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Global and finitistic dimension of Hopf-Galois extensions

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Abstract: Let H be a Hopf algebra over a field k and A/B a right H-Galois extension. Then in this note a spectral sequence for Ext will be constructed which yields the estimate for global dimension of A in terms of the corresponding data for H and B. As an application, we obtain the Maschke-type theorems for crossed products and twisted smash products. Finally, the relationship of finitistic dimensions between A and B will be given, if H is semisimple.

Key words: Hopf-Galois extension, spectral sequence, global dimension, finitistic dimension

1. Introduction and preliminaries

The definition of Hopf-Galois extension has its roots in the Chase-Harrison-Rosenberg approach to Galois theory for groups acting on commutative rings (see [4]). In 1969 Chase and Sweedler extended these ideas to coaction of a Hopf algebra H acting on a commutative k-algebra, for k a commutative ring (see [5]); the general definition appears in [7] in 1981. Hopf-Galois extensions also generalize strongly graded algebras (here H is a group algebra) and certain inseparable field extensions (here the Hopf algebra is the restricted enveloping algebra of a restricted Lie algebra), twisted group rings $R*G$ of a group G acting on a ring R and so on.

Let H be a Hopf algebra over a field k and A a right H-comodule algebra, i.e., A is a k-algebra together with an H-comodule structure $\rho_A : A \to A\otimes H$ (with notation $a \mapsto a_0 \otimes a_1$) such that ρ_A is an algebra map. Let B be the subalgebra of the H-coinvariant elements, $B := A^{coH} := \{a \in A| \ \rho_A(a) = a\otimes 1\}$. Then the extension A/B is right H-Galois if the map $\beta : A\otimes_B A \to A\otimes H$, given by $a\otimes_B b \mapsto (a\otimes 1)\rho(b)$, is bijective.

The aim of this paper is to study the relationship of homological dimensions of Hopf-Galois extensions.

Let A/B be a right H-Galois extension for a Hopf algebra H. If H is finite dimensional, or A is projective as a left B-module, then in Section 2 we prove that the left global dimension of A is less than or equal to the sum of the left global dimension of the subalgebra B and the right global dimension of the Hopf algebra H regarded as an algebra. The result is a consequence of a certain spectral sequence (see Theorem 2.5), which generalizes the main result in [8].

In Section 3, we prove that the left finitistic dimension (which is defined to be the supremum of the projective dimensions of all finitely generated left modules which have finite projective dimension) of A is less

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than or equal to that of B, if H is a semisimple Hopf algebra.

Throughout this paper, k denotes a fixed field, and we will always work over k. The tensor product $\otimes = \otimes_k$ and Hom is always assumed to be over k. For an algebra A, denote by A-Mod and Mod-A the categories of left A-modules and of right A-modules, respectively, and denote by A-mod by the categories of finitely generated left A-modules. Let lgl.dim A and rgl.dim A denote the left global dimension and the right global dimension of A, respectively. For a left A-module M, let proj.dim$M$ denote the projective dimension of M. The reader is referred to [9] and [12] as general references about Hopf algebras. If C is a coalgebra, we use the Sweedler-type notation for the comultiplication: $\Delta(c) = c_1 \otimes c_2$, for all $c \in C$.

By [11], if A/B is right H-Galois, then for any $h \in H$ there are $r_i(h), l_i(h) \in A, i \in I, I$ being a finite set such that, for $a \in A$ and $b \in B$, one has:

$$\beta(\Sigma\, r_i(h) \otimes_B l_i(h)) = 1 \otimes h, \tag{1.1}$$

$$\Sigma\, br_i(h) \otimes_B l_i(h) = \Sigma\, r_i(h) \otimes_B l_i(h)b, \tag{1.2}$$

$$\Sigma\, a_0 r_i(a_1) \otimes_B l_i(a_1) = 1 \otimes_B a, \tag{1.3}$$

$$\Sigma\, r_i(h) l_i(h) = \varepsilon(h) l, \tag{1.4}$$

$$\Sigma\, r_i(h) \otimes_B l_i(h)_0 \otimes l_i(h)_1 = \Sigma\, r_i(h_1) \otimes_B l_i(h_1) \otimes h_2. \tag{1.5}$$

Also by [11], the canonical map $\beta : A \otimes_B A \to A \otimes H$, $\beta(a \otimes_B b) = ab_0 \otimes b_1$, is a morphism of modules in A-Mod and in Mod-A, where

(1) $A \otimes_B A$ and $A \otimes H$ are left A-modules with structures

$$a(x \otimes_B y) := ax \otimes_B y, \quad a(x \otimes h) := ax \otimes h; \tag{1.6}$$

(2) $A \otimes_B A$ and $A \otimes H$ are right A-modules with structures

$$(x \otimes_B y)a := x \otimes_B ya, \quad (x \otimes h)a := xa_0 \otimes ha_1, \tag{1.7}$$

for $a, x, y \in A$ and $h \in H$.

2. The spectral sequence for Hopf-Galois extensions

Let A/B be a right H-Galois extension for a Hopf algebra H. In this section, we construct a Grothendieck spectral sequence for certain Ext groups to estimate the global dimensions of Hopf-Galois extensions.

Lemma 2.1 *Let V and W be left A-modules. For all $\phi \in Hom_B(V, W)$ and $h \in H$ define $\phi h : V \to W$ by $(\phi h)(v) = \Sigma r_i(h)\phi(l_i(h)v), v \in V$. Then $Hom_B(V, W)$ is a right H-module, and there is a canonical k-linear isomorphism*

$$Hom_H(k, Hom_B(V, W)) \cong Hom_A(V, W),$$

where k is the trivial right H-module (i.e., H acts via the counit).

Proof By [11, Corollary 3.5], $\mathrm{Hom}_B(V, W)$ is a right H-module via above definition and $\mathrm{Hom}_A(V, W) = \mathrm{Hom}_B(V, W)^H$ $(:= \{\phi \in \mathrm{Hom}_B(V, W) \mid \phi h = \phi\varepsilon(h), \text{for all}\, h \in H\})$. And there is a natural isomorphism: $\mathrm{Hom}_H(k, \mathrm{Hom}_B(V, W)) \cong \mathrm{Hom}_B(V, W)^H$. So we get the isomorphism. □

Lemma 2.2 *Let W be a left A-module. Then*

$$Hom_B(A, W) \cong Hom(H, W)$$

as right H-modules, where H acts on the right-hand side by $(\psi h)(l) = \psi(hl)$ for $\psi \in Hom(H, W)$ and $h, l \in H$.

Proof The restriction functor ${}_B(-)$ can be written as ${}_BA \otimes_A (-)$. Therefore

$$\begin{aligned} \mathrm{Hom}_B({}_BA, {}_BW) &\cong \mathrm{Hom}_B({}_BA, {}_BA \otimes_A W) \cong \mathrm{Hom}_A(A \otimes_B A, W) \\ &\overset{\beta^*}{\cong} \mathrm{Hom}_A(A \otimes H, W) \cong \mathrm{Hom}(H, W). \end{aligned}$$

So we get the isomorphism. □

Let A/B be a right H-Galois extension. Consider the following two functors

$$\begin{aligned} A \otimes_B - : B\text{-Mod} \to A\text{-Mod}, \qquad & M \mapsto A \otimes_B M, \\ {}_B(-) : A\text{-Mod} \to B\text{-Mod}, \qquad & M \mapsto M, \end{aligned}$$

where ${}_B(-)$ is the restriction functor. Let (F, G) be an adjoint pair of functors of abelian categories. If G is exact, then F preserves projective objects; if F is exact, then G preserves injective objects.

Lemma 2.3 *Let A/B be a right H-Galois extension for a finite dimensional Hopf algebra H. Then $(A \otimes_B -, {}_B(-))$ and $({}_B(-), A \otimes_B -)$ are both adjoint pairs.*

Proof By adjoint isomorphism theorem, $(A \otimes_B -, {}_B(-))$ is an adjoint pair. By [6, Theorem 5], $({}_B(-), A \otimes_B -)$ is also an adjoint pair. □

Lemma 2.4 *Let A/B be a right H-Galois extension for a Hopf algebra H. Let V, V' and W, W' be left A-modules. If $f : V \to V'$ and $g : W \to W'$ are A-module maps, then $g_* \circ f^* : Hom_B(V', W) \to Hom_B(V, W')$ is an H-module map. Furthermore, if H is a finite dimensional Hopf algebra or A is projective as a left B-module, then $Ext^*_B(V, W)$ is a right H-module.*

Proof The first part can be checked straightforwardly. Let V and W be left A-modules and let

$$\mathcal{P}^\bullet : \cdots \xrightarrow{f_{n+1}} P_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} 0$$

be a projective resolution of V, so $H_n(\mathcal{P}^\bullet) = 0$ for $n \neq 0$ and $H_0(\mathcal{P}^\bullet) \cong V$. Since H is a finite dimensional Hopf algebra, by [9, Theorem 8.3.3] or [7, 1.7], A is a finitely generated projective right B-module. This implies that $A \otimes_B -$ is exact. And together with the fact that $({}_B(-), A \otimes_B -)$ is an adjoint pair, one can obtain that ${}_B(-)$ preserves projective objects. So the restriction of $\mathcal{P}^\bullet$ to B-Mod is a projective resolution of ${}_BV$ and we have $\mathrm{Ext}^*_B(V, W) = H^*(\mathrm{Hom}_B(\mathcal{P}^\bullet, W))$. By Lemma 2.1 and the first part of this lemma, the components of the complex $\mathrm{Hom}_B(\mathcal{P}^\bullet, W)$ are right H-modules and the differential $(f_n^*)_n$ is H-linear. Thus the cohomology $H^*(\mathrm{Hom}_B(\mathcal{P}^\bullet, W))$ is a right H-module and hence so is $\mathrm{Ext}^*_B(V, W)$. □

Now we obtain the main result of this section as follows.

Theorem 2.5 *Let A/B be a right H-Galois extension for a Hopf algebra H. Let V and W be left A-modules. If H is finite dimensional or A is projective as a left B-module, then there is a third quadrant cohomological spectral sequence*

$$E_2^{p,q} = Ext_H^p(k, Ext_B^q(V,W)) \underset{p}{\Rightarrow} Ext_A^n(V,W).$$

Proof By Lemma 2.4, $\mathrm{Ext}_H^*(k, \mathrm{Ext}_B^*(V,W))$ makes sense. This spectral sequence can be obtained as applications of the Grothendieck spectral sequence (cf. [10], Chapter 10). The proof is similar to that of the proposition in [8]. For the integrity of the paper, we write out the proof. Let W be a left A-module. Define functors

$$G : A\text{-Mod} \to \text{Mod-}H, \quad G(V) = \mathrm{Hom}_B(V,W)$$

and

$$F : \text{Mod-}H \to \text{Mod-}k, \quad F(X) = \mathrm{Hom}_H(k,X).$$

By Lemma 2.1, FG is equivalent with the functor $\mathrm{Hom}_A(-,W)$ and so the right derived functors $R^n(FG)$ are equivalent with $\mathrm{Ext}_A^n(-,W)$. It is easy to prove that F and G satisfy the conditions of Theorem 10.49 in [10], hence the required spectral sequence exists. □

The above theorem directly implies the following estimate for the projective dimension of modules.

Corollary 2.6 *Let A/B be a right H-Galois extension for a Hopf algebra H. Let V be a left A-module. If H is finite dimensional or A is projective as a left B-module, then $proj.dim_A V \le proj.dim_H(k) + proj.dim_B V$. Consequently, $lgl.dimA \le rgl.dimH + lgl.dimB$. In particular, if B and H are both semisimple, then so is A.*

Next we recall some notations on crossed products (see [2]). A Hopf algebra H is said to measure an algebra A if there is a k-linear map $H \otimes A \to A$ given by $h \otimes a \mapsto h \cdot a$ such that $h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b)$ and $h \cdot 1 = \varepsilon(h)1$, for all $a, b \in A$ and $h \in H$. A map σ in $Hom(H \otimes H, A)$ is said to be convolution invertible if there exists a map τ in $Hom(H \otimes H, A)$ such that $(\sigma * \tau)(h \otimes g) = \sigma(h_1, g_1)\tau(h_2, g_2) = \varepsilon(h)\varepsilon(g)1_A$ and $(\tau * \sigma)(h \otimes g) = \tau(h_1, g_1)\sigma(h_2, g_2) = \varepsilon(h)\varepsilon(g)1_A$, for all $h, g \in H$.

Let H be a Hopf algebra and A an algebra. Assume that H measures A and σ is a convolution invertible map in $\mathrm{Hom}(H \otimes H, A)$. The crossed product $A\#_\sigma H$ of A with H is the set $A \otimes H$ as a vector space, with multiplication

$$(a\#_\sigma h)(b\#_\sigma k) = a(h_1 \cdot b)\sigma(h_2, k_1)\#_\sigma h_3 k_2$$

for $h, k \in H, a, b \in A$. Here we write $a\#_\sigma h$ for the tensor product $a \otimes h$. Then $A\#_\sigma H$ is an associative algebra with identity element $1\#_\sigma 1$ if and only if the following two conditions are satisfied:

(1) A is a twisted H-module; that is, $1 \cdot a = a, \forall\, a \in A$, and

$$h \cdot (k \cdot a) = \sigma(h_1, k_1)(h_2 k_2 \cdot a)\sigma^{-1}(h_3, k_3),$$

for all $h, k \in H, a \in A$.

(2) σ is a cocycle; that is, $\sigma(h, 1) = \sigma(1, h) = \varepsilon(h)1, \forall\, h \in H$, and

$$(h_1 \cdot \sigma(k_1, m_1))\sigma(h_2, k_2 m_2) = \sigma(h_1, k_1)\sigma(h_2 k_2, m),$$

for all $h, k, m \in H$.

Note that if σ is trivial, that is, $\sigma(h,k) = \varepsilon(h)\varepsilon(k)1$, for $h, k \in H$, then (1) of above simply says that A is an H-module, and (2) of above is trivial. Thus A is a left H-module algebra. Moreover, the definition of multiplication as defined above reduces to the multiplication in a smash product, and so $A\#_\sigma H = A\#H$ is just the smash product (see Definition 4.1.3 of [9]).

Let $A\#_\sigma H$ be a crossed product. Then $A\#_\sigma H/A$ is a right H-Galois extension (in fact, $A\#_\sigma H/A$ is exactly a right H-Galois extension and has the normal basis property) (see [3, Theorem 1.18]). Also, $A\#_\sigma H$ is projective (in fact, free) as a left A-module, so by Corollary 2.6 we get the Maschke-type theorem for crossed products as follows.

Corollary 2.7 *Let $A\#_\sigma H$ be a crossed product. Then $lgl.dimA\#_\sigma H \leq rgl.dimH + lgl.dimA$. In particular, if A and H are both semisimple (of gl.dim 0), then so is $A\#_\sigma H$.*

We now give a new example of Hopf-Galois extension and get the Maschke-type theorem for this example which generalizes the main result in [13].

Let H be a Hopf algebra, and let A be an H-bimodule algebra (i.e., A is an H-bimodule, a left H-module algebra and a right H-module algebra) with the left H-module action $\rightarrow$ and the right H-module action $\leftarrow$. So we can form the twisted smash product $A * H$ with the multiplication on $A \otimes H$ as

$$(a * h)(b * g) = a(h_1 \rightarrow b \leftarrow S(h_3)) * h_2 g$$

for all $a, b \in A, h, g \in H$. Then $A * H$ is an algebra with the unit $1 * 1$ (see [14]). The twisted smash product contains the usual smash product and the Drinfeld double (see [9, Chapter 10]), so it plays an important role in quantum group theory. We can check easily that $A * H$ is a right H-comodule algebra with the comodule structure $a * h \mapsto a * h_1 \otimes h_2$ and $(A * H)^{coH} = A * 1 \cong A$.

Proposition 2.8 *Let $A * H$ be a twisted smash product. Then $A * H/A$ is a right H-Galois extension.*

Proof To see that $A * H/A$ is a right H-Galois extension, we recall the Galois map

$$\begin{aligned} \beta : A * H \otimes_A A * H &\rightarrow A * H \otimes H, \\ (a * h) \otimes_A (b * g) &\mapsto (a * h \otimes 1)(b * g_1 \otimes g_2) \\ &= (a * h)(b * g_1) \otimes g_2, \end{aligned}$$

and we construct an inverse map α for β defined by

$$\begin{aligned} \alpha : A * H \otimes H &\rightarrow A * H \otimes_A A * H, \\ a * h \otimes g &\mapsto (a * h)(1 * S(g_1)) \otimes_A (1 * g_2). \end{aligned}$$

We only need prove that α is the inverse of β. For all $a, b \in A, h, g \in H$ we compute

$$\begin{aligned} \beta\alpha[(a * h) \otimes g] &= \beta[(a * h)(1 * S(g_1)) \otimes_A (1 * g_2)] \\ &= (a * h)(1 * S(g_1))(1 * g_2) \otimes g_3 \\ &= (a * h)(1 * 1) \otimes g = (a * h) \otimes g, \end{aligned}$$

and

$$\begin{aligned}
\alpha\beta[(a*h)\otimes_A(b*g)] &= \alpha[(a*h)(b*g_1)\otimes g_2]\\
&= (a*h)(b*g_1)(1*S(g_2))\otimes_A(1*g_3)\\
&= (a*h)[b(g_1\rightarrow 1\leftarrow S(g_3))*g_2S(g_4)]\otimes_A(1*g_5)\\
&= (a*h)(b*1)\otimes_A(1*g)\\
&= (a*h)\otimes_A(b*1)(1*g)\\
&= (a*h)\otimes_A(b*g).
\end{aligned}$$

Thus $A*H/A$ is a right H-Galois. □

In what follows, we obtain a Maschke-type theorem for the twisted smash product generalizing Theorem 5.3 of [13] in which the authors need additional condition.

Corollary 2.9 *Let $A*H$ be a twisted smash product. Then $lgl.dimA*H \leq rgl.dimH + lgl.dimA$. In particular, if A and H are both semisimple, then so is $A*H$.*

3. Finitistic dimension of Hopf-Galois extensions

Let A/B be a right H-Galois extension for a semisimple Hopf algebra H. In this section, we consider relationships between the finitistic dimensions of A and B.

Recall from [1] that the finitistic dimension of an algebra A is defined to be

$$\text{fin.dim}(A) = \sup\{\text{proj.dim}(M) \mid M \in A\text{-mod and proj.dim}(M) < \infty\}.$$

H. Bass in [1] conjectured that $\text{fin.dim}(A) < \infty$ for any finite dimensional algebra A. This conjecture is still open.

Lemma 3.1 *Let A/B be a right H-Galois extension for a semisimple Hopf algebra H. Then for any A-module M, M is an A-direct summand of $A \otimes_B M$.*

Proof First define $f : A\otimes_B M \to M$ by $a\otimes_B m \mapsto am$ for $a \in A$ and $m \in M$. Obviously, f is an A-epimorphism.

Since H is semisimple, there exists a right integral t (i.e., $th = \varepsilon(h)t$, for all $h \in H$) satisfying $\varepsilon(t) = 1$ (see [9, Theorem 2.2.1]). Define $g : M \to A \otimes_B M$ via $g(m) = \Sigma r_i(t) \otimes_B l_i(t)m$ for $m \in M$. Next, we show that g is A-linear. It suffices to show that $\Sigma r_i(t)\otimes_B l_i(t)a = \Sigma a r_i(t)\otimes_B l_i(t), \forall\, a \in A$. In fact, for $a \in A$, we have

$$\begin{aligned}
\beta(\Sigma r_i(t)\otimes_B l_i(t)a) &\overset{(1.7)}{=} \beta(\Sigma r_i(t)\otimes_B l_i(t))a\\
&\overset{(1.1)}{=} (1\otimes t)a \overset{(1.7)}{=} a_0\otimes ta_1\\
&= a_0\otimes t\varepsilon(a_1) = a\otimes t\\
&\overset{(1.6)}{=} a(1\otimes t) \overset{(1.1)}{=} a\beta(\Sigma r_i(t)\otimes_B l_i(t))\\
&\overset{(1.6)}{=} \beta(\Sigma a r_i(t)\otimes_B l_i(t)).
\end{aligned}$$

Since β is bijective, we get $\Sigma r_i(t) \otimes_B l_i(t)a = \Sigma a r_i(t) \otimes_B l_i(t), \forall\, a \in A$.

Finally, for any $m \in M$,

$$fg(m) = f(\Sigma r_i(t) \otimes_B l_i(t)m) = \Sigma r_i(t) l_i(t) m \overset{(1.4)}{=} \varepsilon(t)m = m.$$

Thus M is an A-direct summand of $A \otimes_B M$. □

Lemma 3.2 *Let A/B be a right H-Galois extension for a semisimple Hopf algebra H. Then for each A-module M, proj.dim $({}_AM) =$ proj.dim $({}_BM)$.*

Proof Any semisimple Hopf algebra H is finite dimensional, since any semisimple Hopf algebra is separable (Let k be a field. An associative k-algebra A is said to be separable if for every field extension E/k the algebra $A \otimes_k E$ is semisimple), and a separable algebra over a field is finite dimensional (see [9, Corollary 2.2.2]). Combining the proof of Lemma 2.4, we obtain that any projective resolution of M as an A-module is also a projective resolution of M as a B-module. It implies that proj.dim $({}_BM) \leq$ proj.dim $({}_AM)$.

Conversely, since $(A \otimes_B -, {}_B(-))$ is an adjoint pair and ${}_B(-)$ is exact, we have $A \otimes_B P$ is a projective A-module for each projective B-module P. We may assume that proj.dim $({}_BM) = n < \infty$, and let $\mathcal{P}$ be a projective resolution of M as a B-module of length n. Then $A \otimes_B \mathcal{P}$ is a projective resolution of $A \otimes_B M$ as an A-module. The exactness of this sequence is determined by the projectiveness of A as a right B-module. It implies proj.dim $({}_A(A \otimes_B M)) \leq$ proj.dim $({}_BM)$. Also by Lemma 3.1, M is an A-direct summand of $A \otimes_B M$, it follows that proj.dim $({}_AM) \leq$ proj.dim $({}_A(A \otimes_B M))$. Thus proj.dim $({}_AM) \leq$ proj.dim $({}_BM)$. The proof is completed. □

Following from lemma 3.2, we immediately obtain the main result of this section as follows.

Theorem 3.3 *Let A/B be a right H-Galois extension for a semisimple Hopf algebra H. Then fin.dim$(A) \leq$ fin.dim(B).*

We now apply the above theorem to crossed products and twisted smash products.

Corollary 3.4 *Let H be a semisimple Hopf algebra as well as its dual H^*, and $A\#_\sigma H$ be a crossed product. Then fin.dim$(A\#_\sigma H) =$ fin.dim(A).*

Proof First, $A\#_\sigma H/A$ is a right H-Galois extension and H is semisimple, by Theorem 3.3 we have fin.dim$(A\#_\sigma H) \leq$ fin.dim(A).

Note that $A\#_\sigma H$ is a left H^*-module algebra via $\varphi \cdot (a\#_\sigma h) = a\#_\sigma(\varphi \rightharpoonup h) =< \varphi, h_2 > a\#_\sigma h_1$, for $a\#_\sigma h \in A\#_\sigma H$ and $f \in H^*$. Thus $A\#_\sigma H$ and H^* form a smash product algebra $(A\#_\sigma H)\#H^*$. It is well known that the smash product is a special case of crossed products. So $(A\#_\sigma H)\#H^*/A\#_\sigma H$ is also a right H^*-Galois extension. Combining the semisimplicity of H^*, we have fin.dim$((A\#_\sigma H)\#H^*) \leq$ fin.dim$(A\#_\sigma H)$. By [3, Theorem 2.2], $(A\#_\sigma H)\#H^* \cong M_n(A)$, where $n = \dim H$, so it is Morita equivalent to A. It follows that fin.dim$(A) =$ fin.dim$((A\#_\sigma H)\#H^*)$. Then

$$\text{fin.dim}(A) = \text{fin.dim}((A\#_\sigma H)\#H^*) \leq \text{fin.dim}(A\#_\sigma H) \leq \text{fin.dim}(A).$$

Therefore fin.dim$(A\#_\sigma H) =$ fin.dim(A). □

Corollary 3.5 *Let H be a semisimple Hopf algebra, and $A*H$ be a twisted smash product. Then* $fin.dim(A*H) \leq fin.dim(A)$.

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