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A scheme over prime spectrum of modules

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Abstract: Let R be a commutative ring with nonzero identity and let M be an R-module with $X = \mathrm{Spec}(M)$. It is introduced a scheme $\mathcal{O}_X$ on the prime spectrum of M and some of its properties have been investigated.

Key words and phrases: Prime submodule, Zariski topology, primeful module, sheaf of rings, scheme

1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unital. For a submodule N of an R-module M, $(N :_R M)$ denotes the ideal $\{r \in R \mid rM \subseteq N\}$ and annihilator of M, denoted by $\mathrm{Ann}_R(M)$, is the ideal $(\mathbf{0} :_R M)$. If there is no ambiguity we write $(N : M)$ (resp. $\mathrm{Ann}(M)$) instead of $(N :_R M)$ (resp. $\mathrm{Ann}_R(M)$). An R-module M is called faithful if $\mathrm{Ann}(M) = (0)$.

A submodule N of an R-module M is said to be prime if $N \neq M$ and whenever $rm \in N$ (where $r \in R$ and $m \in M$) then $r \in (N : M)$ or $m \in N$. If N is prime, then the ideal $\mathfrak{p} = (N : M)$ is a prime ideal of R. In these circumstances, N is said to be $\mathfrak{p}$-prime (see [2]). The set of all prime submodules of an R-module M is called the prime spectrum of M and denoted by $\mathrm{Spec}(M)$. Similarly, the collection of all $\mathfrak{p}$-prime submodules of R-module M for any $\mathfrak{p} \in \mathrm{Spec}(R)$ is designated by $\mathrm{Spec}_{\mathfrak{p}}(M)$. We remark that $\mathrm{Spec}(\mathbf{0}) = \emptyset$ and that $\mathrm{Spec}(M)$ may be empty for some nonzero R-module module M. For example, the $\mathbb{Z}(p^\infty)$ as a $\mathbb{Z}$-module has no prime submodule for any prime integer p (see [3] and [7]). Such a module is said to be primeless. An R-module M is called primeful if either $M = (\mathbf{0})$ or $M \neq (\mathbf{0})$ and the natural map $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ defined by $\psi(P) = (P : M)/\mathrm{Ann}(M)$ for every $P \in \mathrm{Spec}(M)$, is surjective (see [6]). Let $\mathfrak{p}$ be a prime ideal of R, and $N \leq M$. By the saturation of N with respect to $\mathfrak{p}$, we mean the contraction of $N_{\mathfrak{p}}$ in M and designate it by $S_{\mathfrak{p}}(N)$ (see [5]).

Let M be an R-module. Throughout this paper X denotes the prime spectrum $\mathrm{Spec}(M)$ of M. Let N be a submodule of M. Then $V(N)$ is defined as, $V(N) = \{P \in X \mid (P : M) \supseteq (N : M)\}$ (see [4]). Set $\mathsf{Z}(M) = \{V(N) : N \leq M\}$. Then the elements of the set $\mathsf{Z}(M)$ satisfy the axioms for closed sets in a topological space X (see [4]). The resulting topology is called the Zariski topology relative to M.

We recall some preliminary results.

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Remark 1.1 *(See [4, Theorem 6.1].) The following statements are equivalent:*

1. *X is T_0-space;*
2. *The natural map $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ is injective;*
3. *If $V(P) = V(Q)$, then $P = Q$ for any $P, Q \in \mathrm{Spec}(M)$;*
4. *$|\mathrm{Spec}_{\mathfrak{p}}(M)| \leq 1$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$.*

Remark 1.2 *(See [4].) For any element r of a ring R, the set $D_r = \mathrm{Spec}(R) \setminus V(rR)$ is open in $\mathrm{Spec}(R)$ and the family $F = \{D_r | r \in R\}$ forms a base for the Zariski topology on $\mathrm{Spec}(R)$. Each D_r, in particular, $D_1 = \mathrm{Spec}(R)$ is known to be quasi-compact. For each $r \in R$, we define $X_r = X - V(rM)$. Then every X_r is an open set of X, $X_0 = \emptyset$, and $X_1 = X$. By [4, Corollary 4.2], for any $r, s \in R$, $X_{rs} = X_r \cap X_s$.*

2. Main results

In this section we use the notion of prime spectrum of a module to define a sheaf of rings. Let M be an R-module. For every open subset U of X we define $\mathrm{Supp}(U) = \{(P : M) \mid P \in U\}$.

Definition 2.1 *Let M be an R-module. For every open subset U of X we define $\mathcal{O}_X(U)$ to be a subring of $\prod_{\mathfrak{p} \in \mathrm{Supp}(U)} R_{\mathfrak{p}}$, the ring of functions $s : U \to \coprod_{\mathfrak{p} \in \mathrm{Supp}(U)} R_{\mathfrak{p}}$, where $s(P) \in R_{\mathfrak{p}}$, for each $P \in U$ and $\mathfrak{p} = (P : M)$ such that for each $P \in U$, there is a neighborhood V of P, contained in U, and elements $a, f \in R$, such that for each $Q \in V$, $f \notin \mathfrak{q} := (Q : M)$, and $s(Q) = a/f$ in $R_{\mathfrak{q}}$.*

It is clear that for an open set U of X, $\mathcal{O}_X(U)$ is closed under sum and product. Thus $\mathcal{O}_X(U)$ is a commutative ring with identity (the identity element of $\mathcal{O}_X(U)$ is the function which sends all $P \in U$ to 1 in $R_{(P:M)}$). If $V \subseteq U$ are two open sets, the natural restriction map $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is a homomorphism of rings. It is then clear that $\mathcal{O}_X$ is a presheaf. Finally, it is clear from the local nature of the definition $\mathcal{O}_X$ is a sheaf. Hence

Lemma 2.2 *Let M be an R-module.*

1. *For each open subset U of X, $\mathcal{O}_X(U)$ is a subring of $\prod_{\mathfrak{p} \in \mathrm{Supp}(U)} R_{\mathfrak{p}}$.*
2. *$\mathcal{O}_X$ is a sheaf.*

Next, we find the stalk of the sheaf.

Proposition 2.3 *Let M be an R-module. Then for each $P \in X$, the stalk $\mathcal{O}_P$ of the sheaf $\mathcal{O}_X$ is isomorphic to $R_{\mathfrak{p}}$, where $\mathfrak{p} := (P : M)$.*

Proof Let P be a $\mathfrak{p}$-prime submodule of M and

$$m \in \mathcal{O}_P = \varinjlim_{P \in U} \mathcal{O}_X(U).$$

Then there exists a neighborhood U of P and $s \in \mathcal{O}_X(U)$ such that m is the germ of s at the point P. We define a homomorphism $\varphi : \mathcal{O}_P \to R_{\mathfrak{p}}$ by $\varphi(m) = s(P)$. This is a well-defined homomorphism. Let V be another neighborhood of P and $t \in \mathcal{O}_X(V)$ such that m is the germ of s at the point P. Then there exists an open subset $W \subseteq U \cap V$ such that $P \in W$ and $s|_W = t|_W$. Since $P \in W$, $s(P) = t(P)$. We claim that φ is an isomorphism.

Let $x \in R_{\mathfrak{p}}$. Then $x = a/f$ where $a \in R$ and $f \in R \setminus \mathfrak{p}$. Since $f \notin \mathfrak{p}$, $P \in X_f$. Now we define $s(Q) = a/f$ in $R_{\mathfrak{q}}$, where $\mathfrak{q} := (Q : M)$, for all $Q \in X_f$. Then $s \in \mathcal{O}(X_f)$. If m is the equivalent class of s in $\mathcal{O}_P$, then $\varphi(m) = x$. Hence φ is surjective.

Now, let $m \in \mathcal{O}_P$ and $\varphi(m) = 0$. Let U be an open neighborhood of P and m be the germ of $s \in \mathcal{O}_X(U)$ at P. There is an open neighborhood $V \subseteq U$ of P and elements $a, f \in R$ such that $s(Q) = a/f \in R_{\mathfrak{q}}$, where $\mathfrak{q} := (Q : M)$, for all $Q \in V$, $f \notin \mathfrak{q}$. Thus $V \subseteq X_f$. Then $0 = \varphi(m) = s(P) = a/f$ in $R_{\mathfrak{p}}$. So, there is $h \in R \setminus \mathfrak{p}$ such that $ha = 0$. For $Q \in X_{fh} = X_f \cap X_h$ we have $s(Q) = a/f \in R_{\mathfrak{q}}$. Since $h \notin \mathfrak{q}$, $s(Q) = \frac{a}{f} = \frac{h}{h}\frac{a}{f} = 0$. Thus $s|_{\mathcal{O}(X_{fh})} = 0$. Therefore, $s = 0$ in $\mathcal{O}(X_{fh})$. Consequently $m = 0$. This completes the proof. □

As a direct consequence of Proposition 2.3, we have

Corollary 2.4 *If M is an R-module, then $(\mathrm{Spec}(M), \mathcal{O}_{\mathrm{Spec}(M)})$ is a locally ringed space.*

Proposition 2.5 *Let M and N be R-modules and $\phi : M \to N$ be an epimorphism. Then ϕ induces a morphism of locally ringed spaces*

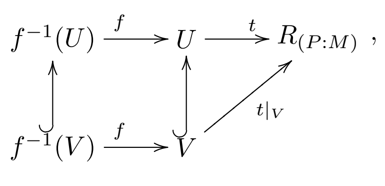
$$(f, f^\sharp) : (\mathrm{Spec}(N), \mathcal{O}_{\mathrm{Spec}(N)}) \to (\mathrm{Spec}(M), \mathcal{O}_{\mathrm{Spec}(M)}).$$

Proof By [4, Proposition 3.9], the map $f : \mathrm{Spec}(N) \to \mathrm{Spec}(M)$ which is defined by $P \mapsto \phi^{-1}(P)$, is continuous. Let U be an open subset of $\mathrm{Spec}(M)$ and $s \in \mathcal{O}_{\mathrm{Spec}(M)}(U)$. Suppose $P \in f^{-1}(U)$. Then $f(P) = \phi^{-1}(P) \in U$. Assume that W is an open neighborhood of $\phi^{-1}(P)$ with $W \subseteq U$ with $a, g \in R$, such that for each $Q \in W$, $g \notin \mathfrak{q} := (Q : M)$, and $s(Q) = a/g$ in $R_{\mathfrak{q}}$. Since $\phi^{-1}(P) \in W$, $P \in f^{-1}(W)$. As we mentioned, f is continuous, so $f^{-1}(W)$ is an open subset of $\mathrm{Spec}(N)$. We claim that for each $Q' \in f^{-1}(W)$, $g \notin (Q' : N)$. Suppose $g \in (Q' : N)$ for some $Q' \in f^{-1}(W)$. Then $\phi^{-1}(Q') = f(Q') \in W$. Since ϕ an epimorphism, $(Q' : N) = (\phi^{-1}(Q') : M)$. So, $g \in (\phi^{-1}(Q') : M)$. This is a contradiction. Therefore, we can define

$$f^\sharp(U) : \mathcal{O}_{\mathrm{Spec}(M)}(U) \to \mathcal{O}_{\mathrm{Spec}(N)}(f^{-1}(U))$$

by $f^\sharp(U)(s) = s \circ f$.

Assume that $V \subseteq U$ and $P \in f^{-1}(V)$. According to the commutativity of the diagram



we have

$$(t \circ f)|_{f^{-1}(V)}(P) = t|_V \circ f(P). \tag{2.1}$$

Consider the diagram

$$\begin{array}{ccc}\mathcal{O}_{\mathrm{Spec}(M)}(U) & \xrightarrow{f^\sharp(U)} & \mathcal{O}_{\mathrm{Spec}(N)}(f^{-1}(U)) \\ \rho_{UV}\downarrow & & \downarrow \rho'_{f^{-1}(U)f^{-1}(V)} \\ \mathcal{O}_{\mathrm{Spec}(M)}(V) & \xrightarrow{f^\sharp(V)} & \mathcal{O}_{\mathrm{Spec}(N)}(f^{-1}(V)).\end{array} \tag{A}$$

Since

$$\begin{aligned}\rho'_{f^{-1}(U)f^{-1}(V)}f^\sharp(U)(t)(P) &= \rho'_{f^{-1}(U)f^{-1}(V)}(t\circ f)(P)\\ &= (t\circ f)|_{f^{-1}(V)}(P)\\ &= t|_V\circ f(P) \qquad \text{by equation 2.1}\\ &= \rho_{UV}(t)\circ f(P)\\ &= f^\sharp(V)\rho_{UV}(t)(P),\end{aligned}$$

for each $t\in\mathcal{O}_{\mathrm{Spec}(M)}(U)$, the diagram (A) is commutative, and it follows that

$$f^\sharp:\mathcal{O}_{\mathrm{Spec}(M)}\longrightarrow f_*\mathcal{O}_{\mathrm{Spec}(N)}$$

is a morphism of sheaves. By Proposition 2.3, the map on stalks

$$f^\sharp_P:\mathcal{O}_{\mathrm{Spec}(M),f(P)}\longrightarrow \mathcal{O}_{\mathrm{Spec}(N),P}$$

is clearly the map of local rings

$$R_{(f(P):M)}\longrightarrow R_{(P:N)}.$$

This implies that

$$(\mathrm{Spec}(N),\mathcal{O}_{\mathrm{Spec}(N)})\xrightarrow{(f,f^\sharp)}(\mathrm{Spec}(M),\mathcal{O}_{\mathrm{Spec}(M)})$$

is a morphism of locally ringed spaces. □

Proposition 2.6 *Let $\Phi:R\to S$ be a ring homomorphism, N a S-module and M a primeful R-module such that $\mathrm{Spec}(M)$ is a T_0-space and $\mathrm{Ann}_R(M)\subseteq\mathrm{Ann}_R(N)$ (here, we consider N as an R-module by means of Φ). Then Φ induces a morphism of locally ringed spaces*

$$(\mathrm{Spec}(N),\mathcal{O}_{\mathrm{Spec}(N)})\xrightarrow{(h,h^\sharp)}(\mathrm{Spec}(M),\mathcal{O}_{\mathrm{Spec}(M)}).$$

Proof Since $\mathrm{Ann}_R(M)\subseteq\mathrm{Ann}_R(N)$, Φ induces the map $\Theta:R/\mathrm{Ann}_R(M)\to S/\mathrm{Ann}_S(N)$. It is well known that the maps $f:\mathrm{Spec}(S)\to\mathrm{Spec}(R)$ by $\mathfrak{p}\mapsto\Phi^{-1}(\mathfrak{p})$ and $d:\mathrm{Spec}(S/\mathrm{Ann}_S(N))\to\mathrm{Spec}(R/\mathrm{Ann}_R(M))$ by $\overline{\mathfrak{p}}\mapsto\Theta^{-1}(\overline{\mathfrak{p}})$ and $\psi_N:\mathrm{Spec}(N)\to\mathrm{Spec}(S/\mathrm{Ann}_S(N))$ with $\psi(P)=(P:_S N)/\mathrm{Ann}_S(N)$ for each $P\in\mathrm{Spec}(N)$ are continuous maps. Also $\psi_M:\mathrm{Spec}(M)\to\mathrm{Spec}(R/\mathrm{Ann}_R(M))$ is homeomorphism by [4, Theorem 6.5]. Therefore the map

$$\begin{array}{rccc}h: & \mathrm{Spec}(N) & \longrightarrow & \mathrm{Spec}(M)\\ & P & \mapsto & \psi_M^{-1}\,d\,\psi_N(P)\end{array}$$

is continuous. For each $P \in \operatorname{Spec}(N)$, we get a local homomorphism

$$\Phi_{(P:_S N)} : R_{f(P:_S N)} \longrightarrow S_{(P:_S N)}.$$

Let U be an open subset of $\operatorname{Spec}(M)$ and let $t \in \mathcal{O}_{\operatorname{Spec}(M)}(U)$. Suppose that $P \in h^{-1}(U)$. Then $h(P) \in U$ and there exists a neighborhood W of $h(P)$ with $W \subseteq U$ and elements $r, g \in R$ such that for each $Q \in W$, $g \notin (Q :_R M)$, and $t(Q) = \frac{r}{g} \in R_{(Q:_R M)}$. Hence $g \notin (h(P) :_R M)$. By definition of h, $(h(P) :_R M) = \Phi^{-1}(P :_S N)$. So, $\Phi(g) \notin (P :_S N)$. Thus $\Phi_{(P:_S N)}(\frac{r}{g})$ define a section on $\mathcal{O}_{\operatorname{Spec}(N)}(h^{-1}(W))$. Since

$$\begin{array}{ccc} R_g & \longrightarrow & S_{\Phi(g)} \\ \downarrow & & \downarrow \\ R_{\Phi^{-1}(P:_S N)} & \longrightarrow & S_{(P:_S N)} \end{array}$$

is commutative, we can define

$$h^\sharp(U) : \mathcal{O}_{\operatorname{Spec}(M)}(U) \longrightarrow h_* \mathcal{O}_{\operatorname{Spec}(N)}(U) = \mathcal{O}_{\operatorname{Spec}(N)}(h^{-1}(U))$$

by $h^\sharp(U)(t)(P) = \Phi_{(P:_S N)}(t(h(P)))$ for each $t \in \mathcal{O}_{\operatorname{Spec}(M)}(U)$ and $P \in h^{-1}(U)$. Assume that $V \subseteq U$ and $P \in h^{-1}(V)$.

According to the commutative diagram

$$\begin{array}{ccccc} h^{-1}(U) & \xrightarrow{h} & U & \xrightarrow{t} & \\ \uparrow & & \uparrow & & \\ h^{-1}(V) & \xrightarrow{h} & V & \xrightarrow{t|_V} & R_{\Phi^{-1}(P:_S N)} \\ & & & & \downarrow \Phi_{(P:_S N)} \\ & \xrightarrow{\Phi_{(P:_S N)} t|_V \circ h} & & & S_{(P:_S N)}, \end{array}$$

we have

$$\Phi_{(P:_S N)} t|_V \circ h(P) = (\Phi_{(P:_S N)} t \circ h)|_{h^{-1}(V)}(P). \tag{2.2}$$

Considering the diagram

$$\begin{array}{ccc} \mathcal{O}_{\operatorname{Spec}(M)}(U) & \xrightarrow{h^\sharp(U)} & \mathcal{O}_{\operatorname{Spec}(N)}(h^{-1}(U)) \\ \rho_{UV} \downarrow & & \downarrow \rho'_{h^{-1}(U)h^{-1}(V)} \\ \mathcal{O}_{\operatorname{Spec}(M)}(V) & \xrightarrow{h^\sharp(V)} & \mathcal{O}_{\operatorname{Spec}(N)}(h^{-1}(V)), \end{array} \tag{B}$$

it is easy to see that

$$\begin{aligned} \rho'_{h^{-1}(U)h^{-1}(V)}h^{\sharp}(U)(t)(P) &= \rho'_{h^{-1}(U)h^{-1}(V)}\Phi_{(P:_S N)}t \circ h(P) \\ &= (\Phi_{(P:_S N)}t \circ h)|_{h^{-1}(V)}(P) \\ &= \Phi_{(P:_S N)}t|_V \circ h(P) \qquad \text{by equation 2.2} \\ &= h^{\sharp}(V)(t|_V)(P) \\ &= h^{\sharp}(V)\rho_{UV}(t)(P). \end{aligned}$$

So, the diagram (B) is commutative, and it follows that

$$h^{\sharp} : \mathcal{O}_{\mathrm{Spec}(M)} \longrightarrow h_* \mathcal{O}_{\mathrm{Spec}(N)}$$

is a morphism of sheaves. By Proposition 2.3, the map on stalks

$$h^{\sharp}_P : \mathcal{O}_{\mathrm{Spec}(M),h(P)} \longrightarrow \mathcal{O}_{\mathrm{Spec}(N),P}$$

is clearly

$$R_{f(P:_S N)} \longrightarrow S_{(P:_S N)}.$$

This implies that

$$(\mathrm{Spec}(N), \mathcal{O}_{\mathrm{Spec}(N)}) \xrightarrow{(h,h^{\sharp})} (\mathrm{Spec}(M), \mathcal{O}_{\mathrm{Spec}(M)})$$

is a morphism of locally ringed spaces. □

Example 2.7 *Let Ω be the set of all prime integers p, $M = \prod_p \frac{\mathbb{Z}}{p\mathbb{Z}}$ and $N = \bigoplus_p \frac{\mathbb{Z}}{p\mathbb{Z}}$ where p runs through Ω. By [6, p.136, Example 1], N is a faithful $\mathbb{Z}$-module and M is a faithful primeful $\mathbb{Z}$-module. It is also shown that*

$$\mathrm{Spec}(M) = \{S_{(0)}(\boldsymbol{0})\} \cup \{pM | p \in \Omega\}.$$

Therefore by Remark 1.1, $\mathrm{Spec}(M)$ *is a T_0-space. Hence by Proposition 2.6, there exists a morphism of locally ringed spaces*

$$(\mathrm{Spec}(\bigoplus_p \frac{\mathbb{Z}}{p\mathbb{Z}}), \mathcal{O}_{\mathrm{Spec}(\bigoplus_p \frac{\mathbb{Z}}{p\mathbb{Z}})}) \to (\mathrm{Spec}(\prod_p \frac{\mathbb{Z}}{p\mathbb{Z}}), \mathcal{O}_{\mathrm{Spec}(\prod_p \frac{\mathbb{Z}}{p\mathbb{Z}})}).$$

Proposition 2.8 *Let M be a faithful and primeful R-module. For any element $f \in R$, the ring $\mathcal{O}_X(X_f)$ is isomorphic to the localized ring R_f.*

Proof We define the map $\Theta : R_f \to \mathcal{O}_X(X_f)$ by

$$\frac{a}{f^m} \mapsto (s : Q \mapsto \frac{a}{f^m} \in R_{(Q:M)}).$$

Indeed Θ sends that $\frac{a}{f^m}$ to the section $s \in \mathcal{O}_X(X_f)$ which assigns to each Q the image of $\frac{a}{f^m} \in R_{(Q:M)}$. It is easy to see Θ is a well-defined homomorphism. We are going to show that Θ is an isomorphism.

We first show that Θ is injective. If $\Theta(\frac{a}{f^n}) = \Theta(\frac{b}{f^m})$, then for every $P \in X_f$, $\frac{a}{f^n}$ and $\frac{b}{f^m}$ have the same image in $R_{\mathfrak{p}}$, where $\mathfrak{p} = (P : M)$. Thus there exists $h \in R \setminus \mathfrak{p}$ such that $h(f^m a - f^n b) = 0$ in R. Let $I = (0 :_R f^m a - f^n b)$. Then $h \in I$ and $h \notin \mathfrak{p}$, so $I \nsubseteq \mathfrak{p}$. This happens for any $P \in X_f$, so we conclude that

$$V(I) \cap \mathrm{Supp}(X_f) = \emptyset$$

hence

$$\mathrm{Supp}(X_f) \subseteq D(I) := \mathrm{Spec}(R) \setminus V(I).$$

Since M is faithful primeful,

$$D_f = \mathrm{Supp}(X_f) \subseteq D(I).$$

Therefore $f \in \sqrt{I}$ and so $f^l \in I$ for some positive integer l. Now we have $f^l(f^m a - f^n b) = 0$ which shows that $\frac{a}{f^n} = \frac{b}{f^m}$ in $R_{\mathfrak{p}}$. Hence Θ is injective.

Let $s \in \mathcal{O}_X(X_f)$. Then we can cover X_f with open subset V_i, on which s is represented by $\frac{a_i}{g_i}$, with $g_i \notin (P : M)$ for all $P \in V_i$, in other words $V_i \subseteq X_{g_i}$. By [4, Proposition 4.3], the open sets of the form X_h form a base for the topology. So, we may assume that $V_i = X_{h_i}$ for some $h_i \in R$. Since $X_{h_i} \subseteq X_{g_i}$, by [4, Proposition 4.1], $h_i \in \sqrt{(g_i)}$. Thus $h_i^n \in (g_i)$ for some $n \in \mathbb{N}$. So, $h_i^n = cg_i$ and

$$\frac{a_i}{g_i} = \frac{ca_i}{cg_i} = \frac{ca_i}{h_i^n}.$$

We see that s is represented by $\frac{b_i}{k_i}$, $(b_i = ca_i, k_i = h_i^n)$ on X_{k_i} and (since $X_{h_i} = X_{h_i^n}$) the X_{k_i} cover X_f. The open cover $X_f = \bigcup X_{k_i}$ has a finite subcover by [4, Proposition 4.4]. Suppose, $X_f \subseteq X_{k_1} \cup \cdots \cup X_{k_n}$. For $1 \le i, j \le n$, $\frac{b_i}{k_i}$ and $\frac{b_j}{k_j}$ both represent s on $X_{k_i} \cap X_{k_j}$. By Remark 1.2, $X_{k_i} \cap X_{k_j} = X_{k_i k_j}$ and by injectivity of Θ, we get $\frac{b_i}{k_i} = \frac{b_j}{k_j}$ in $R_{k_i k_j}$. Hence for some n_{ij},

$$(k_i k_j)^{n_{ij}}(k_j b_i - k_i b_j) = 0.$$

Let $m = \max\{n_{ij} | 1 \le i, j \le n\}$. Then

$$k_j^{m+1}(k_i^m b_i) - k_i^{m+1}(k_j^m b_j) = 0.$$

By replacing each k_i by k_i^{m+1}, and b_i by $k_i^m b_i$, we still see that s is represented on X_{k_i} by $\frac{b_i}{k_i}$, and furthermore, we have $k_j b_i = k_i b_j$ for all i, j. Since $X_f \subseteq X_{k_1} \cup \cdots \cup X_{k_n}$, by [4, Proposition 4.1], we have

$$D_f = \psi(X_f) \subseteq \bigcup_{i=1}^{n} \psi(X_{k_i}) = \bigcup_{i=1}^{n} D_{k_i},$$

where ψ is the natural map $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(R)$. So, there are $c_1, \cdots, c_n$ in R and $t \in \mathbb{N}$, such that $f^t = \sum_i c_i k_i$. Let $a = \sum_i c_i b_i$. Then for each j we have

$$k_j a = \sum_i c_i b_i k_j = \sum_i c_i k_i b_j = b_j f^t.$$

This implies that $\frac{a}{f^t} = \frac{b_j}{k_j}$ on X_{k_j}. So $\Theta(\frac{a}{f^t}) = s$ everywhere, which shows that Θ is surjective. □

Corollary 2.9 *Let M be a faithful and primeful R-module. Then $\mathcal{O}(\mathrm{Spec}(M))$ is isomorphic to R.*

We recall that a scheme X is locally Noetherian if it can be covered by open affine subsets $\mathrm{Spec}(A_i)$, where each A_i is a Noetherian ring. X is Noetherian if it is locally Noetherian and quasi-compact [1].

Theorem 2.10 *Let M be a faithful and primeful R-module such that X is a T_0-space. Then $(X, \mathcal{O}_X)$ is a scheme. Moreover, if R is Noetherian, then $(X, \mathcal{O}_X)$ is a Noetherian scheme.*

Proof Let $g \in R$. Since the natural map $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(R)$ is continuous by [4, Proposition 3.1], the map $\psi|_{X_g} : X_g \to \psi(X_g)$ is also continuous. By assumption and Remark 1.1, $\psi|_{X_g}$ is a bijection. Let E be a closed subset of X_g. Then $E = X_g \cap V(N)$ for some submodule N of M. Hence $\psi(E) = \psi(X_g \cap V(N)) = \psi(X_g) \cap V(N : M)$ is a closed subset of $\psi(X_g)$. Therefore, $\psi|_{X_g}$ is a homeomorphism.

Suppose $X = \bigcup_{i \in I} X_{g_i}$. Since M is faithful primeful and X is a T_0-space, for each $i \in I$

$$X_{g_i} \cong \psi(X_{g_i}) = \mathrm{Supp}(X_{g_i}) = D_{g_i} \cong \mathrm{Spec}(R_{g_i}).$$

Thus by Proposition 2.8, X_{g_i} is an affine scheme and this implies that $(X, \mathcal{O}_X)$ is a scheme. For the last statement, we note that since R is Noetherian, so is R_{g_i} for each $i \in I$. Hence $(X, \mathcal{O}_X)$ is a locally Noetherian scheme. By [4, Theorem 4.4], X is quasi-compact. Therefore, $(X, \mathcal{O}_X)$ is a Noetherian scheme. □

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