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Generalized derivations of prime rings on multilinear polynomials with annihilator conditions

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Abstract: Let K be a commutative ring with unity, R be a prime K-algebra with characteristic not 2, U be the right Utumi quotient ring of R, C the extended centroid of R, I a nonzero right ideal of R and a a fixed element of R. Let g be a generalized derivation of R and $f(X_1,\ldots,X_n)$ a multilinear polynomial over K.

If $ag(f(x_1,\ldots,x_n))f(x_1,\ldots,x_n)=0$ for all $x_1,\ldots,x_n\in I$, then one of the following holds:

(1) $aI=ag(I)=0$;

(2) $g(x)=bx+[c,x]$ for all $x\in R$, where $b,c\in U$. In this case either $[c,I]I=0=abI$ or $aI=0=a(b+c)I$;

(3) $[f(X_1,\ldots,X_n),X_{n+1}]X_{n+2}$ is an identity for I.

Key words: Prime ring, derivation, generalized derivation, right Utumi quotient ring, differential identity, generalized polynomial identity

1. Introduction

Throughout this paper unless specially stated, K will denote a commutative ring with unit, R is always a prime K-algebra with center $Z(R)$ and extended centroid C, U is its right Utumi quotient ring. For $x,y\in R$, the commutator of x and y is denoted by $[x,y]$ and defined by $[x,y]=xy-yx$.

By a derivation of R, we mean an additive mapping d from R into itself satisfying the rule $d(xy)=d(x)y+xd(y)$ for all $x,y\in R$. The study of derivations of prime rings was initiated by E. C. Posner [25]. Later many generalizations of Posner's results have been obtained by a number of authors in the literature (see, [5], [6], [17], [19], [18]).

An additive mapping $g:R\to R$ is called a generalized derivation of R if there exists a derivation d of R such that $g(xy)=g(x)y+xd(y)$ for all $x,y\in R$. The notion of generalized derivation was introduced by M. Brešar [4] and the algebraic study of these mappings was initiated by B. Hvala [15]. Obviously any derivation is a generalized derivation. Moreover, other basic examples of generalized derivations are the mappings of the form $g(x)=ax+xb$, for some $a,b\in R$. Many authors have studied generalized derivations in the context of prime and semiprime rings (see, [1], [11], [15], [21], [22]). Here we will consider some related problems concerning annihilators of generalized derivations in prime rings.

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In [3], M. Brešar proved that if R is a semiprime ring with a nonzero derivation d and $a \in R$ is such that $ad(x)^m = 0$ for all $x \in R$, where m is a fixed positive integer, then $ad(R) = 0$ when R is $(m-1)!$-torsion free.

In [8], C. M. Chang and T. K. Lee proved the following theorem: Let R be a prime ring, I a nonzero right ideal of R, d a nonzero derivation of R and $a \in R$ be such that $ad([x,y])^m \in Z(R)$ $(d([x,y])^m a \in Z(R)$ resp.) for all $x, y \in I$. If $[I,I]I \neq 0$ and $\dim_C RC > 4$, then either $ad(I) = 0$ $(a = 0$ resp.) or d is the inner derivation induced by some $q \in U$ such that $qI = 0$.

In [7], C. M. Chang generalized the above results by proving that if R is a prime ring with extended centroid C, I is a nonzero right ideal of R, d is a nonzero derivation of R, $f(X_1,\dots,X_n)$ is a multilinear polynomial over C, $a \in R$ and $m \geq 1$ is a fixed integer such that $ad(f(x_1,\dots,x_n))^m = 0$ for all $x_1,\dots,x_n \in I$, then either $aI = 0 = d(I)I$ or $[f(X_1,\dots,X_n),X_{n+1}]X_{n+2}$ is an identity for I.

Recently in [12], V. De Filippis investigated the annihilators of power values of generalized derivations on multilinear polynomials and extended Chang's result in [7].

In our recent paper [13], we proved the following theorem. Let K be a commutative ring with unity, R be a prime K-algebra, U its right Utumi quotient ring, C the extended centroid of R, and I a nonzero right ideal of R. Let g be a nonzero generalized derivation of R and $f(X_1,\dots,X_n)$ a multilinear polynomial over K. If

$$g(f(x_1,\dots,x_n))f(x_1,\dots,x_n) = 0$$

for all $x_1,\dots,x_n \in I$, then either $f(X_1,\dots,X_n)X_{n+1}$ is an identity for I or $g(x) = ax + [b,x]$, for suitable $a, b \in U$ and one of the following holds:

(1) $aI = 0$ and $[f(X_1,\dots,X_n),X_{n+1}]X_{n+2}$ is an identity for I;

(2) $aI = 0$ and $(b-\beta)I = 0$ for a suitable $\beta \in C$.

In this paper we will continue the investigation by studying the properties of a subset S of R related to its left annihilator $\mathrm{Ann}_R(S) = \{x \in R \mid xS = (0)\}$. More precisely we will study the case when

$$S = \{g(f(x_1,\dots,x_n))f(x_1,\dots,x_n) \mid x_1,\dots,x_n \in R\},$$

where g is a generalized derivation on R, $f(X_1,\dots,X_n)$ is a multilinear polynomial in n non-commuting variables over K. We prove the following theorem.

Main Theorem. *Let K be a commutative ring with unity, R be a prime K-algebra with characteristic not 2, U be its right Utumi quotient ring, C the extended centroid of R, and I a nonzero right ideal of R. Let g be a nonzero generalized derivation of R, $a \in R$ and $f(X_1,\dots,X_n)$ a multilinear polynomial over K. If*

$$ag(f(x_1,\dots,x_n))f(x_1,\dots,x_n) = 0$$

for all $x_1,\dots,x_n \in I$, then one of the following holds:

(1) *$aI = 0 = ag(I)$;*

(2) *$g(x) = bx + [c,x]$ for all $x \in R$, where $b, c \in U$. In this case, either $[c,I]I = (0) = abI$ or $aI = 0 = a(b+c)I$;*

(3) *$[f(X_1,\dots,X_n),X_{n+1}]X_{n+2}$ is an identity for I.*

2. Preliminaries

In all that follows, unless stated otherwise, R will be a prime K-algebra and $f(X_1,\ldots,X_n)$ a multilinear polynomial over K. For any ring S, $Z(S)$ will denote its center.

The related object we need to mention is the right Utumi quotient ring U of R (sometimes, as in [2], U is called the maximal right ring of quotients). The definitions, the axiomatic formulations and the properties of this quotient ring U can be found in [2].

In any case, when R is a prime ring, all we will need to know about U is that

1. $R\subseteq U$;
2. U is a prime ring with identity;
3. The center of U, denoted by C, is a field which is called the extended centroid of R.

We will also frequently make use of the theory of generalized polynomial identities and differential identities (see [2], [16], [20], [24]). In particular, we need to recall the following facts.

Fact 1. Denote by $T=U*_C C\{X\}$ the free product over C of the C-algebra U and the free C-algebra $C\{X\}$, with X a countable set consisting of non-commuting indeterminates $x_1,\ldots,x_n,\ldots$. The elements of T are called generalized polynomials with coefficients in U. Recall that if B is a basis of U over C, then any element of T can be written in the form $g=\sum_i \alpha_i m_i$, where $\alpha_i\in C$ and m_i are B-monomials, that is $m_i=q_0y_1\ldots y_nq_n$, with $q_i\in B$ and $y_i\in\{x_1,\ldots,x_n,\ldots\}$. In [9] it is shown that a generalized polynomial $g=\sum_i\alpha_i m_i$ is the zero element of T if and only if each α_i is zero. As a consequence, if $a_1,a_2\in U$ are linearly independent over C and $a_1g_1(x_1,\ldots,x_n)+a_2g_2(x_1,\ldots,x_n)=0\in T$, where $g_1(x_1,\ldots,x_n)=\sum_{i=1}^n x_ih_i(x_1,\ldots,x_n)$ and $g_2(x_1,\ldots,x_n)=\sum_{i=1}^n x_ik_i(x_1,\ldots,x_n)$ for $h_i(x_1,\ldots,x_n),k_i(x_1,\ldots,x_n)\in T$, then both $g_1(x_1,\ldots,x_n)$ and $g_2(x_1,\ldots,x_n)$ are the zero element of T.

Fact 2. If R is prime and I is a non-zero right ideal of R, then I, IR and IU satisfy the same generalized polynomial identities with coefficients in U [9].

Fact 3. If R is prime and I is a non-zero right ideal of R, then I, IR and IU satisfy the same differential polynomial identities with coefficients in U [20].

Fact 4. In [21], T. K. Lee extended the definition of a generalized derivation as follows. By a generalized derivation he means an additive mapping $g:I\to U$ such that $g(xy)=g(x)y+xd(y)$ for all $x,y\in I$, where I is a dense right ideal of R and d is a derivation from I into U. He also proved that every generalized derivation g on a dense right ideal of a semiprime ring R can be uniquely extended to a generalized derivation of U and assumes the form $g(x)=ax+d(x)$ for all $x\in U$, for some $a\in U$ and a derivation d on U (Theorem 4 in [21]).

Fact 5. Every derivation d of R can be uniquely extended to a derivation of U (see Proposition 2.5.1 in [2]). Moreover, since R is a prime ring, we may assume $K\subseteq C$ and so for any $\alpha\in K$ one has $d(\alpha.1)\in C$.

Fact 6. We will use the following notation:

$$f(x_1,\ldots,x_n)=\alpha x_1\ldots x_n+\sum_{1\neq\sigma\in S_n}\alpha_\sigma x_{\sigma(1)}\ldots x_{\sigma(n)}$$

for some $\alpha,\alpha_\sigma\in K$ and moreover we denote by $f^d(x_1,\ldots,x_n)$ the polynomial obtained from $f(x_1,\ldots,x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma.1)$. Thus we write $d(f(x_1,\ldots,x_n))=f^d(x_1,\ldots,x_n)+$
$\sum_{i=1}^n f(x_1,\ldots,d(x_i),\ldots,x_n)$ for all $x_1,\ldots,x_n\in R$.

Fact 7. We will also write multilinear polynomial $f(x_1,\ldots,x_n)$ as follows:

$$f(x_1,\ldots,x_n)=\sum_{i=1}^{n} t_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)x_i$$

where t_i are multilinear polynomials in $n-1$ variables, and x_i never appears in any monomials in t_i.

Fact 8. We will need the following fact in the proof of Lemma 1: Let R be a prime ring, $a,b\in R$ and $f(X_1,\ldots,X_n)$ be a multilinear polynomial over C, which is not vanishing on R. Suppose $(af(x_1,\ldots,x_n)+f(x_1,\ldots,x_n)b)f(x_1,\ldots,x_n)=0$ for all $x_1,\ldots,x_n\in R$. Then either $a=-b\in C$ or $f(X_1,\ldots,X_n)$ is central valued on R and $a+b=0$ (Lemma 1 in [13]).

3. Results

We need the following lemmas.

Lemma 1 *Let $R=M_2(F)$ where F is a field, $f(X_1,\ldots,X_n)$ a multilinear polynomial over F, $a,b,c\in R$ be fixed elements, and I a nonzero right ideal of R. If*

$$a(bf(x_1,\ldots,x_n)+f(x_1,\ldots,x_n)c)f(x_1,\ldots,x_n)=0$$

for all $x_1,\ldots,x_n\in I$, then one of the following holds:

(i) $a=0$,

(ii) $c\in F$ *and* $a(b+c)=0$ *unless* $F\cong GF(2)$,

(iii) $[f(X_1,\ldots,X_n),X_{n+1}]X_{n+2}$ *is an identity for* I.

Proof Assume first that $I\neq R$. Since every proper right ideal of R is minimal, we conclude that $[I,I]I=0$. Then clearly $[f(X_1,\ldots,X_n),X_{n+1}]X_{n+2}$ is an identity for I, and we are done. Therefore, we may assume that $I=R$. If now $a=0$, then there is nothing to prove. We assume throughout that $a\neq 0$. Moreover, if $f(X_1,\ldots,X_n)$ is central valued on R, then (iii) holds. So we also assume that $f(X_1,\ldots,X_n)$ is not central valued on R. Let e_{ij} denote the matrix unit with 1 in the (i,j)-th position, and zero elsewhere. Note that

$$Ra(bf(x_1,\ldots,x_n)+f(x_1,\ldots,x_n)c)f(x_1,\ldots,x_n)=0$$

for all $x_1,\ldots,x_n\in R$. Since R is von Neumann regular, there exists an idempotent $e\in R$ such that $Ra=Re$. Hence we may assume that a is an idempotent. Now if a is invertible then $a=1$, and thus $b=-c\in F$ by Fact 8, and we are done. Hence we may consider the case when $Ra=Re$ is a proper left ideal of R. Since any two proper left ideals J and L of R are conjugate, there exists an invertible element $u\in R$ such that $J=uLu^{-1}$. Then $Re_{11}=uRau^{-1}=Ruau^{-1}$, and so replacing a by uau^{-1} we may assume further that $a=e_{11}$.

Now for any nonzero $\alpha\in F$, there exist elements $r_1,\ldots,r_n\in R$ such that $f(r_1,\ldots,r_n)=\alpha e_{12}$ by [23]. Let $c=\sum_{i,j=1}^{2}c_{ij}e_{ij}$. By our assumption we once get that

$$\begin{aligned}0&=a(bf(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)c)f(r_1,\ldots,r_n)\\&=e_{11}(b\alpha e_{12}+\alpha e_{12}c)\alpha e_{12}\\&=\alpha^2c_{21}e_{12}.\end{aligned}$$

Hence $c_{21}=0$. We proceed to show that c is central unless $F\cong GF(2)$. We have seen that c has the form $\begin{pmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{pmatrix}$. We note that $f(R)=\{f(r_1,\ldots,r_n): r_1,\ldots,r_n\in R\}$ is invariant under all F-automorphisms of R. Let $\beta\in F$ and define $\varphi(x)=(1-\beta e_{21})x(1+\beta e_{21})$ for all $x\in R$, an automorphism of R. Then

$$\begin{aligned}\varphi(f(r_1,\ldots,r_n)) &= (1-\beta e_{21})f(r_1,\ldots,r_n)(1+\beta e_{21})\\ &= (1-\beta e_{21})\alpha e_{12}(1+\beta e_{21})\\ &= \alpha(e_{12}+\beta e_{11}-\beta e_{22}-\beta^2 e_{21})\in f(R).\end{aligned}$$

Now by our assumption

$$\begin{aligned}0=\alpha e_{11}\Big(& b(e_{12}+\beta e_{11}-\beta e_{22}-\beta^2 e_{21})\\ &+(e_{12}+\beta e_{11}-\beta e_{22}-\beta^2 e_{21})c\Big)(e_{12}+\beta e_{11}-\beta e_{22}-\beta^2 e_{21}),\end{aligned}$$

and so

$$(e_{12}+\beta e_{11})c(e_{12}+\beta e_{11}-\beta e_{22}-\beta^2 e_{21})=0,$$

since $\alpha\neq 0$. By direct calculation, we see that

$$\beta(c_{11}-c_{22}-\beta c_{12})(e_{11}+\beta e_{12})=0$$

for all $\beta\in F$. If $\beta\neq 0$, then we have

$$c_{11}-c_{22}-\beta c_{12}=0.$$

In particular, for $\beta=1$, one has $c_{11}-c_{22}-c_{12}=0$. Comparing these last two equations, we get $(\beta-1)c_{12}=0$ for all $\beta\in F-\{0\}$. Then $c_{12}=0$ and $c_{11}=c_{22}$, and so $c\in F$ unless $F\cong GF(2)$. Therefore

$$a(b+c)f(x_1,\ldots,x_n)^2=0$$

for all $x_1,\ldots,x_n\in R$. By Lemma 2 in [10], $a(b+c)=0$ since f is not an identity for R. This completes the proof. □

Lemma 2 *Let $R=M_m(F)$, where $m\geq 3$ and F is a field of characteristic not 2, $I=eR=(e_{11}+\cdots+e_{ll})R$, $f(X_1,\ldots,X_n)$ be a multilinear polynomial over F, and $a,b,c\in R$ be fixed elements. If*

$$a(bf(x_1,\ldots,x_n)+f(x_1,\ldots,x_n)c)f(x_1,\ldots,x_n)=0$$

for all $x_1,\ldots,x_n\in I$, then one of the following holds:

(i) *$aI=0$ and either $abI=0$ or $f(X_1,\ldots,X_n)X_{n+1}$ is an identity for I,*

(ii) *$[c,I]I=0$ and either $a(b+c)I=0$ or $f(X_1,\ldots,X_n)X_{n+1}$ is an identity for I,*

(iii) *$[f(X_1,\ldots,X_n),X_{n+1}]X_{n+2}$ is an identity for I.*

Proof If $aI = 0$, then $abf(x_1,\ldots,x_n)^2 = 0$ for all $x_1,\ldots,x_n \in I$. Then by [10], either $abI = 0$ or $f(X_1,\ldots,X_n)X_{n+1}$ is an identity for I. On the other hand, if $[c,I]I = 0$ then $a(b+c)f(x_1,\ldots,x_n)^2 = 0$ for all $x_1,\ldots,x_n \in I$. Hence we deduce again by [10] that either $a(b+c)I = 0$ or $f(X_1,\ldots,X_n)X_{n+1}$ is an identity for I. Therefore we may assume throughout that $aI \neq 0$ and $[c,I]I \neq 0$. Notice that if $f(X_1,\ldots,X_n)$ is central valued on eRe, then $[f(X_1,\ldots,X_n),X_{n+1}]X_{n+2}$ is an identity for I and thus we are done. So we may also assume that $f(X_1,\ldots,X_n)$ is not central valued on eRe. Set $A = \{f(x_1,\ldots,x_n) \mid x_1,\ldots,x_n \in I\}$. In the present case, for any $s \leq l$ and $s \neq t$, there exist $r_1,\ldots,r_n \in I$ such that $f(r_1,\ldots,r_n) = e_{st} \in A$ by Lemma 3 in [7]. By our assumption

$$\begin{aligned} 0 &= a(be_{st} + e_{st}c)e_{st} \\ &= c_{ts}ae_{st}. \end{aligned}$$

Assume that $c_{i_0j_0} \neq 0$ for some $j_0 \leq l$ and $j_0 \neq i_0$. Then since $c_{i_0j_0}ae_{j_0i_0} = 0$ we see that $ae_{j_0i_0} = 0$ which in turn implies that $ae_{j_0j_0} = 0$. Take another $j \leq l$ with $j \neq i_0$. If $c_{i_0j} \neq 0$, we get $ae_{jj} = 0$ as above. Consider now the case $c_{i_0j} = 0$. By Lemma 3 in [7], $e_{j_0i_0} + e_{ji_0} \in A$ and by hypothesis we have

$$\begin{aligned} 0 &= a(b(e_{j_0i_0} + e_{ji_0}) + (e_{j_0i_0} + e_{ji_0})c)(e_{j_0i_0} + e_{ji_0}) \\ &= c_{i_0j_0}a(e_{j_0i_0} + e_{ji_0}). \end{aligned}$$

Since $c_{i_0j_0} \neq 0$ and $ae_{j_0i_0} = 0$, we deduce that $ae_{ji_0} = 0$, whence $ae_{jj} = ae_{ji_0}e_{i_0j} = 0$. Thus we have shown that $ae_{jj} = 0$ for all $j \leq l$ and $j \neq i_0$. We note that if $i_0 > l$, then $ae_{jj} = 0$ for all $j \leq l$ and so $aI = 0$, a contradiction. Thus we may assume that $i_0 \leq l$. If $c_{ki_0} \neq 0$ for some $k \neq i_0$, then we conclude as above that $ae_{i_0i_0} = 0$. But we then arrive at the contradiction $aI = 0$. So we may assume that $c_{ki_0} = 0$ for all $k \neq i_0$.

Consider the following of R,

$$\begin{aligned} \varphi(x) &= (1 + e_{i_0j_0})x(1 - e_{i_0j_0}) \\ \psi(x) &= (1 - e_{i_0j_0})x(1 + e_{i_0j_0}), \end{aligned}$$

and notice that $\varphi(I), \psi(I) \subseteq I$. Therefore I satisfies the following two generalized identities:

$$\begin{aligned} &\varphi(a)\Big(\varphi(b)f(X_1,\ldots,X_n) + f(X_1,\ldots,X_n)\varphi(c)\Big)f(X_1,\ldots,X_n), \\ &\psi(a)\Big(\psi(b)f(X_1,\ldots,X_n) + f(X_1,\ldots,X_n)\psi(c)\Big)f(X_1,\ldots,X_n). \end{aligned}$$

By calculation $\varphi(c)_{i_0j_0} = c_{i_0j_0} - c_{i_0i_0} + c_{j_0j_0}$ and $\psi(c)_{i_0j_0} = c_{i_0j_0} + c_{i_0i_0} - c_{j_0j_0}$ since $c_{j_0i_0} = 0$. If now $\varphi(c)_{i_0j_0} = \psi(c)_{i_0j_0}$, then we see that $c_{i_0i_0} - c_{j_0j_0} = 0$ since $char(F) \neq 2$. Therefore, $\varphi(c)_{i_0j_0} = \psi(c)_{i_0j_0} = c_{i_0j_0} \neq 0$. On the other hand, if $\varphi(c)_{i_0j_0} \neq \psi(c)_{i_0j_0}$, then either $\varphi(c)_{i_0j_0} \neq 0$ or $\psi(c)_{i_0j_0} \neq 0$. By our previous arguments either $\varphi(a)e_{jj} = 0$ for all $j \leq l$ and $j \neq i_0$ or $\psi(a)e_{jj} = 0$ for all $j \leq l$ and $j \neq i_0$. If $\varphi(a)e_{jj} = 0$ for all $j \leq l$ and $j \neq i_0$, then in particular $\varphi(a)e_{j_0j_0} = 0$. So by calculation we see that $(a + a_{j_0i_0})e_{i_0j_0} = 0$ whence $(a + a_{j_0i_0})e_{i_0i_0} = 0$. Now since

$$\begin{aligned} 0 &= e_{i_0j_0}(a + a_{j_0i_0})e_{i_0i_0} \\ &= a_{j_0i_0}e_{i_0i_0} \end{aligned}$$

we see that $ae_{i_0i_0} = 0$. But then we again arrive at the contradiction $aI = 0$. So we must have $\psi(a)e_{jj} = 0$ for all $j \leq l$ and $j \neq i_0$. As above this leads to the contradiction $aI = 0$.

From now on we may assume that $c_{ij} = 0$ for all $j \leq l$ and $j \neq i$. Define now $\tau(x) = (1 + e_{ij})x(1 - e_{ij})$ for $i, j \leq l$ and $i \neq j$. Since $\tau(I) \subseteq I$, we see that

$$\tau(a)\Big(\tau(b)f(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)\tau(c)\Big)f(x_1, \ldots, x_n) = 0$$

for all $x_1, \ldots, x_n \in I$. The (i, j)-entry of $\tau(c)$ is $\tau(c)_{ij} = c_{jj} - c_{ii}$. If now $\tau(c)_{ij} \neq 0$ for some $i, j \leq l$ and $i \neq j$, then we can proceed as before and show that $\tau(a)I = 0$. But then $\tau(aI) = \tau(a)I = 0$ which then leads to the contradiction $aI = 0$. Hence $\tau(c)_{ij} = 0$ for all $i, j \leq l$ and $i \neq j$. Hence $c_{ii} = c_{jj} = \lambda$ for all $i, j \leq l$ and $i \neq j$. Then $(c - \lambda)I = 0$, that is $[c, I]I = 0$ which is again a contradiction. This proves the lemma. □

Lemma 3 *Let R be a prime ring, $a, b, c \in R$ and $f(X_1, \ldots, X_n)$ a nonzero multilinear polynomial over C and I a nonzero right ideal of R such that*

$$a(bf(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)c)f(x_1, \ldots, x_n) = 0$$

for all $x_1, \ldots, x_n \in I$. If R does not satisfy any nontrivial generalized polynomial identity, then one of the following holds:

(i) $aI = 0 = abI$;

(ii) $[c, I]I = 0 = a(b + c)I$.

Proof If $aI = 0$, then we have $abf(x_1, \ldots, x_n)^2 = 0$ for all $x_1, \ldots, x_n \in I$. Then by [10], we have either $abI = 0$ or $f(x_1, \ldots, x_n)x_{n+1} = 0$ for all $x_1, \ldots, x_{n+1} \in I$. If $u \in I$ is nonzero and $abI \neq 0$, then since R does not satisfy any nontrivial generalized polynomial identity (GPI for short)

$$f(uX_1, \ldots, uX_n)uX_{n+1}$$

is the zero element in T. But then we must have $u = 0$, a contradiction. Therefore when $aI = 0$ we also have $abI = 0$, and we are done. On the other hand, if $[c, I]I = 0$, then $a(b + c)f(x_1, \ldots, x_n)^2 = 0$ for all $x_1, \ldots, x_n \in I$. This yields $a(b+c)I = 0$ as above, and we are done again. So we may assume that $aI \neq 0$ and $[c, I]I \neq 0$. Since R does not satisfy any non-trivial GPI by the hypothesis,

$$a(bf(uX_1, ..., uX_n) + f(uX_1, ..., uX_n)c)f(uX_1, ..., uX_n)$$

is the zero element in T, that is

$$a(bf(uX_1, ..., uX_n) + f(uX_1, ..., uX_n)c)f(uX_1, ..., uX_n) = 0 \in T \tag{3.1}$$

for all $u \in I$.

Suppose that there exists $u \in I$ such that abu and au are linearly independent over C. By Fact 1 and (3.1)

$$abf(uX_1, \ldots, uX_n)^2 = 0 \in T,$$

which implies that $abu = 0$ since R does not satisfy any nontrivial GPI, a contradiction. Thus we have abu and au are C-dependent for all $u \in I$. We claim that there exists $\lambda \in C$, independent of u, such that $abu = \lambda au$. If $av = 0$ for some $v \in I$, then since $a(u+v)$ and $ab(u+v)$ are C-dependent, we once see that au and $abu + abv$ are C-dependent. Now we have

$$\alpha au + \beta abu = 0 \tag{3.2}$$

for some $\alpha, \beta \in C$, not both zero, and

$$\gamma au + \mu(abu + abv) = 0 \tag{3.3}$$

for some $\gamma, \mu \in C$, not both zero. Comparing (3.2) and (3.3), we get

$$(\beta\gamma - \alpha\mu)au + \mu\beta abv = 0.$$

If $\mu\beta \neq 0$ then one gets au and abv are C-dependent. If $\mu\beta = 0$, then either $\gamma \neq 0$ or $\alpha \neq 0$. Thus $au = 0$ by (3.2) and (3.3), and again au and abv are C-dependent. Now if $abv \neq 0$, then $au \in Cabv$, and thus aI is a commutative right ideal of R, which is a contradiction since $aI \neq 0$. Hence we have $abv = 0$ whenever $av = 0$. Let $u, v \in I$ be any elements. If $a(u+v) = 0$ then we have seen above that $ab(u+v) = 0$. So we assume that $a(u+v) \neq 0$. Then $ab(u+v) = \lambda_{u+v}a(u+v)$, and so

$$\lambda_u au + \lambda_v av = \lambda_{u+v}au + \lambda_{u+v}av.$$

Notice that the above relation holds even if $au = 0$ (or $av = 0$). Hence we get

$$(\lambda_u - \lambda_{u+v})au + (\lambda_v - \lambda_{u+v})av = 0.$$

Now if $\lambda_u - \lambda_{u+v} = 0 = \lambda_v - \lambda_{u+v}$, then we are done. For otherwise, we conclude that au and av are C-dependent. Therefore, in any case we see that aI is a commutative right ideal of R, a contradiction. Hence we have shown that there exists $\lambda \in C$ such that $abu = \lambda au$ for all $u \in I$, that is $a(b-\lambda)I = 0$. Now for any $u \in I$, we have

$$af(uX_1, \ldots, uX_n)(c+\lambda)f(uX_1, \ldots, uX_n) = 0 \in T$$

implying that either $au = 0$ or $(c+\lambda)u = 0$ for all $u \in I$. Now as an additive group, I is the union of two subgroups $\{u \in I \mid au = 0\}$ and $\{u \in I \mid (c+\lambda)u = 0\}$. Since a group cannot be the union of two proper subgroups, we see that either $aI = 0$ or $(c+\lambda)I = 0$. But we are assuming $aI \neq 0$, and so we must have $(c+\lambda)I = 0$. Thence we see that $[c, I]I = 0$. This contradiction finishes the proof. □

Lemma 4 *Let R be a prime ring of characteristic not 2, $a, b, c \in R$, $f(X_1, \ldots, X_n)$ a multilinear polynomial over C and I a nonzero right ideal of R such that*

$$a(bf(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)c)f(x_1, \ldots, x_n) = 0 \tag{3.4}$$

for all $x_1, \ldots, x_n \in I$. Then one of the following holds:

(i) *$aI = 0$ and either $abI = 0$ or $f(X_1, \ldots, X_n)X_{n+1}$ is an identity for I;*

(ii) *$[c, I]I = 0$ and either $a(b+c)I = 0$ or $f(X_1, \ldots, X_n)X_{n+1}$ is an identity for I;*

(iii) *$[f(X_1, \ldots, X_n), X_{n+1}]X_{n+2}$ is an identity for I.*

Proof If R is not a GPI-ring, then we are done by Lemma 3. Thus suppose that R is a GPI-ring. Since U and R satisfy the same generalized polynomial identities, U is also a GPI-ring. Then by [24], U is a primitive ring with a non-zero socle H. Note that (3.4) also holds for all $x_1, \ldots, x_n \in IU$. Hence replacing R and I by U and IU, respectively, we may assume that R is a primitive ring with a nonzero socle H, $IC = I$ and C is just the center of R. Note that

$$a(bf(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)c)f(x_1, \ldots, x_n) = 0$$

for all $x_1, \ldots, x_n \in J = IH$ by [9]. Thus by replacing R by H and I with $J = IH$, we may assume without loss of generality that R is a simple ring and is equal to its own socle and $I = IR$. Now if $a = 0$, there is nothing to prove. Therefore $Ia \neq 0$, and by replacing a by some $0 \neq ua \in I$ we may assume further that $a \in I$. Suppose that the conclusions of the lemma do not hold. Hence there exist $a_0, c_1, c_2, b_1, \ldots, b_{n+2} \in I$ such that

- $aa_0 \neq 0$ and
- $[c, c_1]c_2 \neq 0$ and
- $[f(b_1, \ldots, b_n), b_{n+1}]b_{n+2} \neq 0$.

Let F be the algebraic closure of C or C itself according to the cases either C is infinite or finite. Note that $I \otimes_C F$ is a completely irreducible right $H \otimes_C F$-module which satisfies the GPI

$$a(bf(X_1, \ldots, X_n) + f(X_1, \ldots, X_n)c)f(X_1, \ldots, X_n) = 0.$$

Thus there exists an idempotent $e \in I \otimes_C F$ such that $a_0, c_1, c_2, b_1, \ldots, b_{n+2} \in e(H \otimes_C F)$. By Litoff's theorem (see [14]) there exists $h^2 = h \in H \otimes_C F$ such that

$$e, eb, be, ec, ce, a, a_0, c_1, c_2, b_1, \ldots, b_{n+2} \in h(H \otimes_C F)h$$

and, moreover, $h(H \otimes_C F)h \cong M_k(F)$ for some $k \geq 2$.

Now for all $x_1, \ldots, x_n \in eh(H \otimes_C F)h \subseteq (I \otimes_C F) \cap h(H \otimes_C F)h$, we have

$$\begin{aligned} 0 &= ha\Big(bef(x_1, \ldots, x_n) + ef(x_1, \ldots, x_n)c\Big)ef(x_1, \ldots, x_n) \\ &= (hah)\Big((hbh)f(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)(hch)\Big)f(x_1, \ldots, x_n). \end{aligned}$$

By Lemmas 1 and 2, one of the following holds:

- $haheh(H \otimes_C F)h = 0$, which leads to the contradiction $0 \neq aa_0 = (hah)eha_0h = 0$;
- $[hch, eh(H \otimes_C F)h]eh(H \otimes_C F)h = 0$, by which we arrive at the contradiction $0 \neq [c, c_1]c_2 = [hch, ehc_1h]ehc_2h = 0$;
- $[f(eh(H \otimes_C F)h), eh(H \otimes_C F)h]eh(H \otimes_C F)h = 0$ which, too, yields the contradiction

$$0 \neq [f(b_1, \ldots, b_n), b_{n+1}]b_{n+2} = [f(ehb_1h, \ldots, ehb_nh), ehb_{n+1}h]ehb_{n+2}h = 0.$$

□

We are now in a position to prove our main theorem.

The Proof of Main Theorem. If $f(X_1,\ldots,X_n)X_{n+1}$ is an identity for I, then (3) holds and we are done. So we may assume that $f(X_1,\ldots,X_n)X_{n+1}$ is not an identity for I and proceed to show that (1)–(3) hold. Now by Fact 4, every generalized derivation g on a dense right ideal of R can be uniquely extended to U and assumes the form $g(x)=bx+d(x)$, for some $b\in U$ and a derivation d on U. Then

$$a\Big(bf(x_1,\ldots,x_n)+d(f(x_1,\ldots,x_n))\Big)f(x_1,\ldots,x_n)=0$$

for all $x_1,\ldots,x_n\in I$. Therefore, for any $u\in I$, U satisfies the following differential identity

$$a\Big(bf(uX_1,\ldots,uX_n)+d(f(uX_1,\ldots,uX_n))\Big)f(uX_1,\ldots,uX_n).$$

If $d=0$, then $abf(x_1,\ldots,x_n)^2=0$ for all $x_1,\ldots,x_n\in I$. Then by [8], we have $abI=0$ and this case is contained in conclusion (2). Hence we may assume that $d\neq 0$. Then I satisfies

$$a(bf(X_1,\ldots,X_n)+f^d(X_1,\ldots,X_n)+\sum_{i=1}^{n}f(X_1,\ldots,d(X_i),\ldots,X_n))f(X_1,\ldots,X_n).$$

In the light of Kharchenko's theory [16], we divide the proof into two cases.

Case 1. If d is an inner derivation induced by an element $c\in U-C$, that is $d(x)=[c,x]$ for all $x\in U$, then $g(x)=bx+d(x)=(b+c)x-xc$ and I satisfies

$$a((b+c)f(X_1,\ldots,X_n)-f(X_1,\ldots,X_n)c)f(X_1,\ldots,X_n).$$

Then by Lemma 4 we have that one of the following conclusions occur:

(a) $aI=0=a(b+c)I$;

(b) $[c,I]I=0=abI$;

(c) $[f(X_1,\ldots,X_n),X_{n+1}]X_{n+2}$ is an identity for I.

In this case we have either the conclusion (2) or (3).

Case 2. Let now d be an outer derivation of U. Now I and IU satisfy the same differential identities in view of Fact 3, and hence

$$a(bf(X_1,\ldots,X_n)+d(f(X_1,\ldots,X_n)))f(X_1,\ldots,X_n)$$

is an identity for IU, that is, for any $u\in I$,

$$a(bf(uX_1,\ldots,uX_n)+d(f(uX_1,\ldots,uX_n)))f(uX_1,\ldots,uX_n)$$

is an identity for U. Then U satisfies the following identity

$$a\Big(bf(uX_1,\ldots,uX_n)+f^d(uX_1,\ldots,uX_n)+\sum_{i=1}^{n}f(uX_1,\ldots,d(u)X_i+ud(X_i),\ldots,uX_n)\Big)f(uX_1,\ldots,uX_n).$$

Since d is an outer derivation, by Kharchenko's results in [16], U satisfies the identity

$$a\Big(bf(uX_1,\dots,uX_n)+f^d(uX_1,\dots,uX_n) + \sum_{i=1}^{n} f(uX_1,\dots,d(u)X_i+uY_i,\dots,uX_n)\Big)f(uX_1,\dots,uX_n). \tag{3.5}$$

It is clear that U satisfies the blended component

$$af(uX_1,\dots,uY_i,\dots,uX_n)f(uX_1,\dots,uX_i,\dots,uX_n).$$

In particular, U satisfies $af(uX_1,\dots,uX_i,\dots,uX_n)^2$. This means either $aI=0$ or $f(uX_1,\dots,uX_n)uX_{n+1}$ is a nontrivial generalized identity for U. We suppose first that $aI=0$ and prove also in this case that U is a GPI-ring. In order to this, as in Fact 7, we write the multilinear polynomial $f(X_1,\dots,X_n)$ as

$$f(X_1,\dots,X_n)=\sum_{i=1}^{n} X_i t_i(X_1,\dots,X_{i-1},X_{i+1},\dots,X_n),$$

where $t_i(X_1,\dots,X_{i-1},X_{i+1},\dots,X_n)$ are multilinear polynomials in $n-1$ variables, and X_i never appears in any monomials in t_i. Then since $au=0$, U satisfies

$$a\Big(b\sum_{i=1}^{n} uX_i t_i(uX_1,\dots,uX_{i-1},uX_{i+1},\dots,uX_n) + \sum_{i=1}^{n} d(u)X_i t_i(uX_1,\dots,uX_{i-1},uX_{i+1},\dots,uX_n)\Big)f(uX_1,\dots,uX_n),$$

that is, U satisfies

$$a(bu+d(u))\sum_{i=1}^{n} X_i t_i(uX_1,\dots,uX_{i-1},uX_{i+1},\dots,uX_n)f(uX_1,\dots,uX_n).$$

In other words,

$$ag(u)\sum_{i=1}^{n} X_i t_i(uX_1,\dots,uX_{i-1},uX_{i+1},\dots,uX_n)f(uX_1,\dots,uX_n)$$

is an identity for U. Since this holds for all $u\in I$, we have either $ag(I)=0$ (and in this case, we are done) or there exists $u\in I$ such that $ag(u)\neq 0$. If the latter holds, then the above identity is a nontrivial generalized polynomial identity for U. In light of this fact, we may always assume that U is a GPI-ring. Finally, we want to show that either conclusion (1) or conclusion (3) holds. By contradiction, in all that follows we suppose that there exists $v\in I$ such that either $av\neq 0$ or $ag(v)\neq 0$, if not conclusion (1) of the Theorem holds. Since $f(X_1,\dots,X_n)X_{n+1}$ is not an identity for I by our assumption, there exist $u_1,\dots,u_{n+1}\in I$ such that $f(u_1,\dots,u_n)u_{n+1}\neq 0$. Now since U is a GPI-ring, U is a primitive ring with socle $H=Soc(U)\neq 0$ by [24]. We note that (3.5) holds for all $x_1,\dots,x_n\in IH$, and so replacing I with IH we may also assume that $I\subseteq H$. By the regularity of H, there exists an idempotent $e\in I=IH$ such that $eH=vH+\sum_{i=1}^{n+1}u_iH$ and $v=ev$,

$u_i = eu_i$ for all $i = 1, \ldots, n+1$. By (3.5), we have

$$a\Big(bf(ex_1, \ldots, ex_n) + f^d(ex_1, \ldots, ex_n) + \sum_{i=1}^{n} f(ex_1, \ldots, d(e)x_i + ed(x_i), \ldots, ex_n)\Big)f(ex_1, \ldots, ex_n) = 0$$

for all $x_1, \ldots, x_n \in H$, and also for all $x_1, \ldots, x_n \in U$. As above, since d is an outer derivation, we get

$$a\Big(bf(ex_1, \ldots, ex_n) + f^d(ex_1, \ldots, ex_n) + \sum_{i=1}^{n} f(ex_1, \ldots, d(e)x_i + ey_i, \ldots, ex_n)\Big)f(ex_1, \ldots, ex_n) = 0.$$

Hence U satisfies the blended component

$$af(eX_1, \ldots, eY_i, \ldots, eX_n)f(eX_1, \ldots, eX_i, \ldots, eX_n).$$

In particular, U satisfies $af(eX_1, \ldots, eX_n)^2$. Then either $ae = 0$ or eU satisfies the identity $f(X_1, \ldots, X_n)X_{n+1}$. In case $ae = 0$, we get the contradiction $0 = aev = av \neq 0$. For the latter case, we have $0 = f(eu_1, \ldots, eu_n)eu_{n+1} = f(u_1, \ldots, u_n)u_{n+1} \neq 0$. These contradictions prove that either $aI = 0 = ag(I)$ or $f(X_1, \ldots, X_n)X_{n+1}$ is an identity for I. □

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