Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2013) 37: 218 – 230
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doi:10.3906/mat-1012-549

Strongly Gorenstein flat and Gorenstein FP-injective modules

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Received: 07.12.2010 • **Accepted:** 12.01.2012 • **Published Online:** 19.03.2013 • **Printed:** 22.04.2013

Abstract: In this paper, we first study the properties of strongly Gorenstein flat (resp. Gorenstein FP-injective) modules which are special Gorenstein projective (resp. Gorenstein injective) modules, and use them to prove that the global strongly Gorenstein flat dimension and the global Gorenstein FP-injective dimension of a ring R are identical when R is n-FC or commutative coherent. Finally, we show that if R is a commutative Noetherian ring, then, for any R-module M, $\mathrm{SGfd}_R M = \mathrm{Gpd}_R M$, and hence $\mathrm{SGfd}_R M < \infty$ if and only if $\mathrm{Gfd}_R M < \infty$, where $\mathrm{SGfd}_R M$ denotes the strongly Gorenstein flat dimension of M.

Key words: Strongly Gorenstein flat modules, global strongly Gorenstein flat dimension, Gorenstein FP-injective modules, global Gorenstein FP-injective dimension

1. Introduction

Throughout this paper, R denotes an associative ring with identity and all modules are, if not specified otherwise, left R-modules. We use $\mathrm{Mod}R$ to denote the class of left R-modules, and use $\mathrm{pd}_R M$, $\mathrm{id}_R M$, $\mathrm{fd}_R M$ to denote, respectively, the projective, injective and flat dimensions of a module M in $\mathrm{Mod}R$. Given a class $\mathcal{X}$ of R-modules, a sequence is $\mathrm{Hom}(-,\mathcal{X})$-exact if it is exact after applying the functor $\mathrm{Hom}(-,X)$ for all $X \in \mathcal{X}$. The sequence is $\mathrm{Hom}(\mathcal{X},-)$-exact if it is exact after applying the functor $\mathrm{Hom}(X,-)$ for all $X \in \mathcal{X}$.

Let $\mathcal{P}$ stand for the class of all projective R-modules, and $\mathcal{I}$ stand for the class of all injective R-modules. Recall that a module M in $\mathrm{Mod}R$ is called Gorenstein projective [11] if there exists a $\mathrm{Hom}(-,\mathcal{P})$-exact exact sequence

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

of projective modules such that $M = \mathrm{Ker}(P_0 \to P^0)$. Dually, M is Gorenstein injective if there exists a $\mathrm{Hom}(\mathcal{I},-)$-exact exact sequence

$$\cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

of injective modules such that $M = \mathrm{Ker}(E_0 \to E^0)$.

Gorenstein projective and Gorenstein injective modules were introduced by Enochs and Jenda [11] and further studied by many authors (see, e.g., [3]–[6], [11]–[13], [15]–[16]). These modules have nice properties when the ring in question is n-Gorenstein (that is, the ring is a left and right Noetherian ring with left and right

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2000 *AMS Mathematics Subject Classification:* 16E10, 16E30.

self-injective dimension at most n). Following [17], a module is called FP-injective if $\mathrm{Ext}^1(N,M)=0$ for any finitely present R-module N. The FP-injective dimension of M, denoted by FP-$\mathrm{id}_R M$, is defined similarly to the classical injective dimension. A ring is called an *n-FC ring* [7] if R is a left and right coherent ring with FP-$\mathrm{id}_R R \le n$ and FP-$\mathrm{id} R_R \le n$. Let $\mathcal{F}$ be the class of flat R-modules, and let $\mathcal{FI}$ be the class of FP-injective R-modules. In [9], a particular case of Gorenstein projective modules which is called strongly Gorenstein flat modules was introduced. An R-module M is called *strongly Gorenstein flat* if there exists a $\mathrm{Hom}(-,\mathcal{F})$-exact exact sequence

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

of projective modules such that $M=\mathrm{Ker}(P_0\to P^0)$. Dual to the definition of strongly Gorenstein flat modules, in [10], a particular case of Gorenstein injective modules which is called Gorenstein FP-injective modules was introduced. An R-module M is called *Gorenstein FP-injective* if there exists a $\mathrm{Hom}(\mathcal{FI},-)$-exact exact sequence

$$\cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

of injective modules such that $M=\mathrm{Ker}(E_0\to E^0)$. We notice that strongly Gorenstein flat and Gorenstein FP-injective modules are also called Ding projective and Ding injective modules in [14], respectively. These two classes of modules have been treated by different authors (see, e.g., [9, 10, 14]). In this paper, we continue to study strongly Gorenstein flat and Gorenstein FP-injective modules. We use $\mathrm{SGfd}_R M$ to denote the strongly Gorenstein flat dimension of a module M in Mod$R$, which is defined as the smallest non-negative integer n such that there exists an exact sequence

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

with each G_i strongly Gorenstein flat. If no such n exists, set $\mathrm{SGfd}_R M=\infty$. We use GFP-$\mathrm{id}_R M$ to denote the Gorenstein FP-injective dimension of a module M in Mod$R$, which is defined as the smallest non-negative integer n such that there exists an exact sequence

$$0 \longrightarrow M \longrightarrow T_0 \longrightarrow \cdots \longrightarrow T_{n-1} \longrightarrow T_n \longrightarrow 0$$

with each T_i Gorenstein FP-injective. If no such n exists, set GFP-$\mathrm{id}_R M=\infty$.

This paper is organized as follows. In Section 2, we study some homological properties of strongly Gorenstein flat modules over a general ring. For example, we prove that the class of strongly Gorenstein flat R-modules is projectively resolving, and for any module M with finite strongly Gorenstein flat dimension $n\ge 1$, then there is an exact sequence $0\longrightarrow T\longrightarrow N \xrightarrow{\ \varphi\ } M\longrightarrow 0$ such $\mathrm{pd}_R T=n-1$ and N is strongly Gorenstein flat, and furthermore, φ is a strongly Gorenstein flat precover. We also obtain some criteria for computing the strongly Gorenstein flat dimension of modules, and prove that if M is an R-module of finite flat dimension then $\mathrm{SGfd}_R M=\mathrm{pd}_R M$. This result generalizes [9, Corollary 2.5] which shows that an R-module M is projective if and only if M is flat and strongly Gorenstein flat. We remark that all results concerning strongly Gorenstein flat dimension in this section have a Gorenstein FP-injective counterpart.

In Section 3, we introduce the notions of the global strongly Gorenstein flat dimension and the global Gorenstein FP-injective dimension of rings. We give some characterizations of strongly Gorenstein flat and Gorenstein FP-injective modules over the rings of finite global strongly Gorenstein flat dimension and finite global Gorenstein FP-injective dimension, respectively. We also prove that these two dimensions of a ring R are identical when R is n-FC or commutative coherent by using the results shown in Section 2.

Section 4 is devoted to studying the strongly Gorenstein flat dimension over a commutative Noetherian ring. In particular, we show that if R is a commutative Noetherian ring of finite Krull dimension, then, for any R-module M, $\mathrm{SGfd}_R M = \mathrm{Gpd}_R M$, and hence $\mathrm{SGfd}_R M < \infty$ if and only if $\mathrm{Gfd}_R M < \infty$.

2. Homological properties of strongly Gorenstein flat modules

In this section we give some properties of strongly Gorenstein flat modules. Notice that all the results in this section have a Gorenstein FP-injective counterpart. These results will be used to study the global strongly Gorenstein flat dimension and the global Gorenstein FP-injective dimension (see Section 3).

Let $\mathcal{M}$ be a class of R-modules. We use $\mathcal{M}^{\perp}$ to denote the class of all modules N such that $\mathrm{Ext}^i(M,N) = 0$ for any $M \in \mathcal{M}$ and $i \geq 1$, and use ${}^{\perp}\mathcal{M}$ to denote the class of modules N such that $\mathrm{Ext}^i(N,M) = 0$ for any $M \in \mathcal{M}$ and $i \geq 1$.

Lemma 2.1 *An R-module M is strongly Gorenstein flat if and only if $M \in {}^{\perp}\mathcal{F}$ and there exists an exact sequence*

$$0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

such that it is $\mathrm{Hom}(-,\mathcal{F})$-exact, where P^i is projective for any $i \geq 0$. In particular, if M is strongly Gorenstein flat, then $\mathrm{Ext}^i(M,H) = 0$ for any R-module H with finite flat dimension and $i \geq 1$.

Proof Immediately from the definition and a dimension-shifting argument. □

Definition 2.2 *([1]) A class $\mathcal{X}$ is called projectively resolving if $\mathcal{P} \subseteq \mathcal{X}$, and for every short sequence $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ in $\mathrm{Mod}R$ with $X'' \in \mathcal{X}$ the conditions $X' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent.*

We use $\mathcal{SGF}$ to denote the class of strongly Gorenstein flat R-modules. By [9, Proposition 2.10], $\mathcal{SGF}$ is projectively resolving over a right coherent ring. In the following, we show that it is true over any ring.

Lemma 2.3 *$\mathcal{SGF}$ is projectively resolving, and closed under arbitrary direct sums and direct summands.*

Proof Routine arguments show that $\mathcal{SGF}$ contains all projective R-modules and is closed under arbitrary direct sums. Also by [15, Proposition 1.4], $\mathcal{SGF}$ is closed under direct summands. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence in $\mathrm{Mod}R$. If M' and M'' are strongly Gorenstein flat, then so is M by Lemma 2.1, using the similar method as in the proof of the "horseshoe lemma". Now assume that M'' and M are strongly Gorenstein flat. We will show that M' is also strongly Gorenstein flat in the following. Notice that ${}^{\perp}\mathcal{F}$ is projectively resolving, so $M' \in {}^{\perp}\mathcal{F}$. Thus, by Lemma 2.1, we only need to prove that M' admits an exact sequence $0 \longrightarrow M' \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$ in $\mathrm{Mod}R$ with each P^i projective, such that it is $\mathrm{Hom}(-,\mathcal{F})$-exact. Since M is strongly Gorenstein flat, there is an exact sequence $0 \longrightarrow M \longrightarrow P^0 \longrightarrow C \longrightarrow 0$ with P^0 projective and C strongly Gorenstin flat. Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & P^0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C & = & C & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Then A is strongly Gorenstein flat by the preceding proof since M'' and C are strongly Gorenstein flat. Thus there is an exact sequence $0 \longrightarrow A \longrightarrow P^1 \longrightarrow P^2 \longrightarrow \cdots$ with each P^i projective such that it is $\mathrm{Hom}(-,\mathcal{F})$-exact. Now, note that the exact sequence $0 \longrightarrow M' \longrightarrow P^0 \longrightarrow A \longrightarrow 0$ is $\mathrm{Hom}(-,\mathcal{F})$-exact by Lemma 2.1 since A is strongly Gorenstein flat, we get an exact sequence $0 \longrightarrow M' \longrightarrow P^0 \longrightarrow P^1 \longrightarrow P^2 \longrightarrow \cdots$, which is the desired sequence. □

Lemma 2.4 *Let $M \in \mathrm{Mod}R$, and let*

$$0 \longrightarrow K_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow K'_n \longrightarrow G'_{n-1} \longrightarrow \cdots \longrightarrow G'_0 \longrightarrow M \longrightarrow 0$$

be exact sequences in $\mathrm{Mod}R$ with all G_i and G'_i strongly Gorenstein flat. Then K_n is strongly Gorenstein flat if and only if K'_n is strongly Gorenstein flat.

Proof It follows from Lemma 2.3 and [1, Lemma 3.12]. □

Let $\mathcal{C}$ be a class of R-modules and M an R-module. Following [12], we say that a homomorphism $\varphi : C \longrightarrow M$ is a $\mathcal{C}$-precover if $C \in \mathcal{C}$ and the abelian group homomorphism $\mathrm{Hom}(C',\varphi) : \mathrm{Hom}(C',C) \longrightarrow \mathrm{Hom}(C',M)$ is surjective for each $C' \in \mathcal{C}$. Dually we have the definition of a $\mathcal{C}$-preenvelope.

Lemma 2.5 *Let $M \in \mathrm{Mod}R$ with finite strongly Gorenstein flat dimension $n \geq 1$. Then there is an exact sequence $0 \longrightarrow T \longrightarrow N \xrightarrow{\varphi} M \longrightarrow 0$ such that $\mathrm{pd}_R T = n-1$ and N is strongly Gorenstein flat. In particular, φ is a $\mathcal{SGF}$-precover of M.*

Proof Let $0 \longrightarrow K' \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ be an exact sequence in $\mathrm{Mod}R$ with each P_i projective. Then K' is strongly Gorenstein flat by Lemma 2.4. Hence there exists an exact sequence

$$0 \longrightarrow K' \longrightarrow Q^0 \longrightarrow \cdots \longrightarrow Q^{n-1} \longrightarrow G \longrightarrow 0$$

in Mod$R$ such that it is Hom$(-,\mathcal{F})$-exact, where Q^i is projective for any $0 \leq i \leq n-1$ and G is strongly Gorenstein flat. Thus we can complete the following commutative diagram with exact rows

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & K' & \longrightarrow & Q^0 & \xrightarrow{\tau^0} & Q^1 & \longrightarrow & \cdots & \longrightarrow & Q^{n-1} & \xrightarrow{\tau^{n-1}} & G & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow \lambda^0 & & \downarrow \lambda^1 & & & & \downarrow \lambda^{n-1} & & \downarrow \lambda & & \\
0 & \longrightarrow & K' & \longrightarrow & P_{n-1} & \xrightarrow{\sigma_{n-1}} & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \xrightarrow{\sigma_0} & M & \longrightarrow & 0.
\end{array}$$

Let $\mathbb{C} \cdots \longrightarrow 0 \longrightarrow Q^0 \xrightarrow{\delta_n} P_{n-1} \oplus Q^1 \xrightarrow{\delta_{n-1}} P_{n-2} \oplus Q^2 \longrightarrow \cdots$

$$\longrightarrow P_1 \oplus Q^{n-1} \xrightarrow{\delta_1} P_0 \oplus G \xrightarrow{\delta_0} M \longrightarrow 0,$$

where $\delta_n(x) = (\lambda^0(x), -\tau^0(x)), \delta_0(a,b) = \sigma_0(a) + \lambda(b)$, and

$$\delta_i(a,b) = (\sigma_i(a) + \lambda^{n-i}(b), -\tau^{n-i}(b))$$

for $i = 1, 2, \ldots, n-1$. Then one can check that $\mathbb{C}$ is exact. Let $K = \text{Ker}\delta_0$, then $0 \longrightarrow K \longrightarrow P_0 \oplus G \xrightarrow{\delta_0} M \longrightarrow 0$ is exact, where $P_0 \oplus G$ is strongly Gorenstein flat and $\text{pd}_R K \leq n-1$. Furthermore, $\text{pd}_R K = n-1$ since $\text{SGfd}_R M = n$. Now, by Lemma 2.1, one can check that δ_0 is a $\mathcal{SGF}$-precover of M. □

Proposition 2.6 *Let* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *be an exact sequence in* Mod$R$. *If* M' *and* M *are strongly Gorenstein flat modules. Then the following statements are equivalent.*

1. M'' *is strongly Gorenstein flat.*
2. M'' *is Gorenstein projective.*
3. $\text{Ext}^1(M'', P) = 0$ *for any projective module* P.
4. $\text{Ext}^1(M'', F) = 0$ *for any flat module* F.

Proof (1) ⇒ (4) by Lemma 2.1.

(4) ⇒ (3) is trivial.

(3) ⇒ (1) Since $\text{SGfd}_R M'' \leq 1$, we have an exact sequence $0 \longrightarrow P \longrightarrow G \longrightarrow M'' \longrightarrow 0$ in Mod$R$ by Lemma 2.5, where P is projective and G is strongly Gorenstein flat. Then M'' is a direct summand of G since $\text{Ext}^1(M'', P) = 0$, and hence M'' is strongly Gorenstein flat by Lemma 2.3.

(2) ⇒ (3) follows from [15, Proposition 2.3].

(3) ⇒ (2) holds by [15, Corollary 2.11] since strongly Gorenstein flat modules are Gorenstein projective. □

Proposition 2.7 *Let* $0 \longrightarrow K \longrightarrow G \longrightarrow M \longrightarrow 0$ *be an exact sequence in* $\mathrm{Mod}R$*, then the following statements hold.*

(1) *If any two of the modules* K, G *and* M *have finite strongly Gorenstein flat dimension, then so has the third.*

(2) *If* G *strongly Gorenstein flat and* $\mathrm{SGfd}_R M \geq 1$*, then* $\mathrm{SGfd}_R K = \mathrm{SGfd}_R M - 1$*.*

Proof (1) Using [2, Proposition 3.4] and the dual versions of Lemmas 2.3 and 2.4, we can complete the proof.

(2) We may assume that $\mathrm{SGfd}_R M = n < \infty$ since $\mathrm{SGfd}_R K = \infty$ whenever $\mathrm{SGfd}_R M = \infty$. If $n = 1$, then the assertion holds obviously. Assume that $n \geq 2$ and

$$\cdots \longrightarrow G_{n-1} \longrightarrow G_{n-2} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow K \longrightarrow 0$$

is a strongly Gorenstein flat resolution of K. Then

$$\cdots \longrightarrow G_{n-1} \longrightarrow G_{n-2} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow G \longrightarrow M \longrightarrow 0$$

is a strongly Gorenstein flat resolution of M. Thus we have that

$$K_{n-1} = \mathrm{Ker}(G_{n-2} \to G_{n-3}),$$

where $G_{-1} = G$ is strongly Gorenstein flat by Lemma 2.4, and therefore $\mathrm{SGfd}_R K \leq n-1$. Furthermore, $\mathrm{SGfd}_R K = n-1$ since $\mathrm{SGfd}_R M = n$. □

Proposition 2.8 *Let* $M \in \mathrm{Mod}R$ *with finite strongly Gorenstein flat dimension and* $n \in \mathbb{N}$*. Then the following statements are equivalent.*

1. $\mathrm{SGfd}_R M \leq n$.
2. $\mathrm{Ext}^i(M, H) = 0$ *for any* R*-module* H *with finite flat dimension and* $i > n$*.*
3. $\mathrm{Ext}^i(M, F) = 0$ *for any flat* R*-module* F *and* $i > n$*.*
4. *For every exact sequence* $0 \longrightarrow K_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$ *with* G_i *strongly Gorenstein flat,* K_n *is strongly Gorenstein flat.*

Proof (1) $\Rightarrow$ (2) Assume that $\mathrm{SGfd}_R M \leq n$. Then there exists an exact sequence

$$0 \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$

in $\mathrm{Mod}R$ with G_i strongly Gorenstein flat for any $0 \leq i \leq n$. Thus we have

$$\mathrm{Ext}^i(M, H) \cong \mathrm{Ext}^{i-n}(G_n, H) = 0$$

for any R-module H with finite flat dimension and $i > n$ by Lemma 2.1.

(3) $\Rightarrow$ (4) Let

$$0 \longrightarrow K_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0 \qquad (*)$$

be an exact sequence in $\mathrm{Mod}R$, where G_i is strongly Gorenstein flat for any $0 \leq i \leq n-1$. Then $\mathrm{Ext}^i(K_n, F) \cong \mathrm{Ext}^{i+n}(M, F) = 0$ for any $i \geq 1$. Decomposing $(*)$ into short exact sequences and applying Proposition 2.7, we have $\mathrm{SGfd}_R K_n < \infty$ since $\mathrm{SGfd}_R M < \infty$. Hence there exists an exact sequence

$$0 \longrightarrow G'_m \longrightarrow \cdots \longrightarrow G'_0 \longrightarrow K_n \longrightarrow 0$$

in Mod$R$ with G'_i strongly Gorenstein flat. We decompose it into short exact sequences

$$0 \longrightarrow C'_j \longrightarrow G'_{j-1} \longrightarrow C'_{j-1} \longrightarrow 0$$

for any $1 \leq j \leq m$, where $C'_m = G'_m$ and $C'_0 = K_n$. Now by Lemma 2.1 we have that $\mathrm{Ext}^1(C'_{j-1}, F) \cong \mathrm{Ext}^j(K_n, F) = 0$ for any flat R-module F and $1 \leq j \leq m$. Thus $C'_0, \cdots, C'_m$ are strongly Gorenstein flat by Proposition 2.6. This shows that $K_n = C'_0$ is strongly Gorenstein flat.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are obvious. □

By [9, Corollary 2.5], an R-module M is projective if and only if M is flat and strongly Gorenstein flat. Here we will give a generalization of this result. On the other hand, one can check easily that $\mathrm{SGfd}_R M \leq \mathrm{pd}_R M$ for any module M, and the equality holds if $\mathrm{pd}_R M < \infty$. In the following we show that the equality holds when $\mathrm{fd}_R M < \infty$.

Proposition 2.9 *If $M \in \mathrm{Mod}R$ with $\mathrm{fd}_R M < \infty$, then $\mathrm{SGfd}_R M = \mathrm{pd}_R M$.*

Proof It suffices to show $\mathrm{pd}_R M \leq \mathrm{SGfd}_R M$. We may assume that $\mathrm{SGfd}_R M < \infty$. If M is strongly Gorenstein flat, then there exists an exact sequence

$$0 \longrightarrow M \longrightarrow P \longrightarrow M' \longrightarrow 0$$

in Mod$R$, where M' is strongly Gorenstein flat and P is projective. Since $\mathrm{fd}_R M < \infty$, $\mathrm{Ext}^1(M', M) = 0$ by Lemma 2.1. This implies that M is a direct summand of P, and hence M is projective.

Now let $\mathrm{SGfd}_R M = n > 0$. By Lemma 2.5, there exists an exact sequence

$$0 \longrightarrow K \longrightarrow G \longrightarrow M \longrightarrow 0$$

in Mod$R$, where G is strongly Gorenstein flat and $\mathrm{pd}_R K = n - 1$. Since G is strongly Gorenstein flat, there exists an exact sequence $0 \longrightarrow G \longrightarrow P \longrightarrow G' \longrightarrow 0$ in Mod$R$, where P is projective and G' is strongly Gorenstein flat. Now consider the following push-out diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & W & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Since P is projective and $\mathrm{pd}_R K = n - 1$, we have $\mathrm{pd}_R W \leq n$ by the exactness of the middle row in the above diagram. Note that G' is strongly Gorenstein flat, then $\mathrm{Ext}^1(G', M) = 0$ by Lemma 2.1, and hence $W \cong M \oplus G'$. This implies that $\mathrm{pd}_R M \leq \mathrm{pd}_R W \leq n$. □

Dually, we have the following result.

Proposition 2.10 *If* $M \in \mathrm{Mod}R$ *with* $\text{FP-id}_R M < \infty$, *then* $\text{GFP-id}_R M = \mathrm{id}_R M$.

3. The global strongly Gorenstein flat dimension and the global Gorenstein FP-injective dimension

In this section, we study the global strongly Gorenstein flat dimension and the global Gorenstein FP-injective dimension. First we introduce the following notions.

Definition 3.1 *The left global strongly Gorenstein flat dimension is defined as*

$$\mathrm{lSGFD}(R) = \sup\left\{\mathrm{SGfd}_R(M) \;\middle|\; M \in \mathrm{Mod}R\right\}.$$

Dually, the left global Gorenstein FP-injective dimension is defined as

$$\mathrm{lGFID}(R) = \sup\left\{\text{GFP-id}_R(M) \;\middle|\; M \in \mathrm{Mod}R\right\}.$$

We get the following criterion for determining whether a given module is strongly Gorenstein flat when $\mathrm{lSGFD}(R)$ is finite.

Proposition 3.2 *If* $\mathrm{lSGFD}(R) \leq n$, *then* M *is strongly Gorenstein flat if and only if there exists an exact sequence*

$$0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots \longrightarrow P^n$$

in $\mathrm{Mod}R$ *with each* P^i *projective.*

Proof The necessity is trivial. For the sufficiency, consider the exact sequence

$$0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots \longrightarrow P^n \longrightarrow C^n \longrightarrow 0$$

in $\mathrm{Mod}R$, where each P^i is projective and $C^n = \mathrm{Coker}(P^{n-1} \longrightarrow P^n)$. Let $H \in \mathrm{Mod}R$ with $\mathrm{fd}_R H < \infty$. By assumption, we have $\mathrm{SGfd}_R(C^n) \leq n$, thus M is strongly Gorenstein flat by Proposition 2.8. □

Dually we have the following proposition.

Proposition 3.3 *If* $\mathrm{lGFID}(R) \leq n$, *then* M *is Gorenstein FP-injective if and only if there exists an exact sequence*

$$E_n \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow M \longrightarrow 0$$

in $\mathrm{Mod}R$ *with each* E_i *injective.*

Lemma 3.4 *If* $\mathrm{lSGFD}(R) < \infty$, *then the following statements are equivalent for any* $n \in \mathbb{N}$.

1. $\mathrm{lSGFD}(R) \leq n$.
2. $\mathrm{id}_R H \leq n$ *for any* $H \in \mathrm{Mod}R$ *with finite flat dimension.*

Proof (1) $\Rightarrow$ (2) Let $H \in \mathrm{Mod}R$ with finite flat dimension. Then by assumption, we get that $\mathrm{Ext}^i(M, H) = 0$ for any R-module M and $i \geq n+1$. Thus $\mathrm{id}_R H \leq n$.

(2) $\Rightarrow$ (1) Let $M \in \mathrm{Mod}R$. Since $\mathrm{Ext}^i(M, H) = 0$ for any R-module H with finite flat dimension and $i \geq n+1$, $\mathrm{SGfd}_R M \leq n$ by Proposition 2.8. Thus $\mathrm{lSGFD}(R) = \sup\{\mathrm{SGfd}_R M \mid M \text{ is an R-module}\} \leq n$. □

The following definitions are analogs of the so called strongly Gorenstein projective and strongly Gorenstein injective modules in [3].

Definition 3.5 *An R-module M is called strongly $*$-Gorenstein flat if there exists an exact sequence of projective R-modules*

$$\cdots \longrightarrow P \xrightarrow{\ f\ } P \xrightarrow{\ f\ } P \xrightarrow{\ f\ } P \longrightarrow \cdots$$

such that $M \cong \mathrm{Ker} f$ and it is $\mathrm{Hom}(-, \mathcal{F})$-exact. Dually, an R-module M is called Gorenstein $*$-FP-injective if there exists an exact sequence of injective R-modules

$$\cdots \longrightarrow E \xrightarrow{\ g\ } E \xrightarrow{\ g\ } E \xrightarrow{\ g\ } E \longrightarrow \cdots$$

such that $M \cong \mathrm{Ker} g$ and it is $\mathrm{Hom}(\mathcal{FI}, -)$-exact.

Remark 3.6 *By definition, strongly $*$-Gorenstein flat modules are strongly Gorenstein flat, and an R-module M is strongly $*$-Gorenstein flat if and only if there exists an exact sequence*

$$0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P projective and there is an $i \geq 1$ such that $\mathrm{Ext}^i(M, F) = 0$ for any flat R-module F. Gorenstein $*$-FP-injective modules are Gorenstein FP-injective, and an R-module M is Gorenstein $*$-FP-injective if and only if there exists an exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow M \longrightarrow 0$$

with E injective and there is an $i \geq 1$ such that $\mathrm{Ext}^i(E, M) = 0$ for any FP-injective R-module E.

For any complexes $\mathbb{X}$ and $\mathbb{Y}$, $\mathcal{H}om(\mathbb{X}, \mathbb{Y})$ denotes the complex of Abelian groups with the degree-n term

$$\mathcal{H}om(\mathbb{X}, \mathbb{Y})^n = \prod_{t \in \mathbb{Z}} \mathrm{Hom}(\mathbb{X}^t, \mathbb{Y}^{n+t})$$

and whose boundary operators are

$$\delta^n((f^t)_{t\in\mathbb{Z}}) = (\delta_{\mathbb{Y}}^{n+t} f^t - (-1)^n f^{t+1} \delta_{\mathbb{X}}^t)_{t\in\mathbb{Z}}.$$

Using a similar method as in [3, Theorem 2.7], we have the following result.

Lemma 3.7 *An R-module is strongly Gorenstein flat if and only if it is a direct summand of a strongly $*$-Gorenstein flat module.*

Proof Let M be a strongly Gorenstein flat R-module. Then there exists an exact sequence of projective R-modules

$$\mathbb{P} \equiv \cdots \longrightarrow P_1 \xrightarrow{d_1^{\mathbb{P}}} P_0 \xrightarrow{d_0^{\mathbb{P}}} P_{-1} \xrightarrow{d_{-1}^{\mathbb{P}}} P_{-2} \longrightarrow \cdots$$

such that $M \cong \mathrm{Im} d_0^{\mathbb{P}}$ and it is $\mathrm{Hom}(-, \mathcal{F})$-exact. For $m \in \mathbb{Z}$, let $\sum^m \mathbb{P}$ be the exact sequence obtained from $\mathbb{P}$ by decreasing all indexes by m, i.e., $(\sum^m \mathbb{P})_i = P_{i-m}$ and $d_i^{\sum^m \mathbb{P}} = d_{i-m}^{\mathbb{P}}$ for all $i \in \mathbb{Z}$. Consider the exact sequence

$$\mathbb{Q} = \oplus(\Sigma^m \mathbb{P}) \equiv \cdots \longrightarrow Q = \oplus P_i \xrightarrow{\oplus d_i^{\mathbb{P}}} Q = \oplus P_i \xrightarrow{\oplus d_i^{\mathbb{P}}} Q = \oplus P_i \longrightarrow \cdots.$$

Since $\mathrm{Im}(\oplus d_i^{\mathbb{P}}) \cong \oplus \mathrm{Im} d_i^{\mathbb{P}}$, M is a direct summand of $\mathrm{Im}(\oplus d_i^{\mathbb{P}})$. Moreover,

$$\mathcal{H}om(\mathbb{Q}, F) = \mathcal{H}om(\oplus(\Sigma^m \mathbb{P}), F) \cong \Pi \mathcal{H}om(\Sigma^m \mathbb{P}, F)$$

is exact for any flat R-module F. Thus $\mathrm{Im}(\oplus d_i^{\mathbb{P}})$ is a strongly $*$-Gorenstein flat module. This shows that M is a direct summand of a strongly $*$-Gorenstein flat module $\mathrm{Im}(\oplus d_i^{\mathbb{P}})$. The converse assertion holds by Lemma 2.3. □

By [8], a ring R is called an n-FC ring if R is a left and right coherent ring with both FP-$\mathrm{id}_R R$ and FP-$\mathrm{id} R_R$ at most n.

Lemma 3.8 *([8, Theorem 3.8]) Let R be an n-FC ring, then $\mathrm{fd}_R E \leq n$ for any FP-injective R-module E, and FP-$\mathrm{id}_R F \leq n$ for any flat R-module F.*

By [8], $\mathrm{lIFD}(R) := \sup\{\mathrm{fd}_R E \mid E$ is an injective R-module$\}$. Dually, $\mathrm{rIFD}(R) := \sup\{\mathrm{fd}_R E \mid E$ is an injective right R-module$\}$.

By [9], $\mathrm{lFID}(R) := \sup\{\mathrm{id}_R F \mid F$ is a flat R-module$\}$.

Lemma 3.9 *([8, Theorems 3.5 and 3.8]) Let R be a ring, then*

$$\mathrm{lIFD}(R) = \sup\{\mathrm{fd}_R E \mid E \text{ is a FP-injective } R\text{-module}\}.$$

Furthermore, if R is a left coherent ring, then

$$\mathrm{rIFD}(R) = \sup\{\text{FP-}\mathrm{id}_R N \mid N \text{ is a flat } R\text{-module}\}.$$

Lemma 3.10 *Let R be a ring, then $\mathrm{lFID}(R) \leq \mathrm{lSGFD}(R)$ and $\mathrm{lIFD}(R) \leq \mathrm{lGFID}(R)$.*

Proof By [9, Proposition 3.2], $\mathrm{lFID}(R) \leq \mathrm{lSGFD}(R)$. In the following we show $\mathrm{lIFD}(R) \leq \mathrm{lGFID}(R)$. We may assume that $\mathrm{lGFID}(R) = n < \infty$. Let C be a FP-injective module and M any R-module, then there exists an exact sequence

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^n \longrightarrow 0$$

with each E^i Gorenstein FP-injective. By the dual version of Lemma 2.1, we have $\mathrm{Ext}^{n+1}(C, M) \cong \mathrm{Ext}^1(C, E^n) = 0$, and so $\mathrm{pd}_R C \leq n$. Hence $\mathrm{lIFD}(R) \leq n$ by Lemma 3.9. □

Lemma 3.11 *Let R be a commutative coherent ring, then the following statements hold:*

1. *If $\mathrm{lSGFD}(R) \leq n$, then $\mathrm{fd}_R E \leq n$ for any FP-injective R-module E.*
2. *If $\mathrm{lGFID}(R) \leq n$, then FP-$\mathrm{id}_R F \leq n$ for any flat R-module F.*

Proof (1) By Lemma 3.10, $\mathrm{lFID}(R) \leq \mathrm{lSGFD}(R) \leq n$, and so $\mathrm{lIFD}(R) \leq n$ by Lemma 3.9. Thus $\mathrm{fd}_R E \leq n$ for any FP-injective R-module E.

(2) If $\mathrm{lGFID}(R) \leq n$, then $\mathrm{rIFD}(R) = \mathrm{lIFD}(R) \leq n$ by Lemma 3.10 and R is a commutative ring, and hence FP-$\mathrm{id}_R F \leq n$ for any flat R-module F by Lemma 3.9. □

We are now in a position to give the following result.

Theorem 3.12 *Let R be a ring, then* $\mathrm{lSGFD}(R) = \mathrm{lGFID}(R)$ *under each of the following conditions:*

1. *R is an m-FC ring; or*
2. *R is a commutative coherent ring.*

Proof We will prove $\mathrm{lGFID}(R) \leq \mathrm{lSGFD}(R)$, and dually one can check that $\mathrm{lSGFD}(R) \leq \mathrm{lGFID}(R)$. We may assume that $\mathrm{lSGFD}(R) = n < \infty$. Let M be an R-module. First assume that M is strongly $*$-Gorenstein flat. Then there exists an exact sequence $0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0$ in Mod$R$ with P projective. Take an injective resolution $0 \longrightarrow M \longrightarrow I_0 \longrightarrow \cdots \longrightarrow I_n \longrightarrow \cdots$ of M, and consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_0 & \longrightarrow & I_0 \oplus I_0 & \longrightarrow & I_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_n & \longrightarrow & E_n & \longrightarrow & K_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Since $\mathrm{id}_R P \leq n$ by Lemma 3.4, we have E_n is injective. Let T be a FP-injective R-module, then, by Lemma 3.8 or Lemma 3.11, $\mathrm{fd}_R T < \infty$. Thus $\mathrm{pd}_R T = \mathrm{SGfd}_R T \leq n$ by Proposition 2.9, and hence $\mathrm{Ext}^i(T, K_n) = 0$ for any $i \geq n+1$. This implies that K_n is Gorenstein $*$-FP-injective by Remark 3.6, and so $\text{GFP-id}_R M \leq n$.

If M is a strongly Gorenstein flat module, then $\text{GFP-id}_R M \leq n$ since every strongly Gorenstein flat R-module is a direct summand of a strongly $*$-Gorenstein flat R-module by Lemma 3.7.

Now we may assume that $\mathrm{SGfd}_R M = k \neq 0$. Let $0 \longrightarrow K \longrightarrow N \longrightarrow M \longrightarrow 0$ be an exact sequence with N projective, then $\mathrm{SGfd}_R K = k-1$ by Proposition 2.7(2). By induction on k we have $\text{GFP-id}_R K \leq n$. Since $\text{GFP-id}_R N \leq n$ by the preceding proof, we have $\text{GFP-id}_R M \leq n$ by the dual versions of Propositions 2.7(1) and 2.8. □

4. Strongly Gorenstein flat dimension over commutative Noetherian rings

Recall that a module M in Mod$R$ is called Gorenstein flat [12] if there exists an exact sequence

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

of flat modules such that it is exact after applying the functor $I \otimes -$ for any injective module I and $M = \mathrm{Ker}(F_0 \to F^0)$. We use $\mathrm{Gfd}_R M$ to denote the Gorenstein flat dimension of a module M in Mod$R$, which is defined as the smallest non-negative integer n such that there exists an exact sequence

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

with each G_i Gorenstein flat. If no such n exists, set $\mathrm{Gfd}_R M = \infty$.

We start with the following lemma.

Lemma 4.1 *Let R be a commutative Noetherian ring of finite Krull dimension and M be an R-module. If $\mathrm{Gfd}_R M < \infty$ and $\mathrm{Ext}^i(M, F) = 0$ for any flat R-module F and any $i \geq 1$, then M is Gorenstein flat, and there is a monic flat preenvelope $M \longrightarrow P$ with P projective.*

Proof Since $\mathrm{Gfd}_R M < \infty$, there is a monomorphism $0 \longrightarrow M \longrightarrow L$ with $\mathrm{fd}_R L < \infty$ by [6, Lemma 2.19]. Let E and E' be any injective modules, then $\mathrm{Hom}(E, E')$ is flat. Thus, for any $i \geq 1$, $\mathrm{Hom}(\mathrm{Tor}_i(E, M), E') \cong \mathrm{Hom}(\mathrm{Tor}_i(M, E), E') \cong \mathrm{Ext}^i(M, \mathrm{Hom}(E, E')) = 0$, and hence $\mathrm{Tor}_i(E, M) = 0$. This implies that M is Gorenstein flat by [15, Theorem 3.14] and there is a monic flat preenvelope $M \longrightarrow P$ with P projective by [18, Theorem 4.2.8] and [13, Lemma 2.4]. □

Theorem 4.2 *Let R be a commutative Noetherian ring of finite Krull dimension, M be an R-module and $n \in \mathbb{N}$. Then the following statements are equivalent.*

1. $\mathrm{SGfd}_R M \leq n$.
2. $\mathrm{Gpd}_R M \leq n$.
3. $\mathrm{Gfd}_R M < \infty$ *and* $\mathrm{Ext}^i(M, F) = 0$ *for all flat R-modules F and all $i \geq n+1$.*
4. $\mathrm{Gfd}_R M < \infty$ *and* $\mathrm{Ext}^i(M, P) = 0$ *for all projective R-modules P and all $i \geq n+1$.*

In particular, $\mathrm{SGfd}_R M = \mathrm{Gpd}_R M$.

Proof (1) ⇒ (2) follows from the fact that every strongly Gorenstein flat module is Gorenstein projective.

(2) ⇒ (4) follows from [13, Theorem 3.2].

(4) ⇒ (3) One can check that $\mathrm{Ext}^i(M, Q) = 0$ for any R-module Q with $\mathrm{pd}_R Q < \infty$ and any $i \geq 1$, thus (3) holds by [18, Theorem 4.2.8].

(3) ⇒ (1) Let $0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ be an exact sequence with each P_i projective. We need to prove that K is strongly Gorenstein flat. Note that $\mathrm{Gfd}_R K < \infty$ and $\mathrm{Ext}^i(K, F) \cong \mathrm{Ext}^{i+n}(M, F) = 0$ for any flat R-module F and any $i \geq 1$, so there is an exact sequence $0 \longrightarrow K \xrightarrow{\varphi^0} P^0 \longrightarrow C^0 \longrightarrow 0$ with P^0 projective and φ^0 a flat preenvelope by Lemma 4.1. Note that $\mathrm{Ext}^i(C^0, F) = 0$ for any flat R-module F and any $i \geq 1$, and $\mathrm{Gfd}_R C^0 < \infty$, so there is an exact sequence $0 \longrightarrow C^0 \xrightarrow{\varphi^1} P^1 \longrightarrow C^1 \longrightarrow 0$ with P^1 projective and φ^1 a flat preenvelope by Lemma 4.1. Continue this process, we can get an exact sequence $0 \longrightarrow K \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$ with each P^i projective such that it is $\mathrm{Hom}(-, \mathcal{F})$-exact. On the other hand, $\mathrm{Ext}^i(K, F) = 0$ for any flat R-module F and any $i \geq 1$, then K is strongly Gorenstein flat by Lemma 2.1. Therefore, $\mathrm{SGfd}_R M \leq n$. □

Corollary 4.3 *Let R be a commutative Noetherian ring of finite Krull dimension and M be an R-module. Then $\mathrm{SGfd}_R M < \infty$ if and only if $\mathrm{Gfd}_R M < \infty$.*

Proof Immediately by Theorem 4.2 and [13, Theorem 3.4]. □

Acknowledgements

This research was partially supported by the Specialized Research Fund for the Doctoral Program of Higher Education (Grant No. 20100091110034) and NSF of Jiangsu Province of China (Grant No. BK2010047). The author would like to express their deep gratitude to Professor Zhaoyong Huang. He gave the author a lot of valuable comments and careful suggestions. The author also thanks the referee for his/her careful reading and useful suggestions.

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