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(p, λ)-Koszul algebras and modules, II

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Abstract: This paper is a continuous work of [14], where the notions of (p, λ)-*Koszul algebra* and (p, λ)-*Koszul module* were first introduced. More precisely, some new criteria for a positively graded algebra to be (p, λ)-Koszul are provided. We also generalize (p, λ)-Koszul objects to the nongraded case and define the so-called quasi-(p, λ)-Koszul objects. Further, the relationships between (quasi-) (p, λ)-Koszul modules and minimal Horseshoe Lemma are established.

Key words: (p, λ)-Koszul algebras and modules, minimal Horseshoe Lemma

It is well known that the noncommutative graded algebras play an important role in algebra, topology, and mathematical physics. Probably the most interesting class of such algebras is the class of Koszul algebras (see [2], [15] and [16]), which give a nice connection of algebraic objects (dual algebra) and homological objects (Yoneda algebras). In the last decade, several extensions of this theory to some more general classes of algebras have been developed. More precisely, motivated by the Artin-Schelter regular algebras of global dimension three (see [1]), Berger introduced *nonquadratic Koszul algebra* (see [3]) in 2001 (many people prefer the name "d-Koszul algebra" to "nonquadratic Koszul algebra" (see [7], [10])), where $d \geq 2$ is an integer. In order to find periodic resolutions for the trivial extension algebras of path algebras of Dynkin quivers in bipartite orientation, the notion of *almost Koszul algebra* was introduced in 2002 (see [4]). In order to study the conditions such that the Ext-algebras of graded algebras are finitely generated, Green and Marcos introduced the notion of δ-*Koszul algebra* in 2005 (see [6]); Green and Snashall introduced the notion of (D, A, B)-*stacked monomial algebra* in 2006 (see [9]). In order to unify the notions of Koszul and d-Koszul algebras, the so-called *piecewise-Koszul algebra* was introduced in 2007 (see [13]); in order to generalize the notion of piecewise-Koszul algebra further, Zhao and the author of the present paper introduced (p, λ)-*Koszul algebra* (see [14]) in 2009. In order to breakthrough the "pure" restrict on the projective resolution, Cassidy and Shelton introduced the notion of $\mathcal{K}_2$-*algebra* in 2008 (see [5]); Si and Lu introduced the notion of *bi-Koszul algebra* in 2009 (see [12]) and so on.

This paper continues the work of [14]. More precisely, the following list the main contents and results of each section. In Section 2, first we recall some notations and definitions. Then we mainly discuss some properties of the category of (p, λ)-Koszul modules, denoted by $\mathcal{K}^p_\lambda(A)$. In particular, we have the following theorem.

Theorem 0.1 *Let A be a standard graded algebra and $\mathcal{K}^p_\lambda(A)$ the category of (p, λ)-Koszul modules.*

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1. *Let* $\xi : 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ *be an exact sequence of finitely generated graded modules. Then*
   (a) *If* K *and* N *are in* $\mathcal{K}^p_\lambda(A)$, *then* $M \in \mathcal{K}^p_\lambda(A)$,
   (b) *If* K *and* M *are in* $\mathcal{K}^p_\lambda(A)$, *then* $M/K \cong N \in \mathcal{K}^p_\lambda(A)$,
   (c) *If* M *and* N *are in* $\mathcal{K}^p_\lambda(A)$, *then* $K \in \mathcal{K}^p_\lambda(A)$ *if and only if* $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ *for all* $i \geq 0$, *if and only if the minimal Horseshoe Lemma is true with respect to* ξ.
2. *Let* $M \in \mathcal{K}^p_\lambda(A)$ *and* $|\lambda|$ *denote the smallest positive period of the periodic function* λ. *Then*
   (a) *All the* $(kp|\lambda|)^{\text{th}}$ *syzygies of* M, $\Omega^{kp|\lambda|}(M)[-\delta^p_\lambda(kp|\lambda|)] \in \mathcal{K}^p_\lambda(A)$;
   (b) *All the* $(kp|\lambda|-1)^{\text{th}}$ *syzygies of* JM, $\Omega^{kp|\lambda|-1}(JM)[-\delta^p_\lambda(kp|\lambda|)] \in \mathcal{K}^p_\lambda(A)$.

In Section 3, we give some new characterizations for (p,λ)-Koszul algebras and obtain this theorem:

Theorem 0.2 *Let* $A = \Bbbk\Gamma/I$ *be a standard graded algebra and*

$$\cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A_0 \longrightarrow 0$$

be a minimal graded projective resolution of the trivial A*-module* A_0. *Then the following statements are equivalent:*

1. A *is a* (p,λ)*-Koszul algebra;*
2. A *is a* (p,λ)*-Koszul module over* A^e;
3. *all the multiplications:* $\mu : \mathrm{Ext}^1_A(A_0,A_0) \otimes \mathrm{Ext}^{n-1}_A(A_0,A_0) \to \mathrm{Ext}^n_A(A_0,A_0)$ $(1 \leq n \leq p-1)$, $\mu : \mathrm{Ext}^p_A(A_0,A_0) \otimes \mathrm{Ext}^{n-p}_A(A_0,A_0) + \mathrm{Ext}^1_A(A_0,A_0) \otimes \mathrm{Ext}^{n-1}_A(A_0,A_0) \to \mathrm{Ext}^n_A(A_0,A_0)$ $(p+1 \leq n \leq 2p-1)$, $\cdots$, $\mu : \mathrm{Ext}^1_A(A_0,A_0) \otimes \mathrm{Ext}^{n-1}_A(A_0,A_0) + \mathrm{Ext}^p_A(A_0,A_0) \otimes \mathrm{Ext}^{n-p}_A(A_0,A_0) + \mathrm{Ext}^{2p}_A(A_0,A_0) \otimes \mathrm{Ext}^{n-2p}_A(A_0,A_0) + \cdots + \mathrm{Ext}^{|\lambda|p}_A(A_0,A_0) \otimes \mathrm{Ext}^{n-|\lambda|p}_A(A_0,A_0) \to \mathrm{Ext}^n_A(A_0,A_0)$ $(n \geq |\lambda|p+1)$ *are surjective, and* $\mathrm{Ext}^{kp}_A(A_0,A_0) = \mathrm{Ext}^{kp}_A(A_0,A_0)_{-\delta^p_\lambda(kp)}$, $k = 1, 2, \cdots, |\lambda|$;
4. *all the comultiplications:* $\triangle : \mathrm{Tor}^A_n(A_0,A_0) \to \mathrm{Tor}^A_1(A_0,A_0) \otimes \mathrm{Tor}^A_{n-1}(A_0,A_0)$ $(1 \leq n \leq p-1)$, $\triangle : \mathrm{Tor}^A_n(A_0,A_0) \to \mathrm{Tor}^A_1(A_0,A_0) \otimes \mathrm{Tor}^A_{n-1}(A_0,A_0) + \mathrm{Tor}^A_p(A_0,A_0) \otimes \mathrm{Tor}^A_{n-p}(A_0,A_0)$ $(p \leq n \leq 2p-1)$, $\cdots$, $\triangle : \mathrm{Tor}^A_n(A_0,A_0) \to \mathrm{Tor}^A_1(A_0,A_0) \otimes \mathrm{Tor}^A_{n-1}(A_0,A_0) + \mathrm{Tor}^A_p(A_0,A_0) \otimes \mathrm{Tor}^A_{n-p}(A_0,A_0) + \cdots + \mathrm{Tor}^A_{|\lambda|p}(A_0,A_0) \otimes \mathrm{Tor}^A_{n-|\lambda|p}(A_0,A_0)(n \geq |\lambda|p+1)$ *are injective, and* $\mathrm{Tor}^A_{kp}(A_0,A_0) = \mathrm{Tor}^A_{kp}(A_0,A_0)_{\delta^p_\lambda(kp)}$, $k = 1, 2, \cdots, |\lambda|$;
5. *if* A *is a standard graded algebra with pure resolution and* M *a* (p,λ)*-Koszul module, then the Ext module* $\bigoplus_{i\geq 0} \mathrm{Ext}^i_A(M,A_0)$ *is generated by* $\mathrm{Ext}^0_A(M,A_0)$ *as a graded* $\bigoplus_{i\geq 0} \mathrm{Ext}^i_A(A_0,A_0)$*-module if and only if* A *is a* (p,λ)*-Koszul algebra.*

In Section 4, we investigate H-Galois graded extension of (p,λ)-Koszul algebras and mainly prove this theorem:

Theorem 0.3 *Let H be a finite dimensional semisimple and cosemisimple Hopf algebra, $A = \bigoplus_{n\geq 0} A_n$ be a graded right H-module algebra such that A_i is finite dimensional for all $i \geq 0$, and let $B = A^{coH}$, the coinvariant subalgebra of A. Suppose that A/B is an H-Galois graded extension. Then B is a (p,λ)-Koszul algebra if and only if A is a (p,λ)-Koszul algebra.*

In 2008, Wang and Li gave some sufficient conditions for the Horseshoe Lemma to be true in the minimal case in [17]. In particular, the authors remark that "Though we have found some sufficient conditions for the minimal Horseshoe Lemma to be true, an interesting but difficult question is how to find some necessary conditions" ([17], Page 384). Theorem 2.8 is the main result of [17]:

- Let $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of nice modules with $JK = K \cap JM$. Then the minimal Horseshoe Lemma holds with respect to such an exact sequence.

In order to generalize and perfect the above theorem, we introduce the notions of quasi-(p,λ)-Koszul algebras and quasi-(p,λ)-Koszul modules in Section 5. Moreover, we mainly establish the relationships between quasi-(p,λ)-Koszul modules and minimal Horseshoe Lemma and prove

Theorem 0.4 *Let R be an augmented Noetherian semiperfect algebra and*

$$\xi : 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence of quasi-(p,λ)-Koszul modules. Then $JK = K \cap JM$ if and only if the minimal Horseshoe Lemma holds with respect to ξ.

1. Some basic properties of (p,λ)-Koszul modules

Throughout, $\mathbb{Z}$ denotes the set of integers, $\mathbb{N} = \{0, 1, 2, \cdots, n, \cdots\}$ the set of natural numbers and $\mathbb{N}^* = \{1, 2, \cdots, n, \cdots\}$ the set of positive integers and $\Bbbk$ will denote an arbitrary ground field; the standard graded algebras are a class of positively graded $\Bbbk$-algebras $A = \bigoplus_{i\geq 0} A_i$ satisfying the following properties: (i) $A_0 = \Bbbk \times \Bbbk \times \cdots \times \Bbbk$, a finite product of $\Bbbk$; (ii) A is generated in degrees 0 and 1; that is, $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$ and (iii) $\dim_{\Bbbk} A_i < \infty$ for all $i \geq 0$. It is easy to see that the graded Jacobson radical of such a graded algebra A is $\bigoplus_{i\geq 1} A_i$, which will be denoted by J.

Let $Gr(A)$ denote the category of graded A-modules, and $gr(A)$, its full subcategory of finitely generated modules. The morphisms in these categories are the A-module maps of degree zero. We denote $Gr_s(A)$ and $gr_s(A)$ the full subcategory of $Gr(A)$ and $gr(A)$ whose objects are generated in degree s respectively. An object in $Gr_s(A)$ or $gr_s(A)$ is called a graded *pure* A-module.

In order to recall the notions of (p,λ)-*Koszul algebra* and (p,λ)-*Koszul module*, let us introduce some set functions:

- Let $\lambda : \mathbb{N}^* \to \mathbb{N}^*$ be a periodic function such that (a) $\lambda(1) \geq 1$, and (b) λ is strictly increasing in the interval $[1, |\lambda|]$, where $|\lambda|$ denotes the smallest positive period of λ.

- Let $\delta_\lambda^p : \mathbb{N} \to \mathbb{N}$ be another set function such that (a) $\delta_\lambda^p(0) = 0$, $\delta_\lambda^p(p) = d$, where $d = \lambda(1) + p - 1$ and $p \geq 2$ are fixed integers; (b) $\delta_\lambda^p(pn+i) - \delta_\lambda^p(pn+i-1) = 1$ for all $1 \leq i \leq p-1$; and (c) $\delta_\lambda^p(pn) - \delta_\lambda^p(pn-1) = \lambda(n)$ for all $n \geq 1$.

Definition 1.1 *Let A be a graded algebra and $M \in gr(A)$. We call M a (p,λ)-Koszul module if it has a minimal graded projective resolution*

$$\mathcal{Q} : \cdots \longrightarrow Q_n \xrightarrow{d_n} \cdots \longrightarrow Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} M \longrightarrow 0$$

such that each Q_n is generated in degree $\delta_\lambda^p(n)$. Let $\mathcal{K}_\lambda^p(A)$ denote the category of (p,λ)-Koszul modules.

In particular, a graded algebra $A = \bigoplus_{i\geq 0} A_i$ is called a (p,λ)-Koszul algebra if the trivial A-module $A_0 \in \mathcal{K}_\lambda^p(A)$.

Now we will prove Theorem 0.1, which is done by several lemmas.

Lemma 1.2 *Let $\xi : 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence in $gr(A)$. Then the following statements are equivalent:*

1. *$K \in gr_s(A)$ provided that $M,\ N \in gr_s(A)$, where $s \in \mathbb{Z}$;*
2. *$JK = K \cap JM$;*
3. *We have the following commutative diagram with exact rows and columns*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(K) & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \longrightarrow & L_0 & \longrightarrow & Q_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Figure 1

such that $P_0 \longrightarrow K \longrightarrow 0$, $L_0 \longrightarrow M \longrightarrow 0$, and $Q_0 \longrightarrow N \longrightarrow 0$ are graded projective covers.

Proof (1) $\Rightarrow$ (2) It suffices to prove $JK \supseteq K \cap JM$ since $JK \subseteq K \cap JM$ is obvious. Let $K = \bigoplus_{i \geq s} K_i$ such that $K_{i+s} = A_i K_s$. Let $x \in K \cap JM$ be any homogeneous element of degree j. Then of course $j \geq s+1$. Thus $x \in K_{s+1} = A_1 K_s \subseteq JK$.

(2) $\Rightarrow$ (3) Obviously, we obtain the exact sequence

$$0 \longrightarrow K/JK \longrightarrow M/JM \longrightarrow N/JN \longrightarrow 0$$

since $JK = K \cap JM$. Note that for any finitely generated graded A-module M, $A \otimes_{A_0} M/JM \longrightarrow M \longrightarrow 0$ is a graded projective cover and the graded cover of a finitely generated graded module is unique up to isomorphisms.

Therefore, we can assume that $P_0 := A \otimes_{A_0} K/JK$, $L_0 := A \otimes_{A_0} M/JM$ and $Q_0 := A \otimes_{A_0} N/JN$. We have the exact sequence

$$0 \longrightarrow P_0 \longrightarrow L_0 \longrightarrow Q_0 \longrightarrow 0$$

since A_0 is semisimple. That is, we have the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \Omega^1(K) & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(N) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \longrightarrow & L_0 & \longrightarrow & Q_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array}$$

such that the columns, the middle and the bottom rows are exact. Now, by the "3×3" Lemma, we get the exact sequence $0 \longrightarrow \Omega^1(K) \longrightarrow \Omega^1(M) \longrightarrow \Omega^1(N) \longrightarrow 0$, which implies the desired diagram

(3) $\Rightarrow$ (1) Suppose that we have Figure 1, then we have $L_0 = P_0 \oplus Q_0$ since the middle row of Figure 1, P_0, L_0 and Q_0, are the graded projective covers of K, M and N, respectively. Note that if $P \longrightarrow K \longrightarrow 0$ is a graded projective cover, then K and P are generated in the same degrees. Thus, by the assumption, L_0 and Q_0 are generated in degree s, which implies that P_0 is generated in degree s. Thus $K \in gr_s(A)$, as desired. $\square$

Corollary 1.3 *Let* $\xi : 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ *be an exact sequence in* $gr(A)$. *Then* $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ *for all* $i \geq 0$ *if and only if for any given commutative diagram*

$$\begin{array}{ccccccccc}
 & & \mathcal{P}_* & & & & \mathcal{Q}_* & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0, \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0, & &
\end{array}$$

Figure 2

where $\mathcal{P}_*$ *and* $\mathcal{Q}_*$ *are minimal projective resolutions of* K *and* N, *respectively. Then we can complete Figure 2 into*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{P}_* & \longrightarrow & \mathcal{L}_* & \longrightarrow & \mathcal{Q}_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array}$$

where $\mathcal{L}_* \longrightarrow M \longrightarrow 0$ *is also a minimal projective resolution and for all* $n \geq 0$, $L_n \cong P_n \oplus Q_n$. *That is, the minimal Horseshoe Lemma holds with respect to such* ξ.

Proof By (2) $\Leftrightarrow$ (3) of Lemma 1.2 again and again, we can obtain a lot of commutative diagrams similar to Figure 1, then putting these diagrams together, we complete the proof. □

Corollary 1.4 *Let* $\xi : 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ *be an exact sequence in* $gr(A)$. *Then*

1. *If* K *and* N *are in* $\mathcal{K}^p_\lambda(A)$, *then* $M \in \mathcal{K}^p_\lambda(A)$,
2. *If* K *and* M *are in* $\mathcal{K}^p_\lambda(A)$, *then* $M/K \cong N \in \mathcal{K}^p_\lambda(A)$,
3. *If* M *and* N *are in* $\mathcal{K}^p_\lambda(A)$, *then* $K \in \mathcal{K}^p_\lambda(A)$ *if and only if* $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ *for all* $i \geq 0$, *if and only if the minimal Horseshoe Lemma is true with respect to* ξ.

Proof (1) By assumption, K, M and N are generated in the same single degree. By (1) $\Leftrightarrow$ (3) of Lemma 1.2, we get Figure 1, which implies that $\Omega^1(K)$, $\Omega^1(M)$ and $\Omega^1(N)$ are generated in the same single degree since K and N are in $\mathcal{K}^p_\lambda(A)$. Repeating the above argument, we have $\Omega^i(K)$, $\Omega^i(M)$ and $\Omega^i(N)$ are generated in the same single degree for all $i \geq 0$. Note that $K,\ N \in \mathcal{K}^p_\lambda(A)$, thus M has a minimal graded projective resolution

$$\mathcal{Q} : \cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

such that each Q_n is generated in degree $\delta^p_\lambda(n)$. That is, $M \in \mathcal{K}^p_\lambda(A)$.

(2) Note that for any exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $gr(A)$, it is trivial that $Z \in gr_s(A)$ provided $Y \in gr_s(A)$. Now the rest of the proof is similar to that of (1) and we omit it.

(3) It is immediate from Lemma 1.2 and Corollary 1.3. □

Proposition 1.5 *Let* A *be a* (p, λ)*-Koszul algebra,* $M \in \mathcal{K}^p_\lambda(A)$ *and* $|\lambda|$ *denote the smallest positive period of the periodic function* λ. *Then*

1. *All the* $(kp|\lambda|)^{\text{th}}$ *syzygies of* M, $\Omega^{kp|\lambda|}(M)[-\delta^p_\lambda(kp|\lambda|)] \in \mathcal{K}^p_\lambda(A)$;
2. *All the* $(kp|\lambda| - 1)^{\text{th}}$ *syzygies of* JM, $\Omega^{kp|\lambda|-1}(JM)[-\delta^p_\lambda(kp|\lambda|)] \in \mathcal{K}^p_\lambda(A)$.

Proof (1) Let $\mathcal{Q} : \cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$ be a minimal graded projective resolution. Then for all $n \geq 0$, Q_n is generated in degree $\delta^p_\lambda(n)$ since M is a (p, λ)-Koszul module. Therefore, $\Omega^{kp|\lambda|}(M)[-\delta^p_\lambda(kp|\lambda|)]$ possesses a minimal graded projective resolution

$$\cdots \longrightarrow Q_{kp|\lambda|+1}(M)[-\delta^p_\lambda(kp|\lambda|)] \longrightarrow Q_{kp|\lambda|}(M)[-\delta^p_\lambda(kp|\lambda|)] \longrightarrow \Omega^{kp|\lambda|}(M)[-\delta^p_\lambda(kp|\lambda|)] \longrightarrow 0.$$

For simplicity, setting $Q_{kp|\lambda|+i}(M)[-\delta_\lambda^p(kp|\lambda|)] := P_i$ for all $i \geq 0$. Then the above resolution becomes

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \Omega^{kp|\lambda|}(M)[-\delta_\lambda^p(kp|\lambda|)] \longrightarrow 0,$$

where each P_n is generated in degree $\delta_\lambda^p(kp|\lambda|+n)-\delta_\lambda^p(kp|\lambda|) = \delta_\lambda^p(n)$, which implies that $\Omega^{kp|\lambda|}(M)[-\delta_\lambda^p(kp|\lambda|)]$ $(k \in \mathbb{N})$ is a (p,λ)-Koszul module.

For (2), we have the exact sequence

$$0 \longrightarrow \Omega^1(M) \longrightarrow \Omega^1(M/JM) \longrightarrow JM \longrightarrow 0$$

such that each term is generated in degree $\delta_\lambda^p(1)$, since the natural exact sequence

$$0 \longrightarrow JM \longrightarrow M \longrightarrow M/JM \longrightarrow 0.$$

Thus, we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^2(M) & \longrightarrow & \Omega^2(M/JM) & \longrightarrow & \Omega^1(JM) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_1 & \longrightarrow & Q_1 \oplus L_0 & \longrightarrow & L_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(M/JM) & \longrightarrow & JM & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array}$$

where the vertical columns are projective covers.

Repeate the above procedures; we get the following exact sequences for all $k \geq 0$:

$$0 \longrightarrow \Omega^{kp|\lambda|}(M) \longrightarrow \Omega^{kp|\lambda|}(M/JM) \longrightarrow \Omega^{kp|\lambda|-1}(JM) \longrightarrow 0,$$

which implies the following exact sequences for all $k \geq 0$:

$$0 \longrightarrow \Omega^{kp|\lambda|}(M)[-\delta_\lambda^p(kp|\lambda|)] \longrightarrow \Omega^{kp|\lambda|}(M/JM)[-\delta_\lambda^p(kp|\lambda|)] \longrightarrow \Omega^{kp|\lambda|-1}(JM)[-\delta_\lambda^p(kp|\lambda|)] \longrightarrow 0.$$

By (1), we have $\Omega^{kp|\lambda|}(M)[-\delta_\lambda^p(kp|\lambda|)]$ and $\Omega^{kp|\lambda|}(M/JM)[-\delta_\lambda^p(kp|\lambda|)]$ are (p,λ)-Koszul modules since M and M/JM are (p,λ)-Koszul modules. Then $\Omega^{kp|\lambda|-1}(JM)[-\delta_\lambda^p(kp|\lambda|)]$ is a (p,λ)-Koszul module by Corollary 1.4. □

Now by Corollary 1.4 and Proposition 1.5, we have proved Theorem 0.1.

2. Some new characterizations of (p,λ)-Koszul algebras

In this section, we will give some new characterizations of (p,λ)-Koszul algebras.

We begin with the following well-known graded version of Gabriel's Theorem.

Lemma 2.1 *Let A be a standard graded algebra. Then there exists a finite quiver $\Gamma = (\Gamma_0, \Gamma_1)$ and a graded ideal I in $\Bbbk\Gamma$ with $I \subset \sum_{n\geq 2}(\Bbbk\Gamma)_n$ such that $A \cong \Bbbk\Gamma/I$ as graded algebras, where Γ_0 denotes the set of vertices of the quiver Γ and Γ_1 denotes the set of arrows of the quiver Γ.*

Lemma 2.2 *Let A be a standard graded algebra and $A^e := A \otimes_\Bbbk A^{op}$ its enveloping algebra. Let $\mathbf{r}$ be the graded Jacobson radical of A^e and $f : P \to Q$ be a homomorphism of finitely generated A^e-projective modules. Then $Imf \subseteq \mathbf{r}Q$ if and only if for each simple A-module S, we have $Im(f \otimes_A 1_S) \subseteq J(Q \otimes_A S)$.*

Proof $(\Leftarrow)$ For the sake of convenience, we may suppose that $Q = Av\otimes_\Bbbk wA$ is an indecomposable A^e-module, where $v, w \in \Gamma_0$ and we use the notations of Lemma 2.1. Assume f is an epimorphism, so

$$P \xrightarrow{\ f\ } Av \otimes_\Bbbk wA \longrightarrow 0$$

is a splittable epimorphism, which implies the exact sequence

$$P \otimes_A M \xrightarrow{f\otimes_A 1_M} Av \otimes_\Bbbk wA \otimes_A M \longrightarrow 0$$

of A-modules for any A-module M. In particular, if we choose $M = Aw/Jw := S$, a simple A-module, then we get the epimorphism

$$P \otimes_A S \xrightarrow{f\otimes_A 1_S} Av \otimes_\Bbbk wA \otimes_A S \cong Av \longrightarrow 0.$$

Now by the hypothesis $Im(f \otimes_A 1_S) \subseteq J(Q \otimes_A S)$, we have that $Imf \subseteq \mathbf{r}Q$.

$(\Rightarrow)$ Suppose that we have the condition $Imf \subseteq \mathbf{r}Q$. Similarly, we may assume that $Q = Av \otimes_\Bbbk wA$ is an indecomposable A^e-module. Note that for each simple A-module $S \neq Aw/Jw$, we have $Q \otimes_A S = 0$. Thus it suffices to prove the case of $S = Aw/Jw$. Consider the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\ f\ } & Q \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ P \otimes_A S & \xrightarrow{f\otimes_A 1_S} & Q \otimes_A S, \end{array}$$

where α and β are the splittable A-epimorphisms given by the split exact sequences in the category of finitely generated A-modules. More precisely, taking β for example, β is determined by the following split exact sequence

$$0 \longrightarrow Av \otimes_\Bbbk wJ \longrightarrow Av \otimes_\Bbbk wA \xrightarrow{\ \beta\ } Av \longrightarrow 0.$$

Not that $\beta^{-1}(v) = v \otimes w + Av \otimes wJ$, thus each element in the preimage of v is an A^e-generator for the module $Q = Av \otimes_\Bbbk wA$. If $f \otimes_A 1_S$ is an epimorphism, then βf is an epimorphism and $\beta^{-1}(v) \cap Imf \neq 0$, which implies that Imf contains an A^e-generator of the cyclic module Q, so f is an epimorphism. Therefore, we have $Im(f \otimes_A 1_S) \subseteq J(Q \otimes_A S)$, as desired. □

Proposition 2.3 *Let A be a standard graded algebra and A^e its enveloping algebra. Then A is a (p, λ)-Koszul algebra if and only if A is a (p, λ)-Koszul module over A^e.*

Proof If $P = Av \otimes_{\Bbbk} wA$ is an indecomposable A^e-projective module and M an A-module, then $P \otimes_A M = (Av)^{\dim wM}$ as an A-module since $Av \otimes_{\Bbbk} wA \otimes_A M \cong Av \otimes_{\Bbbk} wM$. In particular, if $M = S$ a simple A-module, then as A-modules we have $P \otimes_A S \cong Av$ if $wS \neq 0$ and $P \otimes_A S = 0$ otherwise. Let

$$\mathcal{P}_* : \cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

be a graded projective A^e-resolution of A. Then by Lemma 2.2, $\mathcal{P}_*$ is minimal if and only if $\mathcal{P}_* \otimes_A A_0$:

$$\cdots \longrightarrow P_n \otimes_A A_0 \longrightarrow \cdots \longrightarrow P_1 \otimes_A A_0 \longrightarrow P_0 \otimes_A A_0 \longrightarrow A \otimes_A A_0 \cong A_0 \longrightarrow 0$$

is a minimal graded projective resolution of A_0. Further, for all $i \geq 0$, P_i is generated in degree s as a graded A^e-module if and only if $P_i \otimes_A A_0$ is generated in degree s as a graded A-module. Now we finish the proof. □

Let A be a standard graded algebra. Then A_0, the trivial A-module, possesses a canonical graded projective resolution:

$$\cdots \longrightarrow Bar^n(A) \xrightarrow{\partial'_n} \cdots \longrightarrow Bar^1(A) \xrightarrow{\partial'_1} Bar^0(A) \xrightarrow{\partial'_0} A_0 \longrightarrow 0,$$

where for all $n \geq 0$, $Bar^n(A) := A \otimes_{A_0} J^{\otimes n}$ and the differential $\partial'_n : A \otimes_{A_0} J^{\otimes n} \longrightarrow A \otimes_{A_0} J^{\otimes n-1}$ is defined by

$$\partial'_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, \ (a_0 \in A, a_1, \cdots, a_n \in J).$$

Note that with $A_0 \otimes_A Bar^n(A) = A_0 \otimes_A A \otimes_{A_0} J^{\otimes n} \cong J^n$ for all $n \geq 0$, we get the complex

$$\cdots \longrightarrow J^{\otimes n} \xrightarrow{\partial_n} \cdots \longrightarrow J^{\otimes 2} \xrightarrow{\partial_2} J^{\otimes 1} \xrightarrow{\partial_1} J^0 \longrightarrow 0$$

with

$$\partial_n(a_1 \otimes a_2 \otimes \cdots \otimes a_n) := \sum_{i=1}^{n-1} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, \ (a_1, \cdots, a_n \in J).$$

Now it is trivial that

$$\mathrm{Tor}_n^A(A_0, A_0) = \ker \partial_n / Im \partial_{n+1}.$$

Proposition 2.4 *Using the above notations.* $T(A) := \bigoplus_{n \geq 0} \mathrm{Tor}_n^A(A_0, A_0)$ *is a bigraded coalgebra with the comultiplication* $\overline{\triangle} = \sum_{n,i} \overline{\triangle}_{n,i}$, *where* $\overline{\triangle}_{n,i}$ *is induced by* $\triangle_{n,i} : J^{\otimes n} \to J^{\otimes i} \otimes J^{\otimes n-i}$ *via* $\triangle_{n,i}(a_1 \otimes \cdots \otimes a_n) = (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n)$.

Proof It is easy to check that $\triangle = \sum_{n,i} \triangle_{n,i}$ provides a comultiplicative structure for the complex $J^{\otimes \bullet}$ and preserves kernels and images. Thus $(J^{\otimes \bullet}, \partial, \triangle)$ is a differential graded coalgebra and $T(A)$ a graded coalgebra. Note that now A is a standard graded algebra, which implies that $T(A)$ a bigraded coalgebra. □

The cobar complex is the cochain complex $\mathrm{Cob}^\bullet(A)$ defined by $\mathrm{Cob}^n(A) := \mathrm{Hom}_A(J^{\otimes n}, A_0)$ for all $n \geq 0$, where the differential $\partial^*_{n+1} : \mathrm{Cob}^n(A) \to \mathrm{Cob}^{n+1}(A)$ is the pullback of ∂. Clearly, for all $n \geq 0$, we have

$$\mathrm{Ext}_A^n(A_0, A_0) = \ker \partial^*_{n+1} / Im \partial^*_n.$$

Proposition 2.5 *Using the above notations.* $E(A) := \bigoplus_{n\geq 0}\mathrm{Ext}_A^n(A_0,A_0)$ *is a bigraded algebra with the multiplication* $\widetilde{\mu} = \sum_{i,n}\widetilde{\mu}_{i,n-i}$*, where* $\widetilde{\mu}_{i,n-i}$ *is induced by* $\mu_{i,n-i} : \mathrm{Cob}^i(A)\otimes \mathrm{Cob}^{n-i}(A) \to \mathrm{Cob}^n(A)$ *via* $\mu_{i,n-i}(f\otimes g)(a_1\otimes a_2) := f(a_1)\otimes g(a_2)$.

Proof It is easy to check that $\mu = \sum_{n,i}\mu_{i,n-i}$ provides a multiplicative structure for the complex $\mathrm{Cob}^{\bullet}(A)$ and preserves kernels and images. Thus $(\mathrm{Cob}^{\bullet}(A),\partial^*,\mu)$ is a differential graded algebra and $E(A)$ a graded algebra. Note that now A is a standard graded algebra, which implies that $E(A)$ a bigraded algebra. □

We usually call $T(A)$ the *Yoneda coalgebra* of A, and $E(A)$ the *Yoneda algebra* of A.

Proposition 2.6 *The map* $\mu_{n,i} : \mathrm{Cob}^{n-i}(A)\otimes\mathrm{Cob}^i(A)\to \mathrm{Cob}^n(A)$ *and* $\triangle_{n,i} : J^{\otimes n}\to J^{\otimes n-i}\otimes J^{\otimes i}$ *are dual to one another.*

Proof Let $f_1\otimes\cdots\otimes f_i\in \mathrm{Cob}^i(A)$, $g_1\otimes\cdots\otimes g_{n-i}\in\mathrm{Cob}^{n-i}(A)$ and $a_1\otimes\cdots\otimes a_n\in J^{\otimes n}$. Then

$$\begin{aligned}
&\triangle^*((f_1\otimes\cdots\otimes f_i)\otimes(g_1\otimes\cdots\otimes g_{n-i}))(a_1\otimes\cdots\otimes a_n)\\
&=((f_1\otimes\cdots\otimes f_i)\otimes(g_1\otimes\cdots\otimes g_{n-i}))\triangle(a_1\otimes\cdots\otimes a_n)\\
&=(f_1\otimes\cdots\otimes f_i)(a_1\otimes\cdots\otimes a_i)(g_1\otimes\cdots\otimes g_{n-i})(a_{i+1}\otimes\cdots\otimes a_n)\\
&=\mu((f_1\otimes\cdots\otimes f_i)\otimes(g_1\otimes\cdots\otimes g_{n-i}))(a_1\otimes\cdots\otimes a_n).
\end{aligned}$$

Therefore, we are done. □

Lemma 2.7 *[14] Let* A *be a graded algebra. Then* A *is a* (p,λ)*-Koszul algebra if and only if* $E(A)$ *is minimally generated in the ext-degrees* $1,\ p,\ 2p,\ \cdots,\ |\lambda|p$*, and* $\mathrm{Ext}_A^i(A_0,A_0) = \mathrm{Ext}_A^i(A_0,A_0)_{-\delta_\lambda^p(i)}$ *for all* $i = p,\ 2p,\ \cdots,\ |\lambda|p$.

Proposition 2.8 *Let* A *be a standard graded algebra. Then the following statements are equivalent:*

1. A *is a* (p,λ)*-Koszul algebra;*

2. *all the multiplications:* $\mu : \mathrm{Ext}_A^1(A_0,A_0)\otimes\mathrm{Ext}_A^{n-1}(A_0,A_0)\to\mathrm{Ext}_A^n(A_0,A_0)$ $(1\leq n\leq p-1)$, $\mu : \mathrm{Ext}_A^p(A_0,A_0)\otimes\mathrm{Ext}_A^{n-p}(A_0,A_0)+\mathrm{Ext}_A^1(A_0,A_0)\otimes\mathrm{Ext}_A^{n-1}(A_0,A_0)\to\mathrm{Ext}_A^n(A_0,A_0)$ $(p+1\leq n\leq 2p-1)$, $\cdots$, $\mu : \mathrm{Ext}_A^1(A_0,A_0)\otimes\mathrm{Ext}_A^{n-1}(A_0,A_0)+\mathrm{Ext}_A^p(A_0,A_0)\otimes\mathrm{Ext}_A^{n-p}(A_0,A_0)+\mathrm{Ext}_A^{2p}(A_0,A_0)\otimes\mathrm{Ext}_A^{n-2p}(A_0,A_0)+\cdots+\mathrm{Ext}_A^{|\lambda|p}(A_0,A_0)\otimes\mathrm{Ext}_A^{n-|\lambda|p}(A_0,A_0)\to\mathrm{Ext}_A^n(A_0,A_0)$ $(n\geq|\lambda|p+1)$ *are surjective, and* $\mathrm{Ext}_A^{kp}(A_0,A_0)=\mathrm{Ext}_A^{kp}(A_0,A_0)_{-\delta_\lambda^p(kp)}$, $k=1,2,\cdots,|\lambda|$*;*

3. *all the comultiplications:* $\triangle : \mathrm{Tor}_n^A(A_0,A_0)\to\mathrm{Tor}_1^A(A_0,A_0)\otimes\mathrm{Tor}_{n-1}^A(A_0,A_0)$ $(1\leq n\leq p-1)$, $\triangle : \mathrm{Tor}_n^A(A_0,A_0)\to\mathrm{Tor}_1^A(A_0,A_0)\otimes\mathrm{Tor}_{n-1}^A(A_0,A_0)+\mathrm{Tor}_p^A(A_0,A_0)\otimes\mathrm{Tor}_{n-p}^A(A_0,A_0)$ $(p\leq n\leq 2p-1)$, $\cdots$, $\triangle : \mathrm{Tor}_n^A(A_0,A_0)\to\mathrm{Tor}_1^A(A_0,A_0)\otimes\mathrm{Tor}_{n-1}^A(A_0,A_0)+\mathrm{Tor}_p^A(A_0,A_0)\otimes\mathrm{Tor}_{n-p}^A(A_0,A_0)+\cdots+\mathrm{Tor}_{|\lambda|p}^A(A_0,A_0)\otimes\mathrm{Tor}_{n-|\lambda|p}^A(A_0,A_0)(n\geq|\lambda|p+1)$ *are injective, and* $\mathrm{Tor}_{kp}^A(A_0,A_0)=\mathrm{Tor}_{kp}^A(A_0,A_0)_{\delta_\lambda^p(kp)}$, $k=1,2,\cdots,|\lambda|$.

Proof By Lemma 2.7, we have that A is a (p,λ)-Koszul algebra if and only if $E(A)$ is minimally generated in the ext-degrees $1,\ p,\ 2p,\ \cdots,\ |\lambda|p$, and $\mathrm{Ext}^i_A(A_0,A_0)=\mathrm{Ext}^i_A(A_0,A_0)_{-\delta^p_\lambda(i)}$ for all $i=p,\ 2p,\ \cdots,\ |\lambda|p$. Therefore, (1) $\Leftrightarrow$ (2) is immediate by induction on n and Proposition 2.6. By Proposition 2.6, we have μ and $\triangle$ are dual to each other, which establishes the equivalence of conditions (2) and (3). $\square$

Proposition 2.9 *Let A be a standard graded algebra with a pure resolution and M a (p,λ)-Koszul module. Then the Ext module $\bigoplus_{i\geq 0}\mathrm{Ext}^i_A(M,A_0)$ is generated by $\mathrm{Ext}^0_A(M,A_0)$ as a graded $E(A)$-module if and only if A is a (p,λ)-Koszul algebra.*

Proof Let $\mathcal{P}_*$ and $\mathcal{Q}_*$ be the minimal graded projective resolutions of A_0 and M, respectively. By hypothesis, for all $n\geq 0$, Q_n is generated in degree $\delta^p_\lambda(n)$.

($\Rightarrow$) By hypothesis, we have $\mathrm{Ext}^n_A(M,A_0)=\mathrm{Ext}^n_A(A_0,A_0)\cdot\mathrm{Ext}^0_A(M,A_0)$ for all $n\geq 1$. Note that A is a positively graded algebra with a pure resolution, which implies that $\mathrm{Ext}^n_A(A_0,A_0)=\mathrm{Ext}^n_A(A_0,A_0)_{-s}$ for some natural number s. Now observing that $\mathrm{Ext}^n_A(M,A_0)=\mathrm{Ext}^n_A(M,A_0)_{-\delta^p_\lambda(n)}$ since M is a (p,λ)-Koszul module. Thus $\mathrm{Ext}^n_A(A_0,A_0)=\mathrm{Ext}^n_A(A_0,A_0)_{-\delta^p_\lambda(n)}$ for all $n\geq 0$, which implies that A is a (p,λ)-Koszul algebra.

($\Leftarrow$) Suppose that A is a (p,λ)-Koszul algebra. Then as a trivial A-module, A_0 admits a minimal graded projective resolution

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A_0 \longrightarrow 0$$

such that each projective module P_n is generated in degree $\delta^d_p(n)$ for all $n\geq 0$. Note that M is a (p,λ)-Koszul module with respect to δ^p_λ. Thus M has a minimal graded projective resolution

$$\cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

such that each projective module Q_n is generated in degree $\delta^d_p(n)$ for all $n\geq 0$. Then by [4, Proposition 3.5], we have $\mathrm{Ext}^i_A(M,A_0)=\mathrm{Ext}^i_A(A_0,A_0)\cdot\mathrm{Ext}^0_A(M,A_0)$ for all $i\geq 0$. That is, $\bigoplus_{i\geq 0}\mathrm{Ext}^i_A(M,A_0)$ is generated by $\mathrm{Ext}^0_A(M,A_0)$. $\square$

Now Theorem 0.2 is immediate from Propositions 2.3, 2.8 and 2.9.

3. H-Galois graded extension of (p,λ)-Koszul algebras

In this section, we will investigate the H-Galois graded extension of (p,λ)-Koszul algebras and Theorem 0.3 is our main result.

Lemma 3.1 *([14]) Let A be a positively graded algebra and $\mathrm{Ext}^*_A(A_0,A_0)$ be its Yoneda algebra. Then A is a (p,λ)-Koszul algebra if and only if $\mathrm{Ext}^i_A(A_0,A_0)=\mathrm{Ext}^i_A(A_0,A_0)_{-\delta^p_\lambda(i)}$ for all $i\geq 0$.*

Lemma 3.2 *([11]) Let H be a finite dimensional semisimple and cosemisimple Hopf algebra and A/B be an H-Galois graded extension. If $A=\bigoplus_{i\geq 0}A_i$ is a positively graded algebra, then A_0/B_0 is an H-Galois extension.*

Lemma 3.3 *([11]) Let H be a finite dimensional semisimple and cosemisimple Hopf algebra, $A = \bigoplus_{n\geq 0} A_n$ be a graded right H-module algebra and $B = A^{coH}$, the coinvariant subalgebra of A. Suppose that A/B is an H-Galois graded extension. Then we have an isomorphism of bigraded algebras*

$$\mathrm{Ext}_B^*(A_0, A_0) \cong \mathrm{Ext}_A^*(A_0, A_0)\#H,$$

where the bigradeding of $\mathrm{Ext}_A^*(A_0, A_0)\#H$ is induced from that of $\mathrm{Ext}_A^*(A_0, A_0)$.

Now we are ready to prove Theorem 0.3.

Proof By the assumption, B_0 is a finite dimensional semisimple algebra. By Lemma 3.2, A_0/B_0 is an H-Galois extension since A/B is an H-Galois graded extension. Now note that $A_0\#H$ and B_0, A_0 and $(A_0\#H)\#H^*$ are both Morita equivalent, and H is a finite dimensional semisimple and cosemisimple Hopf algebra, we have that B_0 is semisimple if and only if A_0 is semisimple. Further, as a right B_0-module, $A_0 = B_0 \oplus S$ for some finite dimensional B_0-module S.

($\Rightarrow$) By assumption, B is a (p,λ)-Koszul algebra, by Lemma 3.1, which is equivalent to that $\mathrm{Ext}_B^i(B_0, B_0) = \mathrm{Ext}_B^i(B_0, B_0)_{-\delta_\lambda^p(i)}$ for all $i \geq 0$. Note that S is a direct summand of a finite sum of B_0, which implies that $\mathrm{Ext}_B^i(B_0, S) = \mathrm{Ext}_B^i(B_0, S)_{-\delta_\lambda^p(i)}$, $\mathrm{Ext}_B^i(S, B_0) = \mathrm{Ext}_B^i(S, B_0)_{-\delta_\lambda^p(i)}$ and $\mathrm{Ext}_B^i(S, S) = \mathrm{Ext}_B^i(S, S)_{-\delta_\lambda^p(i)}$ for all $i \geq 0$. Also observe that we have the isomorphism

$$\mathrm{Ext}_B^i(A_0, A_0) = \mathrm{Ext}_B^i(B_0, B_0) \oplus \mathrm{Ext}_B^i(B_0, S) \oplus \mathrm{Ext}_B^i(S, B_0) \oplus \mathrm{Ext}_B^i(S, S)$$

for all $i \geq 0$, which implies that $\mathrm{Ext}_B^i(A_0, A_0) = \mathrm{Ext}_B^i(A_0, A_0)_{-\delta_\lambda^p(i)}$ for all $i \geq 0$. By Lemma 3.3, we have

$$\mathrm{Ext}_A^i(A_0, A_0)\#H = (\mathrm{Ext}_A^i(A_0, A_0)\#H)_{-\delta_\lambda^p(i)}$$

for all $i \geq 0$. By the definition of the bigrading of $\mathrm{Ext}_A^i(A_0, A_0)\#H$, we obtain that $\mathrm{Ext}_A^i(A_0, A_0) = \mathrm{Ext}_A^i(A_0, A_0)_{-\delta_\lambda^p(i)}$ for all $i \geq 0$. By Lemma 3.1, we get that A is a (p,λ)-Koszul algebra.

($\Leftarrow$) Suppose that A is a (p,λ)-Koszul algebra, by Lemma 3.1, which is equivalent to

$$\mathrm{Ext}_A^i(A_0, A_0) = \mathrm{Ext}_A^i(A_0, A_0)_{-\delta_\lambda^p(i)}$$

for all $i \geq 0$. By Lemma 3.3, we have $\mathrm{Ext}_B^i(A_0, A_0) = \mathrm{Ext}_B^i(A_0, A_0)_{-\delta_\lambda^p(i)}$ for all $i \geq 0$. Note that $A_0 = B_0 \oplus S$ and S is a direct summand of a finite sum of B_0, which imply that

$$\mathrm{Ext}_B^i(A_0, A_0) = \mathrm{Ext}_B^i(B_0, B_0) \oplus \mathrm{Ext}_B^i(B_0, S) \oplus \mathrm{Ext}_B^i(S, B_0) \oplus \mathrm{Ext}_B^i(S, S)$$

for all $i \geq 0$, which of course implies that $\mathrm{Ext}_B^i(B_0, B_0) = \mathrm{Ext}_B^i(B_0, B_0)_{-\delta_\lambda^p(i)}$ for all $i \geq 0$. By Lemma 3.1, we get that B is a (p,λ)-Koszul algebra. □

Remark 3.1 *Example 2.14 of [11] can explain the above theorem clearly since Koszul algebras are a special class of (p,λ)-Koszul algebras in the sense of $p = d \geq 2$ and $\lambda(n) = 1$ for all $n \in \mathbb{N}$.*

4. Quasi-(p,λ)-Koszul algebras and modules

The main aim of this section is to give the definitions of quasi-(p,λ)-Koszul algebras and modules and to give an application of quasi-(p,λ)-Koszul modules: Another necessary and sufficient condition for minimal Horseshoe Lemma to be true is given in the category of quasi-(p,λ)-Koszul modules.

The definitions of quasi-(p,λ)-Koszul algebras and modules are motivated by the following result:

Lemma 4.1 *Let $A=\Bbbk\Gamma/I$ be a standard graded algebra and*

$$\cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A_0 \longrightarrow 0$$

a minimal graded projective resolution of the trivial A-module A_0. Then the following statements are equivalent:

1. *A is a (p,λ)-Koszul algebra;*
2. *$\ker d_n \subseteq J^{\delta^p_\lambda(n+1)-\delta^p_\lambda(n)}P_n$ and $J\ker f_n=\ker f_n\cap J^{\delta^p_\lambda(n+1)-\delta^p_\lambda(n)+1}P_n$ for all $n\geq 0$;*
3. *for any fixed $n\geq 1$ and $1\leq i\leq n$, $P_i=\bigoplus_{l\geq 1}Ae_{i_l}[-\delta^p_\lambda(i)]$, the component of $d_i(e_{i_l})$ in some Ae_{i-1_m} is in $A_{\delta^p_\lambda(i)-\delta^p_\lambda(i-1)}$, $\ker d_n \subseteq J^{\delta^p_\lambda(n+1)-\delta^p_\lambda(n)}P_n$ and $J\ker f_n=\ker f_n\cap J^{\delta^p_\lambda(n+1)-\delta^p_\lambda(n)+1}P_n$.*

Proof It is similar to the proof of Proposition 3.1 of [8] and we omit the details here. □

Similarly, we can get the corollary.

Corollary 4.2 *Let A be a standard graded algebra and $M\in gr_0(A)$. Suppose that*

$$\cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

is a minimal graded projective resolution of M. Then M is a (p,λ)-Koszul module if and only if for all $n\geq 0$, we have $\ker d_n \subseteq J^{\delta^p_\lambda(n+1)-\delta^p_\lambda(n)}P_n$ and $J\ker f_n=\ker f_n\cap J^{\delta^p_\lambda(n+1)-\delta^p_\lambda(n)+1}P_n$.

Now we can give the definitions of quasi-(p,λ)-Koszul algebras and quasi-(p,λ)-Koszul modules.

Definition 4.3 *Let R be a Noetherian semiperfect algebra with Jacobson radical J and M be a finitely generated R-module. Let*

$$\cdots \longrightarrow P_n \xrightarrow{f_n} \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

be a minimal projective resolution of M. Then M will be called a quasi-(p,λ)-Koszul module if we have the following two conditions:

1. *For $i\equiv pn+j \pmod{p|\lambda|}$, $(n\in[0,\ |\lambda|-1],\ j=0,\ 1,\ \cdots,\ p-2)$, we have $\ker f_i\subseteq JP_i$ and $J\ker f_i=J^2P_i\cap\ker f_i$;*
2. *For $i\equiv pn-1 \pmod{p|\lambda|}$, $(n\in[1,\ |\lambda|])$, we have $\ker f_i\subseteq J^{\lambda(n)}P_i$ and $J\ker f_i=J^{\lambda(n)+1}P_i\cap\ker f_i$.*

In particular, we call R a quasi-(p,λ)-Koszul algebra if R/J is a quasi-(p,λ)-Koszul module.

It is easy to see that quasi-Koszul algebras and modules (see [8]) are special quasi-(p, λ)-Koszul algebras and modules.

The following is the main result of this section.

Theorem 4.4 *Let R be an augmented Noetherian semiperfect algebra and*

$$\xi : 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence of quasi-(p, λ)-Koszul modules. Then $JK = K \cap JM$ if and only if the minimal Horseshoe Lemma is true with respect to ξ.

Proof $(\Rightarrow)$ Suppose that $\xi : 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ is an exact sequence of quasi-(p, λ)-Koszul modules such that $JK = K \cap JM$. Then by Lemma 1.2, we have the diagram Figure 1, which easily implies the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(K) & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & JP_0 & \longrightarrow & JL_0 & \longrightarrow & JQ_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & JK & \longrightarrow & JM & \longrightarrow & JN & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Figure 3

Now apply the additive right exact functor $R/J \otimes_R -$ to Figure 3, we get the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & R/J \otimes_R \Omega^1(K) & \xrightarrow{\varepsilon} & R/J \otimes_R \Omega^1(M) & \longrightarrow & R/J \otimes_R \Omega^1(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R/J \otimes_R JP_0 & \longrightarrow & R/J \otimes_R JL_0 & \longrightarrow & R/J \otimes_R JQ_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & R/J \otimes_R JK & \longrightarrow & R/J \otimes_R JM & \longrightarrow & R/J \otimes_R JN & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

which implies that ϵ is a monomorphism and hence $J\Omega^1(K) = \Omega^1(K) \cap J\Omega^1(M)$. Now by the same procedures, we can get $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ for all $i = 2, 3, \cdots, p-1$. For the case of p, replace Figure 3 by

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^p(K) & \longrightarrow & \Omega^p(M) & \longrightarrow & \Omega^p(N) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J^{\lambda(1)}P_{p-1} & \longrightarrow & J^{\lambda(1)}L_{p-1} & \longrightarrow & J^{\lambda(1)}Q_{p-1} & \longrightarrow & 0, \end{array}$$

Figure 4

then apply the additive right exact functor $R/J\otimes_R -$ to Figure 4, similarly, we get $J\Omega^p(K) = \Omega^p(K) \cap J\Omega^p(M)$. Now repeating the above argument, we have $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ for all $i \geq 0$. Now by Corollary 1.3, we finish the proof of the necessity.

($\Leftarrow$) Suppose that the minimal Horseshoe Lemma holds with respect to ξ, then in particular we have Figure 1. Now by Lemma 1.2, we have $JK = K \cap JM$, which completes the proof of the sufficiency. $\square$

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