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New trace formula for the matrix Sturm-Liouville equation with eigenparameter dependent boundary conditions

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Abstract: A regularized trace formula of first order for the matrix Sturm-Liouville equation with eigenparameter in the boundary conditions is obtained.

Key words: Matrix Sturm-Liouville problem, eigenparameter dependent boundary conditions, trace formula

1. Introduction

As is known, the trace of a finite-dimensional matrix is the sum of all the eigenvalues. But in an infinite-dimensional space, in general, ordinary differential operators do not have a finite trace. Gelfand and Levitan [9] firstly obtained a trace formula for a self-adjoint Sturm-Liouville differential equation. After these studies several mathematicians were interested in developing trace formulae for different differential operators. For the scalar Sturm-Liouville problems, there is an enormous literature on estimates of large eigenvalues and regularized trace formulae which may often be computed explicitly in terms of the coefficients of operators and boundary conditions. A detailed list of publications related to the present aspect can be found in [13].

Note that the trace formulae are used in the numerical computation of the first eigenvalue of the Sturm-Liouville problem [6].

As a generalization of the scalar Sturm-Liouville equation, the matrix Sturm-Liouville equations were found to be important in the study of particle physics [16]. Starting with Faddeev's study of the regularized trace formula [7], matrix Sturm-Liouville operators have raised some interesting new problems. Trace formulae for the matrix Sturm-Liouville problems were considered in [3, 4, 5] and for the Sturm-Liouville problems with eigenparameter in boundary conditions in [1, 2, 10], etc. Problems with a spectral parameter in the equation and boundary conditions form an important part of spectral theory of linear differential operators. A bibliography of papers in which such problems were considered in connection with specific physical processes can be found in [8, 14]. However, a trace formula for the matrix Sturm-Liouville equation with eigenparameter in the boundary conditions has never been considered before.

2. Results

The main objective of this paper is to obtain the regularized trace formula for the matrix Sturm-Liouville problem

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$$-Y''(x)+Q(x)Y(x)=\lambda Y(x),\ x\in(0,\pi) \tag{2.1}$$

with eigenparameter dependent boundary conditions

$$\begin{cases} \lambda(Y'(0)-LY(0))=L_1Y'(0)-L_2Y(0) \\ \lambda(Y'(\pi)+HY(\pi))=H_1Y'(\pi)+H_2Y(\pi), \end{cases} \tag{2.2}$$

where the entries of the $d\times d$ matrix-valued function $Q(x)$ belong to the space $C^1[0,\pi]$, L, L_1, L_2, H, H_1 and H_2 are $d\times d$ scalar matrices and

$$L_1L_2=L_2L_1,\ L_1L=LL_1. \tag{2.3}$$

The corresponding scalar problem ($d=1$) was considered by the authors in [2, 8, 10, 14]. The matrix case has required developing a new apparatus, called the theory of V-Bezoutians of matrix polynomials, see details in [12]. If the boundary conditions of problem (2.1) and (2.2) do not contain the eigenvalue parameter λ, then the boundary conditions are reduced to the conditions of self-adjointness of problem (2.1) and (2.2) in the form given by F. S. Rofe-Beketov [11].

Theorem 2.1 *Let $\lambda_n^{(j)}, j=\overline{1,d}, n=0,1,2,\cdots$ be the spectrum of the problem (2.1) and (2.2), then for sufficiently large n*

$$\lambda_n^{(j)}=(n-2)^2+\tfrac{2}{\pi}\omega_{jj}+O\left(\tfrac{1}{n^2}\right), \tag{2.4}$$

where ω_{ij} denotes entry of matrix ω at the i-th row and j-th column, $i,j=1,2,\cdots,d$ and

$$\omega=L+H+\tfrac{1}{2}\int_0^\pi Q(t)dt.$$

It is seen from formula (2.4) that the series

$$\sum_{j=1}^{d}\left(\lambda_0^{(j)}+\lambda_1^{(j)}\right)+\sum_{n=2}^{\infty}\left[\sum_{j=1}^{d}\left(\lambda_n^{(j)}-(n-2)^2\right)-\frac{2}{\pi}\text{tr}\omega\right] \tag{2.5}$$

is absolutely convergent, where $\text{tr}A$ denotes the trace of a matrix A. In this work, we will find formula for the sum of series (2.5), which is so-called a regularized trace.

Theorem 2.2 *We have the trace formula*

$$\begin{aligned}&\textstyle\sum_{j=1}^{d}\left(\lambda_0^{(j)}+\lambda_1^{(j)}\right)+\sum_{n=2}^{\infty}\left[\sum_{j=1}^{d}\left(\lambda_n^{(j)}-(n-2)^2\right)-\frac{2}{\pi}tr\omega\right]\\ &=\tfrac{tr(Q(0)+Q(\pi))}{4}-\tfrac{1}{2\pi}tr\int_0^\pi Q(t)dt+tr(L_1+H_1)-\tfrac{1}{\pi}tr(L+H)-\tfrac{1}{2}tr(L^2+H^2).\end{aligned} \tag{2.6}$$

Remark 2.3

(1) We note that the trace formula (2.6) is a new and natural generalization of the well-known results on the trace theory for the classical Sturm-Liouville operators which were studied in [9] and other works.

(2) For a special case $L_1 = L_2 = H_1 = H_2 = 0_d$*, the trace formula (2.6) implies*

$$\begin{aligned}&\sum_{n=0}^{\infty}\left[\sum_{j=1}^{d}\left(\lambda_n^{(j)} - n^2\right) - \tfrac{1}{\pi} tr\int_0^{\pi} Q(t)dt - \tfrac{2}{\pi} tr(L+H)\right]\\ &= \tfrac{tr(Q(0)+Q(\pi))}{4} - \tfrac{1}{2\pi} tr\int_0^{\pi} Q(t)dt - \tfrac{1}{\pi} tr(L+H) - \tfrac{1}{2} tr(L^2+H^2)\end{aligned}$$

where $\lambda_n^{(j)}$ *are eigenvalues of the matrix Sturm-Liouville problem with the separated boundary conditions*

$$\begin{cases} -Y''(x) + Q(x)Y(x) = \lambda Y(x), \quad 0 < x < \pi \\ Y'(0) - LY(0) = 0 = Y'(\pi) + HY(\pi) \end{cases}$$

and for sufficiently large n

$$\lambda_n^{(j)} = n^2 + \frac{2}{\pi}\omega_{jj} + O\left(\frac{1}{n^2}\right).$$

3. Proofs

Let $\Phi(x,\lambda)$ be the solution of (2.1) satisfying the initial conditions

$$\Phi(0,\lambda) = L_1 - \lambda I_d, \quad \Phi'(0,\lambda) = L_2 - \lambda L,$$

where I_d is a $d \times d$ unit matrix. Then $\Phi(x,\lambda)$ satisfies the integral equation

$$\begin{aligned}\Phi(x,\lambda) \quad &= (L_1 - \lambda I_d)\cos\sqrt{\lambda}x + (L_2 - \lambda L)\tfrac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\\ &+ \int_0^x \tfrac{\sin[\sqrt{\lambda}(x-t)]}{\sqrt{\lambda}} Q(t)\Phi(t,\lambda)dt.\end{aligned}$$

As $|\lambda|$ tends to infinity through any part of the complex plane one can obtain the following representations

$$\Phi(x,\lambda) = -\lambda\cos\sqrt{\lambda}x I_d - \left(L + \frac{1}{2}\int_0^x Q(t)dt\right)\sqrt{\lambda}\sin\sqrt{\lambda}x + O(e^{\tau x}) \tag{3.1}$$

and

$$\begin{aligned}\Phi'(x,\lambda) \quad &= \lambda\sqrt{\lambda}\sin\sqrt{\lambda}x I_d - \left(L + \tfrac{1}{2}\int_0^x Q(t)dt\right)\lambda\cos\sqrt{\lambda}x\\ &-\Big[L_1 + \tfrac{Q(0)+Q(x)}{4} + \tfrac{1}{2}\int_0^x Q(t)dt L\\ &+\tfrac{1}{8}(\int_0^x Q(t)dt)^2\Big]\sqrt{\lambda}\sin\sqrt{\lambda}x + O(e^{\tau x}),\end{aligned} \tag{3.2}$$

where $\tau = |\mathrm{Im}\sqrt{\lambda}|$.

We see that $\Phi(x,\lambda)$ satisfies boundary condition at the point zero in (2.2), thus the general solutions of systems (2.1) have the form

$$\phi(x,\lambda) = \Phi(x,\lambda)C,$$

where $C = (c_1, c_2, \cdots, c_d)^t, c_k \in \mathbf{C}, k = 1, 2, \cdots, d$, and A^t denotes transpose of the matrix A. If $\phi(x,\lambda) = \Phi(x,\lambda)C$ is a nontrivial solution of the problem (2.1) and (2.2) there exists a non-vanishing vector C satisfying the matrix equation (i.e., boundary condition at the point π in (2.2))

$$((\lambda I_d - H_1)\Phi'(\pi,\lambda) + (\lambda H - H_2)\Phi(\pi,\lambda))\,C = 0.$$

Therefore, λ is an eigenvalue of the problem (2.1) and (2.2) if and only if the matrix

$$W(\lambda) = (\lambda I_d - H_1)\Phi'(\pi, \lambda) + (\lambda H - H_2)\Phi(\pi, \lambda)$$

is singular. Define the matrices

$$\begin{array}{c} \omega = L + H + \frac{1}{2}\int_0^\pi Q(t)dt, \\ \omega_1{=}L_1{+}H_1{+}HL{+}\frac{Q(0)+Q(\pi)}{4}{+}\frac{1}{2}\int_0^\pi Q(t)dt(L{+}H){+}\frac{1}{8}\left(\int_0^\pi Q(t)dt\right)^2. \end{array} \tag{3.3}$$

From (3.1) and (3.2) we have

$$W(\lambda) = (\lambda^2\sqrt{\lambda}\sin\sqrt{\lambda}\pi)I_d - (\lambda^2\cos\sqrt{\lambda}\pi)\omega - (\lambda\sqrt{\lambda}\sin\sqrt{\lambda}\pi)\omega_1 + O(|\lambda|e^{\tau\pi}). \tag{3.4}$$

The eigenvalues of the problem (2.1) and (2.2) coincide with the zeros of the function $\det W(\lambda)$. Using the Laplace expansion of determinants, from (3.4) we obtain

$$\begin{array}{rl} \omega(\lambda): & = \det W(\lambda) \\ & = \prod_{i=1}^d \Big[\lambda^2\sqrt{\lambda}\sin\sqrt{\lambda}\pi - \omega_{ii}\lambda^2\cos\sqrt{\lambda}\pi \\ & \quad -\omega_{1,ii}\lambda\sqrt{\lambda}\sin\sqrt{\lambda}\pi + O\left(|\lambda|e^{\tau\pi}\right)\Big] \\ & \quad +a\left(\lambda^2\sqrt{\lambda}\sin\sqrt{\lambda}\pi\right)^{d-2}\lambda^4\cos^2\sqrt{\lambda}\pi \\ & \quad +O\left(\lambda^{2d-1}\sqrt{\lambda}^{d-1}e^{d\tau\pi}\right), \end{array} \tag{3.5}$$

where

$$a = \begin{cases} -\sum_{i<j}\omega_{ij}\omega_{ji} & (d \geq 2) \\ 0 & (d = 1), \end{cases} \tag{3.6}$$

and A_{ij} denotes entry of $d \times d$ matrix A at the i-th row and j-th column, $i, j = 1, 2, \cdots, d$.

Define

$$\omega_0(\lambda) = \left(\lambda^2\sqrt{\lambda}\sin\sqrt{\lambda}\pi\right)^d, \tag{3.7}$$

and denote by μ_n zeros of the function $\omega_0(\lambda)$, then

$$\mu_0^{(j)} = \mu_1^{(j)} = 0, \quad \mu_n^{(j)} = (n-2)^2, \ n \geq 2, \ j = \overline{1, d},$$

and zeros of the function $\omega_0(\lambda)$ are multiplicities d.

Let Γ_{N_0} be the counterclockwise square contours $ABCD$ as in Figure, integer $N_0 = 0, 1, 2, \cdots \to \infty$, with

$$\begin{array}{ll} A = (N_0 - 2 + \frac{1}{2})^2(1 - i), & B = (N_0 - 2 + \frac{1}{2})^2(1 + i), \\ C = (N_0 - 2 + \frac{1}{2})^2(-1 + i), & D = (N_0 - 2 + \frac{1}{2})^2(-1 - i). \end{array}$$

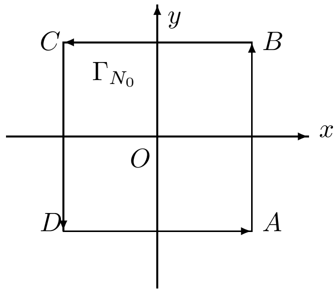

Figure. Contour Γ_{N_0} in λ-complex plane.

To obtain the trace formula we need the following lemma.

Lemma 3.1 *(see [15]) For N_0 large enough, on the contour Γ_{N_0}, there holds uniformly for $t \in [0,\pi]$:*

$$\left|\frac{\sin(\lambda t)}{\sin(\lambda\pi)}\right| \le 4, \qquad \left|\frac{\sin(\lambda t)}{\cos(\lambda\pi)}\right| \le 4. \tag{3.8}$$

Proof

For λ on the side AB, let $\lambda = N_0 - \frac{3}{2} + i\tau,\ -(N_0-\frac{3}{2}) \le \tau \le N_0 - \frac{3}{2}$. Then we have

$$\left|\frac{\sin(\lambda t)}{\sin(\lambda\pi)}\right| = \left|\frac{\sin((N_0-\frac{3}{2}+i\tau)t)}{\sin((N_0-\frac{3}{2}+i\tau)\pi)}\right| = \left|\frac{e^{-\tau t+i(N_0-\frac{3}{2})t}-e^{\tau t-i(N_0-\frac{3}{2})t}}{e^{-\tau\pi+i(N_0-\frac{3}{2})\pi}-e^{\tau\pi-i(N_0-\frac{3}{2})\pi}}\right|.$$

Since

$$\begin{aligned}\left|e^{-\tau t+i(N_0-\frac{3}{2})t}-e^{\tau t-i(N_0-\frac{3}{2})t}\right| &\le \left|e^{-\tau t+i(N_0-\frac{3}{2})t}\right| + \left|e^{\tau t-i(N_0-\frac{3}{2})t}\right| \\ &= e^{-\tau t}+e^{\tau t}, \\ \left|e^{-\tau\pi+i(N_0-\frac{3}{2})\pi}-e^{\tau\pi-i(N_0-\frac{3}{2})\pi}\right| &= e^{\tau\pi}+e^{-\tau\pi},\end{aligned}$$

for λ on the side AB, we get

$$\left|\frac{\sin(\lambda t)}{\sin(\lambda\pi)}\right| \le \frac{e^{-\tau t}+e^{\tau t}}{e^{-\tau\pi}+e^{\tau\pi}} = \frac{e^{|\tau|t}+e^{-|\tau|t}}{e^{|\tau|\pi}+e^{-|\tau|\pi}} = \frac{e^{|\tau|(t-\pi)}+e^{-|\tau|(t+\pi)}}{1+e^{-2|\tau|\pi}} \le e^{|\tau|(t-\pi)}+e^{-|\tau|(t+\pi)} < 2.$$

As λ locates on the side BC, let $\lambda = \sigma + i(N_0-\frac{3}{2}),\ -(N_0-\frac{3}{2}) \le \sigma \le N_0-\frac{3}{2}$. Then we have

$$\left|\frac{\sin(\lambda t)}{\sin(\lambda\pi)}\right| = \left|\frac{\sin(\sigma t+i(N_0-\frac{3}{2})t)}{\sin(\sigma\pi+i(N_0-\frac{3}{2})\pi)}\right| = \left|\frac{e^{-(N_0-\frac{3}{2})t+i\sigma t}-e^{(N_0-\frac{3}{2})t-i\sigma t}}{e^{-(N_0-\frac{3}{2})\pi+i\sigma\pi}-e^{(N_0-\frac{3}{2})\pi-i\sigma\pi}}\right|.$$

Since

$$\begin{aligned}\left|e^{-(N_0-\frac{3}{2})t+i\sigma t}-e^{(N_0-\frac{3}{2})t-i\sigma t}\right| &\le e^{(N_0-\frac{3}{2})t}+e^{-(N_0-\frac{3}{2})t}, \\ \left|e^{-(N_0-\frac{3}{2})\pi+i\sigma\pi}-e^{(N_0-\frac{3}{2})\pi-i\sigma\pi}\right| &\ge e^{(N_0-\frac{3}{2})\pi}-e^{-(N_0-\frac{3}{2})\pi},\end{aligned}$$

we have

$$\left|\frac{\sin(\lambda t)}{\sin(\lambda\pi)}\right| \le \frac{e^{(N_0-\frac{3}{2})t}+e^{-(N_0-\frac{3}{2})t}}{e^{(N_0-\frac{3}{2})\pi}-e^{-(N_0-\frac{3}{2})\pi}} = \frac{e^{(N_0-\frac{3}{2})(t-\pi)}+e^{-(N_0-\frac{3}{2})(t+\pi)}}{1-e^{-(2N_0-3)\pi}}.$$

Since $\lim_{N_0\to\infty} e^{-(2N_0-3)\pi}=0$, there holds $e^{-(2N_0-3)\pi}<\frac{1}{2}$ for N_0 large enough. Thus, for N_0 large enough, $1-e^{-(2N_0-3)\pi}>\frac{1}{2}$. Taking into account $e^{(N_0-\frac{3}{2})(t-\pi)}+e^{-(N_0-\frac{3}{2})(t+\pi)}<2$ as $0\le t\le\pi$, for λ on the side BC, there holds uniformly for $t\in[0,\pi]$:

$$\left|\frac{\sin(\lambda t)}{\sin(\lambda\pi)}\right|\le 4.$$

For λ on the side CD and the side DA, the same conclusions are true. Applying a similar method, for N_0 large enough, on the contour Γ_{N_0}, we obtain $\left|\frac{\cos(\lambda t)}{\sin(\lambda\pi)}\right|\le 4$ for $t\in[0,\pi]$. □

By Rouché's theorem, we can obtain a proof of the theorem 2.1 and omit it. Now we can give a proof of theorem 2.2.

Proof of Theorem 2.2

Asymptotic formula (2.4) imply that, for all sufficiently large N_0, the numbers λ_n which are the zeros of the function $\omega(\lambda)$, with $n\le N_0$, are inside Γ_{N_0} and the number λ_n, with $n>N_0$ are outside Γ_{N_0}.

Obviously, $\mu_n=n^2$, which are the zeros of function $\omega_0(\lambda)$, don't lie on the contour Γ_{N_0}.

Combining (3.5), (3.7) and (3.8), and arranging the terms on the right-hand side in decreasing order of powers of λ gives

$$\begin{aligned}\frac{\omega(\lambda)}{\omega_0(\lambda)}&=\det\left[I_d-\frac{\cot\sqrt{\lambda}\pi}{\sqrt{\lambda}}\omega-\frac{\omega_1}{\lambda}+O\left(\frac{1}{\lambda\sqrt{\lambda}}\right)\right]\\ &=\prod_{i=1}^{d}\left[1-\omega_{ii}\frac{\cot\sqrt{\lambda}\pi}{\sqrt{\lambda}}-\frac{\omega_{1,ii}}{\lambda}+O\left(\frac{1}{\lambda\sqrt{\lambda}}\right)\right]+a\frac{\cot^2\sqrt{\lambda}\pi}{\lambda}+O\left(\frac{1}{\lambda\sqrt{\lambda}}\right)\\ &=1-\sum_{i=1}^{d}\omega_{ii}\frac{\cot\sqrt{\lambda}\pi}{\sqrt{\lambda}}-\sum_{i=1}^{d}\omega_{1,ii}\frac{1}{\lambda}+\sum_{i<j}^{d}\omega_{ii}\omega_{jj}\frac{\cot^2\sqrt{\lambda}\pi}{\lambda}\\ &\quad+a\frac{\cot^2\sqrt{\lambda}\pi}{\lambda}+O\left(\frac{1}{\lambda\sqrt{\lambda}}\right)\qquad\text{on }\Gamma_{N_0}.\end{aligned}$$

Expanding $\log\frac{\omega(\lambda)}{\omega_0(\lambda)}$ by the Maclaurin formula, we find that on Γ_{N_0}

$$\begin{aligned}\log\frac{\omega(\lambda)}{\omega_0(\lambda)}&=-\sum_{i=1}^{d}\omega_{ii}\frac{\cot\sqrt{\lambda}\pi}{\sqrt{\lambda}}-\sum_{i=1}^{d}\omega_{1,ii}\frac{1}{\lambda}\\ &\quad+\left(a-\frac{1}{2}\sum_{i=1}^{d}\omega_{ii}^2\right)\frac{\cot^2\sqrt{\lambda}\pi}{\lambda}+O\left(\frac{1}{\lambda\sqrt{\lambda}}\right).\end{aligned}\tag{3.9}$$

By residue theorem, it follows that

$$\begin{aligned}\sum_{\Gamma_{N_0}}\left(\lambda_n^{(j)}-\mu_n^{(j)}\right)&=\frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\lambda\left[\frac{\omega'(\lambda)}{\omega(\lambda)}-\frac{\omega_0'(\lambda)}{\omega_0(\lambda)}\right]d\lambda\\ &=\frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\lambda d\log\frac{\omega'(\lambda)}{\omega_0(\lambda)}=-\frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\log\frac{\omega(\lambda)}{\omega_0(\lambda)}d\lambda,\end{aligned}\tag{3.10}$$

where $\lambda_n^{(j)}$ are zeros of entire functions $\omega(\lambda)$ inside the contour Γ_{N_0} listed with multiplicity, respectively.

Using well-known formulae

$$\cot z=\frac{1}{z}+2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2\pi^2},\qquad \csc^2 z=\sum_{n=-\infty}^{\infty}\frac{1}{(z+n\pi)^2},$$

we get

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\frac{\cot\sqrt{\lambda}\pi}{\sqrt{\lambda}}d\lambda&=\frac{2(N_0-2)+1}{\pi},\\ \frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\frac{\cot^2\sqrt{\lambda}\pi}{\lambda}d\lambda&=-1+O(\frac{1}{N_0})\end{aligned}\tag{3.11}$$

and for large N_0

$$\left|\oint_{\Gamma_{N_0}} O\left(\frac{1}{\lambda\sqrt{\lambda}}\right) d\lambda\right| = O\left(\frac{1}{N_0}\right). \tag{3.12}$$

Thus, from (3.9), (3.10), (3.11) and (3.12), we have

$$\begin{aligned}
&\sum_{j=1}^{d}\left(\lambda_0^{(j)}+\lambda_1^{(j)}\right)+\sum_{n=2}^{N_0}\sum_{j=1}^{d}\left(\lambda_n^{(j)}-(n-2)^2\right)\\
&=-\tfrac{1}{2\pi i}\oint_{\Gamma_{N_0}}\log\tfrac{\omega(\lambda)}{\omega_0(\lambda)}d\lambda\\
&=\tfrac{1}{2\pi i}\oint_{\Gamma_{N_0}}\left[\sum_{i=1}^{d}\omega_{ii}\tfrac{\cot\sqrt{\lambda}\pi}{\sqrt{\lambda}}+\sum_{i=1}^{d}\omega_{1,ii}\tfrac{1}{\lambda}\right.\\
&\quad\left.-\left(a-\tfrac{1}{2}\sum_{i=1}^{d}\omega_{ii}^2\right)\tfrac{\cot^2\sqrt{\lambda}\pi}{\lambda}+O\left(\tfrac{1}{\lambda\sqrt{\lambda}}\right)\right]d\lambda\\
&=\tfrac{2(N_0-2)+1}{\pi}\sum_{i=1}^{d}\omega_{ii}+\sum_{i=1}^{d}\omega_{1,ii}+a-\tfrac{1}{2}\sum_{i=1}^{d}\omega_{ii}^2+O\left(\tfrac{1}{N_0}\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sum_{j=1}^{d}\left(\lambda_0^{(j)}+\lambda_1^{(j)}\right)+\sum_{n=2}^{N_0}\left[\sum_{j=1}^{d}\left(\lambda_n^{(j)}-(n-2)^2\right)-\tfrac{2}{\pi}\sum_{i=1}^{d}\omega_{ii}\right]\\
&=\tfrac{1}{\pi}\sum_{i=1}^{d}\omega_{ii}+\sum_{i=1}^{d}\omega_{1,ii}+a-\tfrac{1}{2}\sum_{i=1}^{d}\omega_{ii}^2+O\left(\tfrac{1}{N_0}\right).
\end{aligned}\tag{3.13}$$

Passing to the limit as $N_0 \to \infty$ in (3.13), we find that

$$\begin{aligned}
&\sum_{j=1}^{d}\left(\lambda_0^{(j)}+\lambda_1^{(j)}\right)+\sum_{n=2}^{\infty}\left[\sum_{j=1}^{d}\left(\lambda_n^{(j)}-(n-2)^2\right)-\tfrac{2}{\pi}\sum_{i=1}^{d}\omega_{ii}\right]\\
&=\tfrac{1}{\pi}\sum_{i=1}^{d}\omega_{ii}+\sum_{i=1}^{d}\omega_{1,ii}+a-\tfrac{1}{2}\sum_{i=1}^{d}\omega_{ii}^2.
\end{aligned}$$

From (3.3) and (3.6), it yields that

$$\begin{aligned}
&\sum_{i=1}^{d}\omega_{ii}=\mathrm{tr}\left[L+H+\tfrac{1}{2}\int_0^{\pi}Q(t)dt\right],\\
&\sum_{i=1}^{d}\omega_{1,ii}=\mathrm{tr}\left[L_1+H_1+HL+\tfrac{Q(0)+Q(\pi)}{4}\right.\\
&\qquad\qquad\left.+\tfrac{1}{2}\int_0^{\pi}Q(t)dt(L+H)+\tfrac{1}{8}\left(\int_0^{\pi}Q(t)dt\right)^2\right],\\
&\tfrac{1}{\pi}\sum_{i=1}^{d}\omega_{ii}+\sum_{i=1}^{d}\omega_{1,ii}+a-\tfrac{1}{2}\sum_{i=1}^{d}\omega_{ii}^2=\\
&\qquad\qquad\tfrac{\mathrm{tr}(Q(0)+Q(\pi))}{4}-\tfrac{1}{2\pi}\mathrm{tr}\int_0^{\pi}Q(t)dt+\mathrm{tr}\left(L_1+H_1\right)\\
&\qquad\qquad-\tfrac{1}{\pi}\mathrm{tr}(L+H)-\tfrac{1}{2}\mathrm{tr}(L^2+H^2)-\sum_{i<j}\omega_{ij}\omega_{ji},
\end{aligned}$$

hence we find that the formula (2.6) holds. The proof of the theorem is finished. □

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