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On quasiconformal harmonic mappings lifting to minimal surfaces

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Abstract: We prove a growth theorem for a function to belong to the class $\sum(\mu; a)$ and generalize a Weierstrass-Enneper representation type theorem for the minimal surfaces given in [5] to spacelike minimal surfaces which lie in 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$. We also obtain some estimates of the Gaussian curvature of the minimal surfaces in 3-dimensional Euclidean space $\mathbb{R}^3$ and of the spacelike minimal surfaces in $\mathbb{L}^3$.

Key words: Minimal surface, isothermal parameters, Weierstrass-Enneper representation, quasiconformal harmonic mapping

1. Introduction

It is well known that the connection between harmonic mappings and minimal surfaces arises from the fact that Euclidean coordinates of a minimal surfaces are harmonic functions of isothermal (conformal) parameters. The projection of a minimal graph onto its base plane defines a harmonic mapping. Conversely, the harmonic mappings that lift to minimal surfaces have a simple description and corresponding surfaces can be given by explicit formulas. The representation makes harmonic mappings an effective tool in the study of minimal surfaces theory. In this paper we consider the univalent quasiconformal harmonic mappings with starlike analytic part whose second dilatation ω is in the class $\sum(\mu; a)$, and consider the (regular) minimal surfaces with isothermal parameters in $\mathbb{R}^3$ and the (regular) spacelike minimal surfaces with isothermal parameters in $\mathbb{L}^3$. First, we give a growth theorem for a function which belongs to the class $\sum(\mu; a)$ and then we generalize a Weierstrass-Enneper representation type theorem for the minimal surfaces given in [5] to the spacelike minimal surfaces which lie in 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$. Finally, applying the growth theorem, we obtain some estimates of the Gaussian curvature of the minimal surfaces in $\mathbb{R}^3$ and of the spacelike minimal surfaces in $\mathbb{L}^3$ which lift by the (sense-preserving) univalent quasiconformal harmonic mapping with starlike analytic part whose second dilatation ω is in the class $\sum(\mu; a)$. Our work is motivated by studies on the theory of minimal surfaces, especially those that are lifted by harmonic mappings; see [5, 6, 8, 11, 13].

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2. Preliminaries

Minimal surfaces in Euclidean space $\mathbb{R}^3$

Minimal surfaces are most commonly known as those which have the minimum area amongst all other surfaces spanning a given closed curve in $\mathbb{R}^3$. Geometrically, the definition of a minimal surface is that the mean curvature H is zero at every point of the surface. If locally one can write the minimal surface in $\mathbb{R}^3$ as $(x, y, \Phi(x, y))$, the minimal surface equation $H = 0$ is equivalent to

$$(1+\Phi_y^2)\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + (1+\Phi_x^2)\Phi_{yy} = 0.$$

There exists a choice of isothermal parameters $(u, v) \in \Omega \subset \mathbb{R}^2$ so that the surface $X(u, v) = (x(u, v), y(u, v), \Phi(u, v)) \in \mathbb{R}^3$ satisfying the minimal surface equation is given by

$$E = |X_u|^2 = |X_v|^2 = G > 0, \quad F =< X_u, X_v >= 0, \quad \triangle_{(u,v)}X = 0,$$

where Δ denotes the Laplacian operator [3]. In the classical theory of minimal surfaces in $\mathbb{R}^3$, a basic tool is the Weierstrass-Enneper representation. One of the local versions of this can be stated as follows.

Theorem 2.1 *[5, 11] Let $\Omega \subset \mathbb{C}$ be an open set endowed with a complex coordinate $z = u+iv$. Let $X : \Omega \to \mathbb{R}^3$ be an isothermal (conformal) minimal immersion. Then the vector field*

$$\varphi = (\varphi_1, \varphi_2, \varphi_3) = 2\frac{\partial X}{\partial z} := (\frac{\partial X}{\partial u} - i\frac{\partial X}{\partial v})$$

satisfies

1. $|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 > 0$,
2. $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$,
3. $\dfrac{\partial \varphi_k}{\partial \overline{z}} = 0, \quad k = 1, 2, 3.$

Conversely, if Ω is simply connected and $\varphi_k : \Omega \to \mathbb{C}$ $(k = 1, 2, 3)$ are functions satisfying the above conditions, the map

$$X = Re\int \varphi dz$$

is a well-defined conformal immersion. Moreover, the functions φ_k $(k = 1, 2, 3)$ can be described as

$$\varphi_1 = p(1+q^2), \quad \varphi_2 = -ip(1-q^2), \quad \varphi_3 = -2ipq, \tag{2.1}$$

where p (respectively q) is an analytic function (respectively meromorphic function) on Ω such that pq^2 is analytic on Ω and $\varphi_2 \neq i\varphi_1$ for $z \in \Omega$. Then the first fundamental form of $S = X(\Omega)$ is given by

$$ds^2 = |p|^2(1+|q|^2)^2|dz|^2 \tag{2.2}$$

and the Gaussian curvature of the minimal surface $S = X(\Omega)$ is given by

$$K = -\frac{4|q'|^2}{|p|^2(1+|q|^2)^4} \ . \tag{2.3}$$

In Theorem 2.1, the first condition tells us that X is an immersion or equivalently that the surface $S = X(\Omega)$ is regular, the second one that X is conformal and the third one that X is minimal [5, 11].

Minimal Surfaces in Lorentz-Minkowski space $\mathbb{L}^3$

The affine space $\mathbb{R}^3$ endowed with the Lorentzian metric

$$g = dx_1^2 + dx_2^2 - dx_3^2$$

is called Lorentz-Minkowski space and denoted by $\mathbb{L}^3$ in general. Let Ω be an open set in $\mathbb{R}^2$ and $X : \Omega \to \mathbb{L}^3$ be an immersion. We shall say that X is *spacelike* if the induced metric X^*g is positive definite. If X^*g is a symmetric non-degenerate form of index 1, that is, if it is a Lorentzian metric, then we say that X is *timelike.* In the case the induced metric X^*g is positive definite, a surface $S = X(\Omega)$ is called a spacelike surface. A spacelike surface with vanishing mean curvature is called a *spacelike minimal surface.* Some authors also call $S = X(\Omega)$ a *maximal surface* [8, 11].

For spacelike minimal surfaces in $\mathbb{L}^3$, an analogue of Theorem 2.1 was proved by O. Kobayashi in [8] and can be stated as follows.

Theorem 2.2 *[8, 11] Let $\Omega \subset \mathbb{C}$ be an open set endowed with a complex coordinate $z = u+iv$. Let $X : \Omega \to \mathbb{L}^3$ be a conformal (isothermal) minimal immersion. Then the vector field*

$$\varphi = (\varphi_1, \varphi_2, \varphi_3) = 2\frac{\partial X}{\partial z} := (\frac{\partial X}{\partial u} - i\frac{\partial X}{\partial v})$$

satisfies

1. $|\varphi_1|^2 + |\varphi_2|^2 - |\varphi_3|^2 > 0;$
2. $\varphi_1^2 + \varphi_2^2 - \varphi_3^2 = 0;$
3. $\frac{\partial \varphi_k}{\partial \bar{z}} = 0, \quad k = 1, 2, 3.$ *Equivalently, the functions φ_k are analytic.*

Conversely, if Ω is simply connected and $\varphi_k : \Omega \to \mathbb{C}$ $(k = 1, 2, 3)$ are functions satisfying the above conditions, the map

$$X = Re \int \varphi dz = Re \int (p(1+q^2), ip(1-q^2), -2pq)dz$$

is a well-defined conformal immersion, where p $(respectively, q)$ is an analytic function (respectively, meromorphic function) on Ω such that pq^2 is analytic on Ω and $|q(z)| \neq 1$ for $z \in \Omega$. Moreover, the first fundamental form of the minimal surface $S = X(\Omega)$ is given by

$$ds = |p|(1 - |q|^2)dz, \tag{2.4}$$

and the Gaussian curvature of the minimal surface $S = X(\Omega)$ is given by

$$K = \frac{4|q'|^2}{|p|^2(1 - |q|^2)^4}. \tag{2.5}$$

Contrary to the case of minimal surfaces in $\mathbb{R}^3$, a spacelike minimal surface in $\mathbb{L}^3$ has non-negative Gaussian curvature [8].

Quasiconformal harmonic mappings in the plane $\mathbb{C}$

Quasiconformal harmonic mappings in the plane were first studied by O. Martio in [12], and nowadays they are actively investigated both in the planar and the multidimensional setting from several different points of view. Some of the topics considered are the boundary behavior, including Hölder and Lipschitz continuity of quasiconformal harmonic mappings, and more generally moduli of continuity of them. As we have already mentioned above, we will be interested in planar quasiconformal harmonic mappings throughout this paper.

A complex-valued function f which is harmonic in a simply connected domain $\mathbb{D} \subset \mathbb{C}$ has the canonical representation $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$ and $g(z_0) = 0$ for some prescribed point $z_0 \in \mathbb{D}$. According to a theorem of H. Lewy [2], f is locally univalent if and only if its Jacobian $(|f_z|^2 - |f_{\overline{z}}|^2 = |h^{'}(z)|^2 - |g^{'}(z)|^2)$ does not vanish. f is said to be sense-preserving if its Jacobian is positive. In this case, $h^{'}(z)$ does not vanish and the analytic function $\omega(z) = \dfrac{g'(z)}{h'(z)}$, called the second dilatation of f, has the property $|\omega(z)| < 1$ for all $z \in \mathbb{D}$. A univalent harmonic mapping is called *μ-quasiconformal* $(0 \leq \mu < 1)$ if $|\omega(z)| < \mu$ [7]. For general definition of quasiconformal mappings, see [1, 9].

Now, recall that $f = h + \overline{g}$ can be lifted locally to a regular minimal surface in $\mathbb{R}^3$ given by isothermal parameters if and only if its dilatation is the square of an analytic function $\omega(z) = q^2(z)$ for some analytic function$q$ with $|q(z)| < 1$. Equivalently, the requirement is that any zero of ω be of even order, unless $\omega \equiv 0$ on its domain, so that there is no loss of generality in supposing that z ranges over the unit disk $\mathbb{D}$, because any other isothermal representation can be precomposed with a conformal map from the unit disk $\mathbb{D}$ whose existence is guaranteed by the Riemann mapping theorem [5]. Then Theorem 2.1 can be restated as follows.

Theorem 2.3 *[5] Let $f = h + \overline{g}$ be a sense-preserving univalent harmonic mapping of $\mathbb{D}$ onto some domain Ω with $\omega(z) = q^2(z)$ for some function q analytic in $\mathbb{D}$, and let $S = X(\mathbb{D})$ be a regular minimal surface as in Theorem 1.1. If $f = h + \overline{g}$ lifts to the minimal surface $S = X(\mathbb{D})$, then*

$$\varphi_1 = h' + g', \quad \varphi_2 = -i(h' - g'), \quad \varphi_3 = -2ih'\sqrt{w} \tag{2.6}$$

and

$$h' = p, \quad g' = pq^2, \tag{2.7}$$

where ω is the second dilatation of $f = h + \overline{g}$. Moreover, the first fundamental form of the minimal surface $S = X(\mathbb{D})$ is given by

$$ds = \lambda dz, where \lambda = |h'| + |g'| = |h'|(1 + |\omega|) = |p|(1 + |q|^2) \tag{2.8}$$

and the Gaussian curvature of the minimal surface $S = X(\mathbb{D})$ is given by

$$K = -\frac{|\omega'|^2}{|h'g'|(1 + |\omega|)^4}. \tag{2.9}$$

We shall call the functions φ_k $(k = 1, 2, 3)$ as *Weierstrass-Enneper functions* of the regular minimal surface with conformal parameters. Now, we define the following class of harmonic functions [4], which is used throughout this paper.

Let $h(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ and $g(z) = b_0 + b_1 z + b_2 z^2 + \cdots$ be analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The class of all sense-preserving harmonic functions in $\mathbb{D}$ with $a_0 = b_0 = 0$ and $a_1 = 1$ will be denoted by $\mathcal{S}_H$. Thus $\mathcal{S}_H$ contains the standard class $\mathcal{S}$ of analytic functions (See [4, 5]). Let $s(z) = z + c_2 z + c_3 z^2 + \cdots$ be an analytic function in the open unit disk $\mathbb{D}$. If $s(z)$ satisfies the condition

$$\operatorname{Re}\left(z\frac{s'(z)}{s(z)}\right) > 0, \quad (z \in \mathbb{D}), \tag{2.10}$$

then $s(z)$ is called a starlike function in $\mathbb{D}$, and the class of starlike functions in $\mathbb{D}$ is denoted by $\mathcal{S}^*$.

Let $\sum(\mu; a)$ be the family of functions $\omega(z)$ which are regular in $\mathbb{D}$ and satisfy the conditions $\omega(0) = a$ and $|\omega(z)| < \mu$ for all $z \in \mathbb{D}$, where $0 < |a| < 1$ and $0 \leq \mu < 1$.

We denote by $\mathcal{S}^*_{QCH}$ the subclass of $\mathcal{S}_H$ consisting of all univalent quasiconformal harmonic functions whose analytic part is starlike. From now on we will assume that f is a locally univalent, sense-preserving, quasiconformal harmonic function whose second dilatation belongs to class $\sum(\mu; a)$, unless otherwise stated.

3. Main Theorems

Lemma 3.1 *Let ω be an element of $\sum(\mu; a)$. Then*

$$\frac{\mu(|a| - \mu r)}{\mu - |a|r} \leq |\omega(z)| \leq \frac{\mu(|a| + \mu r)}{\mu + |a|r}. \tag{3.1}$$

Proof Since the transformation $\omega(z) = \dfrac{\mu^2(a - z)}{\mu^2 - \overline{a}z}$ maps $|z| = r$ onto the disk with center

$$C(r) = \left(\frac{(\mu^4 - \mu^2 r^2)a_1}{\mu^4 - |a|^2 r^2}; \frac{(\mu^4 - \mu^2 r^2)a_2}{\mu^4 - |a|^2 r^2}\right)$$

and the radius

$$\varrho(r) = \frac{\mu^2(\mu^2 - |a|^2)r}{\mu^4 - |a|^2 r^2},$$

where $a_1 = \operatorname{Re}(a)$ and $a_2 = \operatorname{Im}(a)$, we can write

$$\left|\omega(z) - \frac{\mu^2(1 - r^2)a}{\mu^2 - |a|^2 r^2}\right| \leq \frac{\mu(\mu^2 - |a|^2)r}{\mu^2 - |a|^2 r^2}. \tag{3.2}$$

After simple calculations from (3.2) we get (3.1). □

Corollary 3.2 *If $\omega \in \sum(\mu; a)$, then*

$$\frac{\mu(1 - \mu r) + |a|(\mu - r)}{\mu + |a|r} \leq (1 - |\omega(z)|) \leq \frac{\mu(1 + \mu r) - |a|(\mu + r)}{\mu - |a|r}, \tag{3.3}$$

and

$$\frac{\mu - |a|r + \mu|a| - \mu^2 r}{\mu - |a|r} \leq 1 + |\omega(z)| \leq \frac{\mu + |a|r + \mu|a| + \mu^2 r}{\mu + |a|r}. \tag{3.4}$$

Proof These inequalities are simple consequences of Lemma 3.1. □

Corollary 3.3 *Let* $f = h + \overline{g}$ *be an element of* $\mathcal{S}^*_{QCH}$. *Then*

$$\frac{\mu(1-r)(|a|-\mu r)}{(1+r)^3(\mu-|a|r)} \le |g'(z)| \le \frac{\mu(1+r)(|a|+\mu r)}{(1-r)^3(\mu+|a|r)} \tag{3.5}$$

Proof Recall that if the analytic part h of f is starlike, we then have

$$\frac{1-r}{(1+r)^3} \le |h'(z)| \le \frac{1+r}{(1-r)^3}. \tag{3.6}$$

On the other hand, if we consider Lemma 3.1 and the definition of the second dilatation of f, then we can write

$$\frac{\mu(|a|-\mu r)}{\mu-|a|r} \le \left|\frac{g'(z)}{h'(z)}\right| \le \frac{\mu(|a|+\mu r)}{\mu+|a|r}. \tag{3.7}$$

Considering the inequalities (3.6) and (3.7) together, we obtain (3.5). □

Theorem 3.4 *Let* $f = (h + \overline{g}) \in \mathcal{S}^*_{QCH}$ *lift to a regular minimal surface* S *in* $\mathbb{R}^3$. *If the functions* φ_k *for* $k = 1, 2, 3$ *are the Weierstrass-Enneper functions of the minimal surface* S, *then*

$$\frac{(1-r)[\mu(1-\mu r)+|a|(\mu-r)]}{(1+r)^3(\mu+|a|r)} \le |\varphi_1| \le \frac{(1+r)[\mu+|a|r+\mu|a|+\mu^2 r]}{(1-r)^3(\mu+|a|r)}, \tag{3.8}$$

$$\frac{(1-r)[\mu(1-\mu r)+|a|(\mu-r)]}{(1+r)^3(\mu+|a|r)} \le |\varphi_2| \le \frac{(1+r)[\mu+|a|r+\mu|a|+\mu^2 r]}{(1-r)^3(\mu+|a|r)}, \tag{3.9}$$

and

$$\frac{4\mu(1-r)^2(|a|-\mu r)}{(1+r)^6(\mu-|a|r)} \le |\varphi_3|^2 \le \frac{4\mu(1+r)^2(|a|+\mu r)}{(1-r)^6(\mu+|a|r)}. \tag{3.10}$$

Proof From the equation (2.6) we can write

$$\varphi_1 = h' + g' = h'(1 + \frac{g'}{h'}) = h'(1+\omega), \tag{3.11}$$

$$\varphi_2 = -i(h' - g') = -ih'(1 - \frac{g'}{h'}) = -ih'(1-\omega), \tag{3.12}$$

and

$$\varphi_3^2 = -4\omega(h')^2. \tag{3.13}$$

Hence, we get

$$|\varphi_1| = |h'||(1+\omega)|, \tag{3.14}$$

$$|\varphi_2| = |h'||(1-\omega)|, \tag{3.15}$$

and

$$|\varphi_3|^2 = 4|\omega||h'|^2. \tag{3.16}$$

Using the triangle inequality in (3.14) and (3.15), we have

$$|h'|(1-|\omega|) \leq |\varphi_1| \leq |h'|(1+|\omega|) \tag{3.17}$$

and

$$|h'|(1-|\omega|) \leq |\varphi_2| \leq |h'|(1+|\omega|) \tag{3.18}$$

With the help of the inequalities (3.3), (3.4) and (3.6) from (3.17) and (3.18), we obtain (3.8) and (3.9). Finally, if we use the inequalities (3.1) and (3.6) in the equality (3.16), we obtain (3.10). □

Theorem 3.5 *Let K be the Gaussian curvature of the regular minimal surface S and $f=(h+\overline{g})\in\mathcal{S}^*_{QCH}$ lift to the minimal surface S. Then*

$$|K| \leq \frac{(1+r)^2(\mu-|a|r)[\mu(1+\mu r)-|a|(\mu+r)]^2[\mu+|a|r+\mu|a|+\mu^2 r]^2}{\mu(1-r)^2(|a|-\mu r)(\mu+|a|r)^2[\mu-|a|r+\mu|a|-\mu^2 r]^4} \tag{3.19}$$

Proof From the inequalities (3.5) and (3.6), we get

$$\frac{(1-r)^6(\mu+|a|r)}{\mu(1+r)^2(|a|+\mu r)} \leq \frac{1}{|g'(z)h'(z)|} \leq \frac{(1+r)^6(\mu-|a|r)}{\mu(1-r)^2(|a|-\mu r)}. \tag{3.20}$$

Using the equality (2.9) and the inequality (3.20) we have

$$|K| = \frac{|\omega'(z)|^2}{|g'(z)h'(z)|(1+|\omega(z)|)^4} \leq \frac{|\omega'(z)|^2(1+r)^6(\mu-|a|r)}{(1+|\omega(z)|)^4\mu(1-r)^2(|a|-\mu r)}. \tag{3.21}$$

On the other hand, if we use the Schwarz-Pick's Lemma for the function

$$\phi(z) = \frac{\omega(z)-\omega(0)}{1-\overline{\omega(0)}\omega(z)},$$

we obtain

$$|\omega'(z)|^2 \leq \frac{(1-|\omega(z)|^2)^2}{(1-r^2)^2} = \frac{(1-|\omega(z)|)^2(1+|\omega(z)|)^2}{(1-r)^2(1+r)^2}. \tag{3.22}$$

If we use the inequality (3.22) together with (3.3) and (3.4) in (3.21) we obtain (3.19). □

Example. Consider the function $f(z)=z-\dfrac{\mu}{2}\dfrac{\bar{z}}{2-\bar{z}}$, where $\mu\in(0,1)$ is a constant and $z\in\mathbb{D}$. Since $\triangle f=\dfrac{4\partial^2 f}{\partial z\partial\overline{z}}=0$, f is harmonic. The functions $h(z)=z$ and $g(z)=-\dfrac{\mu}{2}\dfrac{z}{2-z}$, the analytic and co-analytic parts of f, are analytic in $\mathbb{D}$ and satisfy $h(0)=g(0)=0$. As $J_f(z)=|h^{'}(z)|^2-|g^{'}(z)|^2=1-\dfrac{\mu}{|2-z|^2}>0$ on $\mathbb{D}$, it follows that f is sense-preserving and univalent. Furthermore, the analytic part $h(z)=z$ of f is starlike. The second dilatation of f is $\omega(z)=\dfrac{g'(z)}{h'(z)}=-(\dfrac{\sqrt{\mu}}{2-z})^2$. Since $|\omega(0)|=\dfrac{\mu}{4}\in(0,1)$ and $|\omega(z)|<\mu$ on $\mathbb{D}$, we have $\omega\in\sum(\mu;a)$. Therefore f belongs to the class $\mathcal{S}^*_{QCH}$. On the other hand, $\omega(z)$ is the square of the analytic function $q(z) = \dfrac{i\sqrt{\mu}}{2-z}$ in $\mathbb{D}$. Thus the univalent quasiconformal harmonic function f can lift locally to a (regular) minimal surface. Now, using (2.1), we get $p(z) \equiv 1$ and $q(z) = \dfrac{i\sqrt{\mu}}{2-z}$, and it is known that the functions p and q are the Weierstrass-Enneper parameters of the minimal surface Catenoid.

As in $\mathbb{R}^3$, the projection of a regular spacelike minimal surface with isothermal parameters in $\mathbb{L}^3$ onto its base plane defines a harmonic mapping (see [6]). Therefore, we can construct an analogue of Theorem 2.3 for the spacelike minimal surfaces in $\mathbb{L}^3$. Consider a regular minimal graph

$$S = \{(u, v, \Phi(u, v)) : u + iv \in \Omega\}$$

in $\mathbb{L}^3$, over a simply connected domain $\Omega \subset \mathbb{C}$ containing the origin. Suppose that Ω is not the whole plane. In view of Theorem 2.2, the surface has a reparametrization by isothermal parameters $z = x + iy$ in the unit disk $\mathbb{D}$, so that

$$u = Re\{\int_0^z \varphi_1(\xi)d\xi\}, \quad v = Re\{\int_0^z \varphi_2(\xi)d\xi\},$$

$$\Phi(u, v) = Re\{\int_0^z \varphi_3(\xi)d\xi\}, \quad z \in \mathbb{D}.$$

There is no loss of generality in supposing that z ranges over the unit disk $\mathbb{D}$, because any other isothermal representation can be precomposed with a conformal map from the unit disk $\mathbb{D}$ whose existence is guaranteed by the Riemann mapping theorem. Now let $w = u + iv$ and let $w = f(z)$ denote the projection of S onto its base plane:

$$f(z) = \mathrm{Re}\{\textstyle\int_0^z \varphi_1(\xi)d\xi\} + iRe\{\int_0^z \varphi_2(\xi)d\xi\}. \tag{3.23}$$

Then f is a harmonic mapping of $\mathbb{D}$ onto Ω with $f(0) = 0$. Let

$$f = h + \overline{g}, \quad h(0) = g(0) = 0,$$

be the canonical decomposition of f, where h and g are analytic in $\mathbb{D}$. Differentiating from (3.23) we get

$$h' = \frac{1}{2}(\varphi_1 + i\varphi_2), \quad g' = \frac{1}{2}(\varphi_1 - i\varphi_2)$$

or

$$\varphi_1 = h' + g', \quad \varphi_2 = -i(h' - g'). \tag{3.24}$$

Hence

$$\varphi_3^2 = \varphi_1^2 + \varphi_2^2 = 4h'g' = 4\omega(h')^2,$$

where $\omega(z) = \dfrac{g'(z)}{h'(z)}$ is the dilatation of f. This shows that $\omega(z) = \dfrac{1}{4}\dfrac{\varphi_3^2}{(h')^2}$ is the square of a meromorphic function. In other words, the harmonic mappings that result from the projection of minimal graphs have dilatations with single-valued square roots. If f is sense-preserving, this is equivalent to saying that its dilatation function ω has no zeros of odd order.
On the other hand, from Theorem 2.2 we have

$$\varphi_1 = p(1 + q^2), \quad \varphi_2 = ip(1 - q^2), \quad \varphi_3 = -2pq. \tag{3.25}$$

From (3.24) and (3.25) we obtain

$$h' = pq^2, \quad g' = p, \tag{3.26}$$

which gives

$$\omega = \frac{1}{q^2} \tag{3.27}$$

for the dilatation of the projected harmonic mapping f. In particular, it follows that f is sense-preserving if and only if q is analytic and $|q(z)| > 1$ in $\mathbb{D}$.

Theorem 3.6 *If a regular minimal graph*

$$S = \{(u, v, \Phi(u, v)) : u + iv \in \Omega\}$$

in $\mathbb{L}^3$ is parametrized by sense-preserving isothermal parameters $z = x + iy \in \mathbb{D}$, then the projection onto its base plane defines a harmonic mapping $w = u+iv = f(z)$ of $\mathbb{D}$ onto Ω whose second dilatation is the square of an analytic function. Conversely, if $f = h+\overline{g}$ is a sense-preserving harmonic mapping of $\mathbb{D}$ onto some domain Ω with second dilatation $\omega = \frac{1}{q^2}$ for some function q analytic and has the property $|q(z)| > 1$ in $\mathbb{D}$, then the formulas

$$u = Re\{f(z)\}, \quad v = Im\{f(z)\}, \quad t = 2Re \int_0^z \frac{h'(\xi)}{q(\xi)} d\xi. \tag{3.28}$$

define by isothermal parameters a minimal graph whose projection is f. Except for the choice of sign and an arbitrary additive constant in the third coordinate function, this is the only such surface.

Proof The necessity of the condition $\omega = \frac{1}{q^2}$ has already been proved. For the converse it needs only to show that the surface defined by the equations (3.28) is represented by harmonic functions of isothermal parameters. According to Theorem 2.2, this is equivalent to showing that each of the derivatives $\frac{\partial u}{\partial z}$, $\frac{\partial v}{\partial z}$ and $\frac{\partial t}{\partial z}$ are analytic, and that

$$\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 - \left(\frac{\partial t}{\partial z}\right)^2 = 0. \tag{3.29}$$

Taking derivative from (3.28) and using (3.23), we get

$$\frac{\partial u}{\partial z} = \frac{1}{2}(h' + g'), \quad \frac{\partial v}{\partial z} = \frac{1}{2i}(h' - g'), \tag{3.30}$$

and

$$\frac{\partial t}{\partial z} = 2.\frac{1}{2}.\left(\int_0^z \frac{h'(\xi)}{q(\xi)} d\xi + \overline{\int_0^z \frac{h'(\xi)}{q(\xi)} d\xi}\right) = \frac{h'}{q}. \tag{3.31}$$

It is easy to see that $\frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}$ and $\frac{\partial t}{\partial z}$ are analytic and (3.29) holds. Since f is univalent by the hypothesis, the given surface is seen to be a graph: the third coordinate function t is actually just a function of u and v.
To verify the uniqueness assertion, let

$$u = \mathrm{Re}\{f(z)\}, \quad v = \mathrm{Im}\{f(z)\}, \quad t = k(z)$$

represent some other minimal surface in isothermal parameters. Then k is such that $\dfrac{\partial t}{\partial z}$ is analytic. Since the representation is isothermal, the condition (2) in Theorem 2.2 must hold. This implies that

$$\left(\frac{\partial k}{\partial z}\right)^2 = \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 = \left(\frac{h'}{q}\right)^2,$$

so that $\left(\dfrac{\partial k}{\partial z}\right) = \pm\dfrac{h'}{q}$. But the real-valued function k has a unique representation $k = \psi + \overline{\psi} = 2Re\{\psi\}$ for some analytic function ψ. Since $\psi' = \pm\dfrac{h'}{q}$, it follows that

$$\psi(z) = \pm\int_0^z \frac{h'}{q} d\xi + c$$

for some complex constant c, which proves the uniqueness. □

Therefore, we can restate the Weierstrass-Enneper type theorem for spacelike minimal surfaces in $\mathbb{L}^3$.

Theorem 3.7 *Let $f = h + \overline{g}$ be a sense-preserving univalent harmonic mapping of $\mathbb{D}$ onto some domain Ω with $\omega(z) = \dfrac{1}{q^2(z)}$ for some function q analytic with the property $|q(z)| > 1$ in $\mathbb{D}$, and let $S = X(\mathbb{D})$ be a regular spacelike minimal surface as in Theorem 2.1. If $f = h + \overline{g}$ lifts to the minimal surface $S = X(\mathbb{D})$, then*

$$\varphi_1 = h' + g', \quad \varphi_2 = -i(h' - g'), \quad \varphi_3 = 2h'\sqrt{w} \tag{3.32}$$

and

$$g' = p, \quad h' = pq^2, \tag{3.33}$$

where ω is the second dilatation of $f = h + \overline{g}$. Moreover, the first fundamental form of the minimal surface $S = X(\mathbb{D})$ is given by

$$ds = |h'|(1 - |\omega|)dz| \tag{3.34}$$

and the Gaussian curvature of the minimal surface $S = X(\mathbb{D})$ is given by

$$K = \frac{|\omega'|^2}{|h'g'|(1 - |\omega|)^4}. \tag{3.35}$$

Now, we give an estimate for the Gaussian curvature of a spacelike minimal surface S with isothermal parameters which lies in $\mathbb{L}^3$.

Theorem 3.8 *Let K be the Gaussian curvature of a spacelike minimal surface S with isothermal parameters and let $f = h + \overline{g} \in \mathcal{S}^*_{QCH}$ lift to the minimal surface S. Then*

$$K \leq \frac{(1+r)^4(\mu - |a|r)[\mu + |a|r + \mu|a| + \mu^2 r]^2}{(1-r)^2\mu(|a| - \mu r)[\mu(1-\mu r) + |a|(\mu - r)]^2}. \tag{3.36}$$

Proof From (3.20) and (3.33) we have

$$K \leq \frac{|\omega'(z)|^2(1+r)^6(\mu - |a|r)}{(1-|\omega(z)|)^4\mu(1-r)^2(|a|-\mu r)}. \tag{3.37}$$

If we use the inequality (3.22) together with (3.3) and (3.4) in (3.37) we obtain (3.36). □

Example. Consider the function $f(z) = z + \frac{\mu}{2}\bar{z}$, where $\mu \in (0,1)$ is a constant and $z \in \mathbb{D}$. Since $\triangle f = \frac{4\partial^2 f}{\partial z \partial \overline{z}} = 0$, f is harmonic. The functions $h(z) = z$ and $g(z) = \frac{\mu}{2}z$, the analytic and co-analytic parts of f, are analytic in $\mathbb{D}$ and satisfy $h(0) = g(0) = 0$. As $J_f(z) = |h^{'}(z)|^2 - |g^{'}(z)|^2 = 1 - \frac{\mu^2}{4} > 0$ on $\mathbb{D}$, it follows that f is sense-preserving and univalent. Furthermore, the analytic part $h(z) = z$ of f is starlike. The second dilatation of f is $\omega(z) = \frac{g'(z)}{h'(z)} = \frac{\mu}{2}$. Since $|\omega(0)| = \frac{\mu}{4} \in (0,1)$ and $|\omega(z)| = \frac{\mu}{2} < \mu$ on $\mathbb{D}$, we have $\omega \in \sum(\mu; a)$. Therefore f belongs to the class $\mathcal{S}^*_{QCH}$. On the other hand, $\omega(z)$ is the square of the analytic function $\frac{1}{q(z)} = \sqrt{\frac{\mu}{2}}$ with the property $|q(z)| = \sqrt{\frac{2}{\mu}} > 1$ in $\mathbb{D}$. Thus the univalent quasiconformal harmonic function f can lift locally to a (regular) spacelike minimal surface. Now using by the equations (3.26) and (3.27), we get $p(z) = \frac{\mu}{2}$ and $q(z) = \sqrt{\frac{2}{\mu}}$. By setting $p(z) = \frac{\mu}{2}$ and $q(z) = \sqrt{\frac{2}{\mu}}$ in Theorem 2.2, we obtain the *spacelike plane* in $\mathbb{L}^3$ which is the simplest example of spacelike minimal surface (see [8]).

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