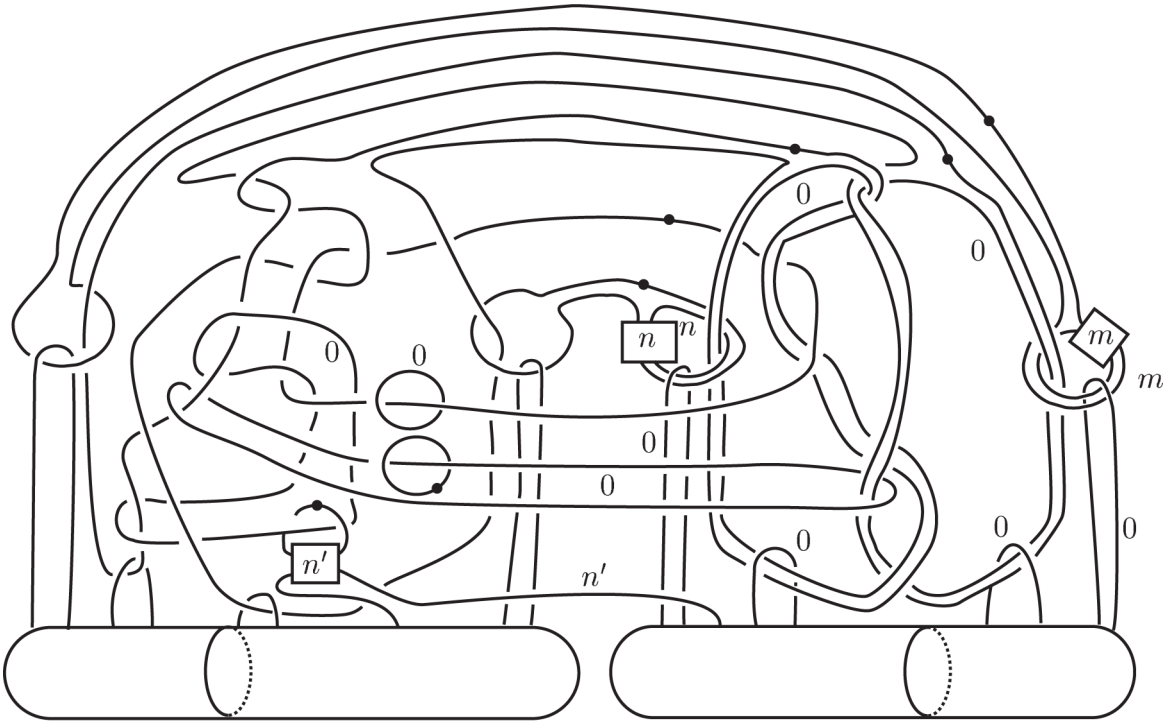


Figure 34. $\mathcal{S}_{m,n,0,n'}$

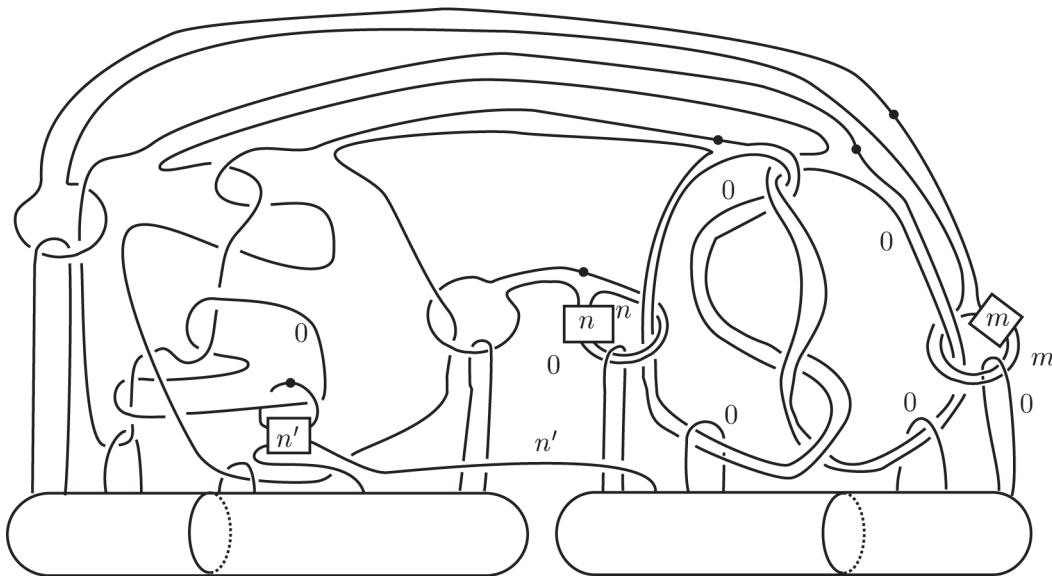


Figure 35. $\mathcal{S}_{m,n,0,n'}$

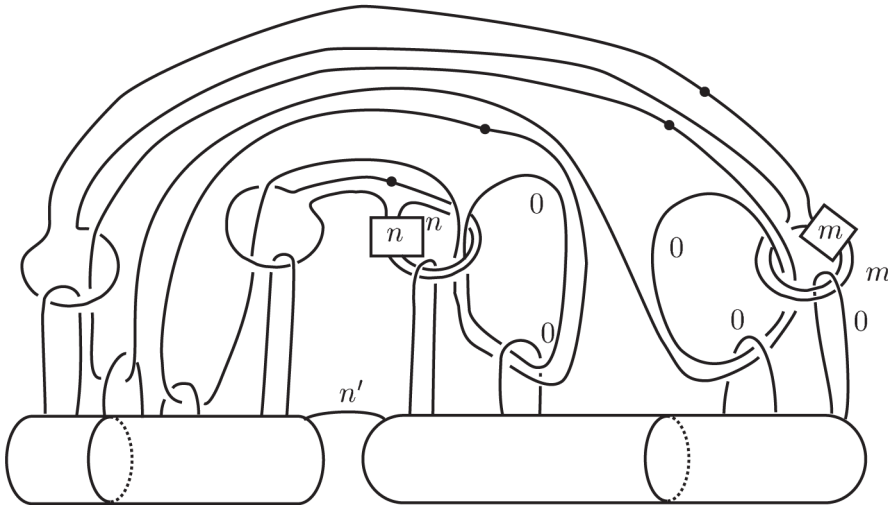


Figure 36. $\mathcal{S}_{m,n,0,n'}$

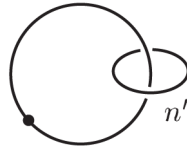


Figure 37. $\mathcal{S}_{m,n,0,n'}$

As a corollary we have the following.

Corollary 1 $\mathcal{S}_{m,n,m',n'}$ are given by one logarithmic transformation along a torus.

Now we are in a position to prove the main theorem.

4.2. Proof of Theorem 1

By Lemma 3 the 2-handlebody of $\mathcal{S}_{m,n,m',n'}$ consists of 4-component framed link, which is the same as $\mathcal{F}_{m,0,m',n'}$ up to several handle slides. Namely $\mathcal{S}_{m,n,m',n'}$ is diffeomorphic to $\mathcal{S}_{m,0,m',n'}$. From Lemma 1 and Proposition 2 we have $\mathcal{S}_{m,n,m',n'} \cong S^4$. □

Corollary 2 The diagram Figure 1 is a framed link presentation of $\#^2 S^2 \times S^1$.

Proof Figure 33 gives a handle decomposition of S^4 :

$$D^4 \cup^2 2\text{-handles} \cup^2 3\text{-handles} \cup 4\text{-handle.}$$

The boundary $\partial(D^4 \cup^2 2\text{-handles})$ is, therefore, $\#^2 S^2 \times S^1$. □

This corollary implies e and h in Figure 1 are candidates of counterexample to generalized Property R conjecture.

Acknowledgements

The work described in this paper was done during my visit to the University of Texas at Austin in 2011. I deeply appreciate Professor Cameron Gordon's acceptance of my visit. Furthermore, I thank Professor John

Luecke and Cagri Karakurt for their hospitality and the many pieces of chalk as well as the wide blackboard there to draw some pictures. Also, this research was supported by Grant-in-Aid for JSPS Fellow (21-1458).

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