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Some properties on the Baer-invariant of a pair of groups and $\mathcal{V}_G$-marginal series

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Abstract: The aim of this paper is to present some properties of the Baer-invariant of a pair of groups with respect to a given variety of groups $\mathcal{V}$. We derive some equalities and inequalities of the Baer-invariant of a pair of finite groups, as long as $\mathcal{V}$ is considered to be a Schur-Baer variety. Moreover, we present a relative version of the concept of lower marginal series and give some isomorphisms among $\mathcal{V}_G$-marginal factor groups. Also, we conclude a generalized version of the Stallings' theorem.

Key words: Baer-invariant, pair of groups, Schur-Baer variety, $\mathcal{V}_G$-nilpotent

1. Introduction and preliminaries

Let F_∞ be the free group freely generated by the countable set $X = \{x_1, x_2, ...\}$, and V be a subset of F_∞. Let $\mathcal{V}$ be the variety of groups defined by the set of laws V. We assume that the reader is familiar with the notions of the *verbal subgroup*, $V(G)$, and the *marginal subgroup*, $V^*(G)$, associated with the variety of groups $\mathcal{V}$ and a given group G (see [14] for more information on varieties of groups). Variety $\mathcal{V}$ is called a *Schur-Baer* variety if for any group G in which the marginal factor group $G/V^*(G)$ is finite, then the verbal subgroup $V(G)$ is also finite. Schur [17] proved that the variety of abelian groups is a Schur-Baer variety and Baer [2] showed that a variety defined by outer commutator words carries this property.

Let G be any group with a normal subgroup N, then we define $[NV^*G]$ to be the subgroup of G generated by the following set:

$$\{\nu(g_1, g_2, ..., g_i n, ..., g_r)\nu(g_1, g_2, ..., g_r)^{-1} \mid 1 \le i \le r,\ \nu \in V,\ g_1, ..., g_r \in G,\ n \in N\}.$$

It is easily checked that $[NV^*G]$ is the smallest normal subgroup T of G contained in N, such that N/T is contained in $V^*(G/T)$.

The following lemma gives the basic properties of the verbal and marginal subgroups of a group G with respect to the variety of groups $\mathcal{V}$ which is useful in our investigation, so you may see [7].

Lemma 1.1 *Let $\mathcal{V}$ be a variety of groups defined by a set of laws V and N be a normal subgroup of a given group G. Then*

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(i) $G \in \mathcal{V} \iff V(G) = 1 \iff V^*(G) = G$;
(ii) $V(G/N) = V(G)N/N$ *and* $V^*(G/N) \supseteq V^*(G)N/N$*;*
(iii) $N \subseteq V^*(G) \iff [NV^*G\,] = 1$;
(iv) $V(N) \subseteq [NV^*G\,] \subseteq N \cap V(G)$. *In particular,* $V(G) = [GV^*G\,]$*;*
(v) $V(V^*(G)) = 1$ *and* $V^*(G/V(G)) = G/V(G)$.

Let $1 \to R \to F \to G \to 1$ be a free presentation of the group G and N be a normal subgroup of G such that $N \cong S/R$, for a normal subgroup S of the free group F. Then we define the *Baer-invariant* of a pair of groups (G, N) with respect to the variety $\mathcal{V}$ denoted by $\mathcal{V}M(G, N)$ to be

$$\frac{R \cap [SV^*F]}{[RV^*F]}.$$

One may check that $\mathcal{V}M(G, N)$ is an abelian group and independent of the choice of the free presentation of G (see [11, 13]). If $N = G$, then the Baer-invariant of the pair (G, G) will be $(R \cap V(F))/[RV^*F]$ which is the usual Baer-invariant of the group G (denoted by $\mathcal{V}M(G)$). The first modern treatment about Baer-invariants is conducted by Fröhlich [4], who considered associative algebras, and named the invariants after Baer′s group-theoretical papers [1]. Furtado-Coelho and Lue [6, 10] worked in the context of Higgins' varieties of Ω-groups [3]. In particular, if $\mathcal{V}$ is the variety of abelian groups, then Baer-invariant of the group G will be $(R \cap [F, F])/[R, F]$ which by Schur [18] is isomorphic to the *Schur multiplier* of G.

It is interesting to know the connection between the Baer-invariant of a pair of finite groups (G, N) and its factor groups with respect to the Schur-Baer variety $\mathcal{V}$. Jones [9] gave some inequalities for the Schur multiplier of a finite group G and its factor group. Moghaddam et al. [13] generalized these inequalities to a pair of finite groups. In the next section, we give generalized version of these inequalities for the Baer-invariant of a pair of groups and its factor groups (Theorem 2.3). We also give some necessary and sufficient conditions for establishing connection between the orders of the Baer-invariants of pair of finite groups (Theorem 2.4). In the final section, we show that under some circumstances there are some isomorphisms among $\mathcal{V}_G$-marginal factor groups (Theorem 3.3). Also, we extend the works of Stallings (Theorem 3.5).

2. Some inequalities

In the following lemma we present some exact sequences for the Baer-invariant of a pair of groups and its factor groups.

Lemma 2.1 *Let G be a group with a free presentation $1 \to R \to F \to G \to 1^\circ$, also S and T are normal subgroups of the free group F such that $T \subseteq S$, $S/R \cong N$ and $T/R \cong K$. Then the following sequences are exact:*

(i) $$1 \to \frac{R \cap [TV^*F]}{[RV^*F]} \to \mathcal{V}M(G, N) \xrightarrow{\alpha} \mathcal{V}M(G/K, N/K) \xrightarrow{\beta} \frac{K \cap [NV^*G]}{[KV^*G]} \to 1;$$

(ii) *If K is contained in $V^*(G)$, then*

$$1 \to \frac{R \cap [SV^*F]}{[TV^*F] \cap [SV^*F]} \to \mathcal{V}M(G/K, N/K) \to K \xrightarrow{\gamma} \frac{N}{[NV^*G]} \xrightarrow{\theta} \frac{N}{[NV^*G]K} \to 1.$$

Proof By the definition of the Baer-invariant of the pair of groups, we can conclude:

$$\mathcal{V}M(G,K)=\frac{R\cap[TV^*F]}{[RV^*F]} \qquad\qquad \mathcal{V}M(G,N)=\frac{R\cap[SV^*F]}{[RV^*F]},$$

$$\frac{K\cap[NV^*G]}{[KV^*G]}=\frac{(T\cap[SV^*F])R}{[TV^*F]R}.$$

One can easily check that the sequence (i) is exact.
(ii) Using the assumption and Lemma 1.1, then we have $[TV^*F]\subseteq R$. One can easily check that the following sequence is exact:

$$1\to\frac{R\cap[SV^*F]}{[TV^*F]\cap[SV^*F]}\to\frac{T\cap[SV^*F]}{[TV^*F]}\to T/R\to\frac{S}{[SV^*F]R}\to\frac{S}{[SV^*F]T}\to 1.$$

□

Now we introduce a relative version of the concept of *lower marginal series.*

Definition 2.2 *Let N be a normal subgroup of a group G. Then we define a series of normal subgroups of N as follows:*

$$N=V_0(N,G)\supseteq V_1(N,G)\supseteq V_2(N,G)\supseteq\cdots\supseteq V_n(N,G)\supseteq\cdots,$$

where $V_i(N,G)=[V_{i-1}(N,G)V^*G]$ for all $n\geq 1$. We call such a series the lower $\mathcal{V}_G$-marginal series of N in G. One may also define the upper $\mathcal{V}_G$-marginal series as in [12].
We say that the normal subgroup N of G is $\mathcal{V}_G$-nilpotent if it has a finite lower $\mathcal{V}_G$-marginal series. The shortest length of such series is called the class of $\mathcal{V}_G$ -nilpotency of N in G.

If $N=G$, then this is called *lower $\mathcal{V}$-marginal series* of G. The group G is said to be *$\mathcal{V}$-nilpotent* iff $V_n(G)=1$, for some positive integer n [5].

In 2002, Moghaddam et al. [12] proved that for finite group G, $\mathcal{V}M(G)$ and hence $\mathcal{V}M(G,N)$ are finite when $\mathcal{V}$ is a Schur-Baer variety. Therefore, throughout the rest of this section we always assume that $\mathcal{V}$ is a variety of groups which enjoys the Schur-Baer property. By the Definition 2.2 and using Lemma 2.1(i) we have the following theorem.

Theorem 2.3 *Let G be a finite group with a free presentation $1\to R\to F\to G\to 1$ and S be a normal subgroup of the free group F such that $S/R\cong N$. If N is a subgroup $\mathcal{V}_G$-nilpotent of G of class $c\geq 2$, then*

(i) $|\,V_{c-1}(N,G)\,||\,\mathcal{V}M(G,N)\,|=\big|\mathcal{V}M(\frac{G}{V_{c-1}(N,G)},\frac{N}{V_{c-1}(N,G)})\big|\Big|\frac{[V_{c-1}(S,F)RV^*F]}{[RV^*F]}\Big|$;

(ii) $d(\mathcal{V}M(G,N))\leq d\Big(\mathcal{V}M(\frac{G}{V_{c-1}(N,G)},\frac{N}{V_{c-1}(N,G)})\Big)+d\Big(\frac{[V_{c-1}(S,F)RV^*F]}{[RV^*F]}\Big)$;

(iii) $e(\mathcal{V}M(G,N))$ divides $e\Big(\mathcal{V}M(\frac{G}{V_{c-1}(N,G)},\frac{N}{V_{c-1}(N,G)})\Big)e\Big(\frac{[V_{c-1}(S,F)RV^*F]}{[RV^*F]}\Big)$.

where e(X) and d(X) are the exponent and the minimal number of generators of a group X, respectively.

Proof By Lemma 2.1(i) we have,

$$|\mathcal{V}M(G,N)| = |L|\Big|\frac{R\cap[V_{c-1}(S,F)RV^*F]}{[RV^*F]}\Big| \quad \textit{and} \quad \frac{\mathcal{V}M(\frac{G}{K},\frac{N}{K})}{L} \cong K,$$

where L is $\mathrm{Im}(\alpha)$ in Lemma 2.1(i) and $K = V_{c-1}(N,G)$. Hence:

$$|K||\mathcal{V}M(G,N)| = |\mathcal{V}M(G/K,N/K)|\Big|\frac{R\cap[V_{c-1}(S,F)RV^*F]}{[RV^*F]}\Big|.$$

But $[KV^*G] = [V_{c-1}(N,G)V^*G] = V_c(N,G) = 1$, then $[V_{c-1}(S,F)RV^*F] \subseteq R$. This implies part (i). We can prove (ii) and (iii) in the same way. □

Let H be the marginal factor group of G and $L = N/V^*(G)$. Finally, some necessary and sufficient conditions for establishing connection between the orders of the Baer-invariants of pair of finite groups (G,N) and (H,L) have been highlighted.

Theorem 2.4 *Let G be a finite group with a normal subgroup N such that $V^*(G)\subseteq N$. Let $H = G/V^*(G)$ be the marginal factor group of G and $L = N/V^*(G)$. Then*

(i) $|[NV^*G]| \le |\mathcal{V}M(H,L)||[LV^*H]| \le |\mathcal{V}M(G,N)||[NV^*G]|$;

(ii) $|[NV^*G]| = |\mathcal{V}M(H,L)||[LV^*H]| \iff \mathcal{V}M(H,L) \cong V^*(G)\cap[NV^*G]$;

(iii) $|\mathcal{V}M(H,L)||[LV^*H]| = |\mathcal{V}M(G,N)||[NV^*G]| \iff \dfrac{\mathcal{V}M(H,L)}{\mathcal{V}M(G,N)} \cong V^*(G)\cap[NV^*G]$.

Proof (i) By considering $K = V^*(G)$ and using the Lemma 2.1(i), we have

$$|\mathcal{V}M(H,L)| = |V^*(G)\cap[NV^*G]||\ker\beta|.$$

But $[LV^*H] = \dfrac{[NV^*G]V^*(G)}{V^*(G)} \cong \dfrac{[NV^*G]}{V^*(G)\cap[NV^*G]}$, so $|V^*(G)\cap[NV^*G]| = \dfrac{|[NV^*G]|}{|[LV^*H]|}$.

Hence:

$|[NV^*G]||\ker\beta| = |\mathcal{V}M(H,L)||[LV^*H]|$, then $|[NV^*G]| \le |\mathcal{V}M(H,L)||[LV^*H]|$.

Moreover, $|\ker\beta| = |\mathrm{Im}\alpha| \le |\mathcal{V}M(G,N)|$, then $|\mathcal{V}M(H,L)||[LV^*H]| \le |\mathcal{V}M(G,N)||[NV^*G]|$ which proves this part. (ii) By considering the first part, we have

$|\ker\beta| = 1$ if and only if $\mathcal{V}M(H,L) \cong V^*(G)\cap[NV^*G]$ and

$|\ker\beta| = 1$ if and only if $|[NV^*G]| = |\mathcal{V}M(H,L)||[LV^*H]|$.

Thus, the result holds.

(iii) By Lemma 2.1(i), $|\ker\alpha||\mathcal{V}M(H,L)||[LV^*H]| = |\mathcal{V}M(G,N)||[NV^*G]|$. Also

$|\ker\alpha| = 1$ if and only if $\dfrac{\mathcal{V}M(H,L)}{\mathcal{V}M(G,N)} \cong V^*(G)\cap[NV^*G]$, which completes the proof. □

The following corollary gives a connection between the order of the Baer-invariant of any finite group with its marginal factor group.

Corollary 2.5 *Let G be a finite group and $H = G/V^*(G)$ be the marginal factor group of G. Then*

(i) $|V(G)| \le |\mathcal{V}M(H)||V(H)| \le |\mathcal{V}M(G)||V(G)|$;

(ii) $|V(G)| = |\mathcal{V}M(H)||V(H)| \iff \mathcal{V}M(H) \cong V^*(G) \cap V(G)$;

(iii) $|\mathcal{V}M(H)||V(H)| = |\mathcal{V}M(G)||V(G)| \iff \dfrac{\mathcal{V}M(H)}{\mathcal{V}M(G)} \cong V^*(G) \cap V(G)$.

3. Some isomorphisms and $\mathcal{V}_G$-marginal series

In this section, we want to show that under some circumstances there are some isomorphisms among $\mathcal{V}_G$-marginal factor groups. The following lemma can really help us to prove the results in our paper. The lemma can bring us an exact sequence of a given group extension. Therefore, it becomes obvious that the production of the lemma generalizes 11.4.17 of [15].

Lemma 3.1 *Let $\mathcal{V}$ be a variety of groups defined by the set of laws V. If $1 \to N \to E \xrightarrow{\pi} G \to 1$ is a group extension, and L is a normal subgroup of E such that $1 \to N \to L \xrightarrow{\overline{\pi}} M \to 1$ is a group extension, then the following sequence is exact:*

$$\mathcal{V}M(E,L) \to \mathcal{V}M(G,M) \to \frac{N}{[NV^*E]} \to \frac{L}{[LV^*E]} \to \frac{M}{[MV^*G]} \to 1\,.$$

Proof We define the following maps

$$\pi' : \frac{L}{[LV^*E]} \longrightarrow \frac{M}{[MV^*G]} \qquad \sigma' : \frac{N}{[NV^*E]} \longrightarrow \frac{L}{[LV^*E]}$$

$$x[LV^*E] \longmapsto \overline{\pi}(x)[MV^*G] \qquad n[NV^*E] \longmapsto n[LV^*E]$$

Clearly, π' is an epimorphism with the kernel $\dfrac{N[LV^*E]}{[LV^*E]}$. The image and the kernel of σ' are $\dfrac{N[LV^*E]}{[LV^*E]}$ and $\dfrac{N \cap [LV^*E]}{[NV^*E]}$, respectively. So the exactness at $\dfrac{L}{[LV^*E]}$ and $\dfrac{M}{[MV^*G]}$ follows immediately. Now let $1 \to R \to F \xrightarrow{\pi_1} E \to 1$ be a free presentation of E and $L \cong T/R$ for a normal subgroup T of the free group F. Then $\pi \circ \pi_1 : F \to G$ is a free presentation of G. Put $\ker\ \pi \circ \pi_1 = S$, therefore, S is the inverse image of N under π_1. Hence $R \subseteq S \subseteq T$, $N \cong S/R$ and $M \cong T/S$. Also:

$$\mathcal{V}M(E,L) = \frac{R \cap [TV^*F]}{[RV^*F]} \qquad \mathcal{V}M(G,M) = \frac{S \cap [TV^*F]}{[SV^*F]}.$$

Now, we define the maps

$$\varphi : \mathcal{V}M(G,M) \longrightarrow \frac{N}{[NV^*E]} \qquad \psi : \mathcal{V}M(E,L) \longrightarrow \mathcal{V}M(G,M)$$

$$x[SV^*F] \longmapsto \pi_1(x)[NV^*E] \qquad x[RV^*F] \longmapsto x[SV^*F].$$

It can be easily checked that the image of φ is $\dfrac{N \cap [LV^*E]}{[NV^*E]}$ which is the same as the kernel of σ'. Also, the kernel of φ is $\dfrac{(R \cap [TV^*F])[SV^*F]}{[SV^*F]}$ which is the same as the image of ψ. Thus, the sequence is exact and the proof is completed. □

The above lemma has the following corollary which is of interest in its own account.

Corollary 3.2 *Let G be a finite group with two normal subgroups K and N such that $K \subseteq N$. Then*

(i) *the following sequence is exact:*

$$\mathcal{V}M(G,N) \to \mathcal{V}M(G/K,N/K) \to \frac{K}{[KV^*G]} \to \frac{N}{[NV^*G]} \to \frac{N}{[NV^*G]K} \to 1;$$

(ii) *The following conditions are equivalent:*

(a) *sequence* $1 \to \mathcal{V}M(G/K,N/K) \to \frac{K}{[KV^*G]} \to \frac{N}{[NV^*G]} \to \frac{N}{[NV^*G]K} \to 1$ *is exact;*

(b) $\mathcal{V}M(G,K) = \mathcal{V}M(G,N)$;

(c) $\mathcal{V}M(G/K,N/K) \cong \frac{K \cap [NV^*G]}{[KV^*G]}$.

Proof (i) This part results from Lemma 3.1, by considering two exact sequences $1 \to K \to G \to G/K \to 1$ and $1 \to K \to N \to N/K \to 1$.

(ii) By the definition of the Baer-invariant of the pair of groups and Lemma 2.1(i), we have the following exact sequence:

$$1 \to \mathcal{V}M(G,K) \to \mathcal{V}M(G,N) \to \mathcal{V}M(G/K,N/K) \to \frac{K \cap [NV^*G]}{[KV^*G]} \to 1.$$

It is easily checked that (b) and (c) are equivalent. Also, by first part sequence $\mathcal{V}M(G/K,N/K) \overset{\alpha}{\to} \frac{K}{[KV^*G]} \to \frac{N}{[NV^*G]} \to \frac{N}{[NV^*G]K} \to 1$ is exact. Now by the technique which has been mentioned in Theorem 2.4, we have $\frac{|\mathcal{V}M(G/K,N/K)|}{|\ker\alpha|} = |\frac{K \cap [NV^*G]}{[KV^*G]}|$. Hence ($a$) and ($c$) are equivalent. □

By using Corollary 3.2(i), we have the following theorem, which generalizes 7.9.1 of [8].

Theorem 3.3 *Let $f : G \to H$ be a group homomorphism and N be a normal subgroup of G and K be a normal subgroup of H such that $f(N) \subseteq K$. Suppose f induces isomorphisms $f_0 : G/N \to H/K$ and $\overline{f}_1 : N/[NV^*G] \to K/[KV^*H]$, and that $f_* : \mathcal{V}M(G,N) \to \mathcal{V}M(H,K)$ is an epimorphism. Then f induces isomorphisms $f_n : G/V_n(N,G) \overset{\cong}{\to} H/V_n(K,H)$ and $\overline{f}_n : N/V_n(N,G) \overset{\cong}{\to} K/V_n(K,H)$ for all $n \geq 0$.*

Proof At first, we want to mention a point that for making it easier to draw the following diagrams, we would like to introduce $P_n = V_n(N,G)$ and $Q_n = V_n(K,H)$. We proceed by induction. For $n = 0$ the assertion is trivial. For $n = 1$, consider the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N/[NV^*G] & \longrightarrow & G/[NV^*G] & \longrightarrow & G/N & \longrightarrow & 1 \\
 & & \big\downarrow \overline{f}_1 & & \big\downarrow f_1 & & \big\downarrow f_0 & & \\
1 & \longrightarrow & K/[KV^*H] & \longrightarrow & H/[KV^*H] & \longrightarrow & H/K & \longrightarrow & 1.
\end{array}$$

By the hypothesis $\overline{f}_1$ and f_0 are isomorphism, hence f_1 is an isomorphism. Assume that $n \geq 2$. By considering Corollary 3.2(i), we can conclude the following commutative diagram:

$$\begin{array}{ccccccccccc}
\mathcal{V}M(G,N) & \rightarrow & \mathcal{V}M(G/P_{n-1}, N/P_{n-1}) & \rightarrow & P_{n-1}/P_n & \rightarrow & N/[NV^*G] & \rightarrow & N/[NV^*G]P_{n-1} & \rightarrow & 1 \\
\downarrow{\alpha_1} & & \downarrow{\alpha_2} & & \downarrow{\alpha_3} & & \downarrow{\alpha_4} & & \downarrow{\alpha_5} & & (*) \\
\mathcal{V}M(H,K) & \rightarrow & \mathcal{V}M(H/Q_{n-1}, K/Q_{n-1}) & \rightarrow & Q_{n-1}/Q_n & \rightarrow & K/[KV^*H] & \rightarrow & K/[KV^*H]Q_{n-1} & \rightarrow & 1.
\end{array}$$

Note that the naturality of the map f induces homomorphisms $\alpha_i, i = 1, 2, ..., 5$ such that $(*)$ is commutative. By hypothesis α_1 is an epimorphism and α_4, α_5 are isomorphisms. Also, by considering the induction hypothesis and definition of the Baer-invariant of the pair of groups, α_2 is an isomorphism. Hence by five lemma of [16], α_3 is an isomorphism. Now consider the following diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & P_{n-1}/P_n & \longrightarrow & N/P_n & \longrightarrow & N/P_{n-1} & \longrightarrow & 1 \\
 & & \downarrow{\alpha_3} & & \downarrow{\overline{f}_n} & & \downarrow{\overline{f}_{n-1}} & & \\
1 & \longrightarrow & Q_{n-1}/Q_n & \longrightarrow & K/Q_n & \longrightarrow & K/Q_{n-1} & \longrightarrow & 1
\end{array}$$

by the above discussion α_3 is an isomorphism and by induction hypothesis $\overline{f}_{n-1}$ is an isomorphism, therefore, $\overline{f}_n$ is an isomorphism. Finally, by the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N/P_n & \longrightarrow & G/P_n & \longrightarrow & G/N & \longrightarrow & 1 \\
 & & \downarrow{\overline{f}_n} & & \downarrow{f_n} & & \downarrow{f_1} & & \\
1 & \longrightarrow & K/Q_n & \longrightarrow & H/Q_n & \longrightarrow & H/K & \longrightarrow & 1
\end{array}$$

and in the same way, f_n is an isomorphism. □

Now we obtain the following corollary.

Corollary 3.4 *Let $(f, f|) : (G, N) \longrightarrow (H, K)$ be group homomorphisms that satisfy the hypotheses of Theorem 3.3. Suppose further that N and K are $\mathcal{V}_G$-nilpotent and $\mathcal{V}_H$-nilpotent, respectively. Then f and $f|$ are isomorphisms.*

Proof The assertion follows from Theorem 3.3 and the remark that there exists $n \geq 0$ such that $V_n(N, G) = \{1\}$ and $V_n(K, H) = \{1\}$. □

As a final result we have the following theorem, which is a generalization of Stallings′ theorem [19].

Theorem 3.5 *Let $\mathcal{V}$ be a variety of groups and $f : G \longrightarrow H$ be an epimorphism. Let N be a $\mathcal{V}_G$-nilpotent normal subgroup of G and K be a normal subgroup of H such that $f(N) = K$. If $\ker f \subseteq [NV^*G]$ and $\mathcal{V}M(H,K)$ is trivial, then f and $f|$ are isomorphisms.*

Proof Put $M = \ker f$, then $\frac{N}{[NV^*G]} \cong \frac{K}{[KV^*H]}$, $\frac{G}{N} \cong \frac{H}{K}$ and $\frac{V_n(N,G)M}{M} = V_n(K,H)$ for all $n \geq 0$. Now the result follows from Corollary 3.4. □

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