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Blow-up phenomena for nonlocal inhomogeneous diffusion problems

Jian-Wen SUN,* Fei-Ying YANG
School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, P. R. China

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Abstract: This paper is concerned with the blow-up of solutions to some nonlocal inhomogeneous dispersal equations subject to homogeneous Neumann boundary conditions. We establish conditions on nonlinearities sufficient to guarantee that solutions exist for all time as well as blow up at some finite time. Moreover, lower bounds for blow-up time of nonlocal problems are obtained.

Key words: Blow-up solutions, bounds on blow-up time, nonlocal dispersal, comparison principle

1. Introduction and main results

During the past twenty years, the nonlocal diffusion equation of the form

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x,y)u(y,t)dy - u(x,t),$$

and variations of it, have been widely used to model diffusion process. The research on the nonlocal diffusion equations has attracted some attention; see [16, 3, 2, 8, 23, 6, 5, 4, 15, 25]. For the study of evolution equations, one of the most remarkable properties is to consider the blow-up problems; that is, the solutions may become unbounded in finite time, and such phenomena are known as blow-up in the literature. If our model describes a physical or chemical reaction process which may become discontinuous before blow-up, then the bounds for blow-up time are very useful and a lower bound gives a safe time interval for operation or reaction. A variety of methods have been used to study the questions like existence and nonexistence of global solutions, blow-up solutions, upper estimates of blow-up solutions, blow-up times, blow-up rates and asymptotic behaviors of classical parabolic problems (see [26, 22, 24]), but there seems to have been relatively little work devoted to the study of the questions mentioned above for nonlocal evolution problems, especially bounds for blow-up time. In this paper, we consider the blow-up problem for a class of nonlocal diffusion equation

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g^N(y)}dy - u(x,t) + f(u(x,t)), & x \in \mathbb{R}^N,\ t \in (0,T), \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.1}$$

Here $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative, bounded, symmetric ($J(z) = J(-z)$) function with unit integral. The reaction term f is a positive $C^2(\mathbb{R})$ function, and the initial datum φ is continuous and non-trivial. Note that

*Correspondence: jiveysean@gmail.com
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the diffusion kernel of form in (1.1) was developed in [10, 12] the authors studied the linear problems. We can see such diffusion kernel is different from convolution type kernel as considered in [8, 23]. Let us notice that

$$f(u(x,t)) = \int_{\mathbb{R}^N} J\left(\frac{x-y}{g(x)}\right)\frac{f(u(x,t))}{g^N(x)}dy; \tag{1.2}$$

therefore, the term $f(u(x,t))$ can be interpreted as a force which increases the dispersal to x from another place.

For the case $g(y)=1$, for all $y \in \mathbb{R}^N$, then (1.1) reduces to the convolution type nonlocal problem

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y,t)dy - u(x,t) + f(u(x,t), \\ u(x,0) = \varphi(x). \end{cases} \tag{1.3}$$

The nonlocal diffusion equation (1.3) is very important in the study of image processing, particle systems, coagulation models etc. (see [7, 3, 14]). In this case, under the homogeneous Neumann boundary conditions on a smooth bounded domain, Boni and Kouakou [6] studied equation (1.3) by means of a semi-discrete form and proved that the semi-discrete blow-up time converges to the blow-up time of (1.3) in some sense. For the case that $g(y)=1$, for all $y\in\mathbb{R}^N$, and $f(s)=s^p$, $p>0$, Pérez Llanos and Rossi [23] studied the blow-up problems and gave some numerical experiments which illustrate their results. However, there is little work about lower bounds for blow-up times of the nonlocal diffusion problems (1.3). We refer to [18, 19, 20, 21, 27] for general traveling wave problems on (1.3).

The purpose of this paper is to study the blow-up phenomena of (1.1) on a smooth bounded domain $\Omega \subset \mathbb{R}^N$, prescribing homogeneous Neumann boundary conditions

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\Omega}\left[J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g^N(y)} - J\left(\frac{x-y}{g(x)}\right)\frac{u(x,t)}{g^N(x)}\right]dy + f(u(x,t)), \\ u(x,0) = \varphi(x), \end{cases} \tag{1.4}$$

where $x\in\Omega$ and $t\in(0,T)$. In this model, we have that diffusion takes place only in Ω. The individuals may not enter or leave Ω. This is the analogous of what is called homogeneous Neumann boundary conditions; see [11, 8]. As we consider the inhomogeneous diffusion problem, the methods used to study the blow-up problem of (1.3) are no longer valid [23, 6]. We consider a comparison principle for the inhomogeneous problem and obtain lower bounds for blow-up time for the nonlocal problem for the first time in this paper. Also, for our nonlocal evolution problem (1.4), methods used to study the classical Laplace diffusion are not valid. For general references on blow-up phenomena of parabolic and hyperbolic partial differential equations as well as systems, one can see [4, 17, 9, 26] and the references therein.

Throughout this paper, we make the following assumptions:

(i) $g\in L^\infty(\mathbb{R}^N)$, $0<\alpha\le g\le\beta<\infty$, for some positive constants α and β;

(ii) J is strictly positive in $B(0,d)$ and vanishes in $\mathbb{R}^N\backslash B(0,d)$.

Our first result states the existence and uniqueness of solutions of (1.4).

Theorem 1.1 *For every $\varphi \in C(\bar{\Omega})$, there exists a time $T > 0$ and a unique solution $u \in C^1([0,T); C(\bar{\Omega}))$ to* (1.4)*. Moreover, if the maximal interval of existence, T, is finite, then the solution blows up in the sense*

$$\lim_{t \to T-} \max_{\bar{\Omega}} |u(x,t)| = \infty.$$

Also, the total mass satisfies

$$\int_\Omega u(x,t)dx = \int_\Omega \varphi(x)dx + \int_0^t \int_\Omega f(u(x,s))dxds.$$

Theorem 1.2 *(Comparison Principle) Let $u, v \in C^1([0,\infty); C(\bar{\Omega}))$ be two continuous solutions of* (1.4) *when $f = 0$, and*

$$u(x,0) \le v(x,0), \; for\, all\, x \in \bar{\Omega}.$$

Then $u(x,t) \le v(x,t), \; for\, all\, (x,t) \in \bar{\Omega} \times [0,T]$.

We have ensured the existence and uniqueness; next we consider the blow-up problem with f satisfies the increasing condition

$$0 \le f(s) \le a_1 + a_2 s^p, \quad s \ge 0, \tag{1.5}$$

for some constants $a_1 \ge 0$, $a_2 > 0$ and $p > 0$.

We say the solution of (1.4) blows up provided there exists a time $0 < T < +\infty$, called blow-up time, such that the solution u is well defined for all time $0 < t < T$, while $\limsup_{t \to T-} |u(x,t)| \to +\infty$. Meanwhile, we say the solution u is global if $T = +\infty$.

Theorem 1.3 *Assume that $\varphi(x)$ is continuous and nonnegative, f satisfies* (1.5)*, if u is a solution of* (1.4) *when $0 < p \le 1$, then u must be global.*

Once the blow-up occurs, the precise blow-up time is very important due to realistic means. We give a lower bound first and then consider a blow-up criterion; meanwhile, an upper bound for blow-up time is also obtained.

Theorem 1.4 *(Lower bounds for blow-up time). Let φ and f satisfy the hypotheses of Theorem 1.3 and $p > 1$, u be a solution of* (1.4) *which blows up at finite time T. Then a lower bound for T is given by*

$$T \ge \frac{1}{(p-1)(C+a_1)} \ln\left(1 + \frac{C + a_1}{a_2(\varphi_M + 1)^{p-1}}\right),$$

for some positive constant C.

Theorem 1.5 *If u is a nonnegative solution of* (1.4)*, and $f(s) = as^p$, for $p > 1$, $a > 0$, then u must blow up at some finite time T. Moreover, when $a > \beta$, an upper bound for T is given by*

$$T \le \frac{2}{p-1} \ln\left(1 + \frac{\beta}{a - \beta}\right).$$

In the case that $0 < a \leq \beta$, then T is bounded above by

$$T \leq \frac{1}{a(p-1)} \left(\frac{|\Omega|}{\int_\Omega \varphi(x,t)dx} \right)^{p-1},$$

where β is a fixed positive constant.
Meanwhile, a lower bound for T is given by

$$T \geq \frac{1}{C(p-1)} \ln \left(1 + \frac{C}{a(\varphi_M + 1)^{p-1}} \right);$$

here the constant C is given in Theorem 1.4.

This paper is organized as follows. In Section 2, we prove the existence and uniqueness and regularity of solutions of (1.2), then we give some comparison principles and preliminaries. By considering auxiliary functions, we prove Theorem 1.3 in Section 3. Section 4 is devoted to Theorem 1.4; different inequalities and auxiliary functions are used to obtain low bounds for blow-up time. Section 5 is devoted to give a criterion for blow-up and the proof of Theorem 1.5. In the last Section, we apply our results to a single population model.

2. Local existence, uniqueness and comparison principles

We first establish the existence and uniqueness of solutions of (1.4) via Banach's fixed point theorem. The problem (1.4) can be written as

$$\begin{cases} u_t(x,t) = \int_\Omega J\left(\frac{x-y}{g(y)}\right) \frac{u(y,t)}{g^N(y)} dy - \int_\Omega J\left(\frac{x-y}{g(x)}\right) \frac{u(x,t)}{g^N(x)} dy + f(u(x,t)), \\ u(x,0) = \varphi(x), \end{cases} \tag{2.1}$$

for all $x \in \bar{\Omega},\ t \in (0,T)$.

Consider the Banach space

$$X_{t_0} = C([0,t_0]; C(\bar{\Omega})),$$

with the norm

$$\|\omega\| = \max_{0 \leq t \leq t_0} \max_{\bar{\Omega}} |\omega(x,t)|.$$

Fix $t_0 > 0$, let $Y = \{\omega | \, \omega \in X_{t_0},\ \|\omega\| \leq 2\|\varphi\|\}$, with the induced metric

$$d(u,v) = \|u - v\|, \ \textit{for every } u, v \in Y.$$

Then it is easy to check that $(Y, \|\cdot\|)$ is a complete metric space.

Define

$$\begin{aligned} \mathfrak{D}(\omega)(x,t) = & \varphi(x) + \int_0^t \int_\Omega J\left(\frac{x-y}{g(y)}\right) \frac{\omega(y,s)}{g^N(y)} dyds \\ & - \int_0^t \int_\Omega J\left(\frac{x-y}{g(x)}\right) \frac{\omega(x,s)}{g^N(x)} dyds + \int_0^t f(\omega(x,s))ds. \end{aligned}$$

By the previous preparations, we will obtain solutions of (2.1) as a fixed point of the operator $\mathfrak{D}(\cdot)$.

Proof [Proof of Theorem 1.1 (Existence and Uniqueness)] We only need to show that the operator $\mathfrak{D}$ maps Y to Y and is strictly contractive. By Banach's fixed point theorem, the solution of (2.1) will be obtained in Y.
Step 1. $\mathfrak{D}$ maps Y to Y. To see this, we take $\omega \in Y$, then

$$\begin{aligned}\|\mathfrak{D}(\omega)\| \leq &\|\varphi\| + \max_{0\leq t\leq t_0} \max_{\bar{\Omega}} \left| \int_0^t \int_\Omega J\left(\frac{x-y}{g(y)}\right) \frac{\omega(y,s)}{g^N(y)} dyds \right| \\ &+ \max_{0\leq t\leq t_0} \max_{\bar{\Omega}} \left| \int_0^t \int_\Omega J\left(\frac{x-y}{g(x)}\right) \frac{\omega(x,s)}{g^N(x)} dyds \right| + \max_{0\leq t\leq t_0} \max_{\bar{\Omega}} \left| \int_0^t f(\omega(x,s))ds \right|, \\ \leq &\|\varphi\| + 4\alpha^{-N}|\Omega|\|J\|_\infty \|\varphi\| t_0 + \|f\|_{max} t_0, \\ = &\|\varphi\| + t_0(4\alpha^{-N}|\Omega|\|J\|_\infty\|\varphi\| + \|f\|_{max}).\end{aligned}$$

Here $\|f\|_{\max} = \{\max_{[-2\|\varphi\|, 2\|\varphi\|]} f(s)\}$ and $|\Omega|$ denote the Lebesgue measure of domain Ω, $\|J\|_\infty = \max_{\bar{\Omega}} J(x)$. For the bounded of $\|f\|_{\max}$ and $|\Omega|$, we take t_0 small enough, say $t_0 < t_1$, where

$$t_1 = t_1(\|\varphi\|) = \frac{\|\varphi\|}{4\alpha^{-N}|\Omega|\|J\|_\infty + \|f\|_{\max}}.$$

Then it is easy to see that $\|\mathfrak{D}(\omega)\| \leq 2\|\varphi\|$.
Step 2. For any $\omega_1, \omega_2 \in Y$, by directly computing, we have

$$\begin{aligned}d(\mathfrak{D}(\omega_1), \mathfrak{D}(\omega_2)) = &\|\mathfrak{D}(\omega_1) - \mathfrak{D}(\omega_2)\| \\ \leq &2\alpha^{-N}|\Omega|\|J\|_\infty\|\omega_1-\omega_2\|t_0 + |f'(\theta)|\|\omega_1 - \omega_2\|t_0 \\ = &(2\alpha^{-N}|\Omega|\|J\|_\infty + |f'(\theta)|)\|\omega_1-\omega_2\|t_0 \\ = &(2\alpha^{-N}|\Omega|\|J\|_\infty + |f'(\theta)|)t_0 d(\omega_1,\omega_2),\end{aligned}$$

where $\theta \leq \max\{\|\omega_1\|, \|\omega_2\|\} \leq 2\|\varphi\|$. For the bounded of $|f'(\theta)|$, we take

$$t_0 < t_2(\|\varphi\|) = \frac{1}{2\alpha^{-N}|\Omega|\|J\|_\infty + |f'(\theta)|},$$

so that $\mathfrak{D}(\omega)$ is strictly contractive. To end the proof, we only need to take $t_0 < \min(t_1, t_2)$. Then we obtain the unique solution of (2.1).

If $u(\cdot, t_0)$ is also finite in Y, we can take it as an initial datum and do as before; it is easy to extend the solution to the maximal existence of the interval, say $[0, T)$. If $T = T(\|\varphi\|)$ is finite, then the solutions of (2.1) must blow up in the following sense:

$$\lim_{t\to T-} \max_{\bar{\Omega}} |u(x,t)| = \infty.$$

Finally, integrating on (2.1) with x and t, respectively, by Fubini's theorem

$$\int_\Omega u(x,t)dx = \int_\Omega \varphi(x)dx + \int_0^t \int_\Omega f(u(x,s))dxds.$$

□

Remark 2.1 *The function* u *is a solution of* (2.1) *if and only if*

$$\begin{aligned} u(x,t) = e^{-H(x)t}\varphi(x) + \int_0^t \int_\Omega e^{-H(x)(t-s)} J\left(\frac{x-y}{g(y)}\right) \frac{u(y,s)}{g^N(y)} dyds \\ + \int_0^t e^{-H(x)(t-s)} f(u(x,s))ds. \end{aligned} \tag{2.2}$$

From our basic assumption (ii), it is easy to see that

$$H(x) = \int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{dy}{g^N(x)} = \int_{\frac{\Omega-x}{g(x)}} J(y)dy \geq \gamma > 0,$$

for a certain constant γ.

Proof [Proof of Theorem 1.1 (Regularity)] We will prove that solutions of (2.1) are differential with respect to time variable. By virtues of existence and uniqueness, we can solve

$$\begin{cases} v_t(x,t) = e^{-H(x)t} \int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{v(y,t)}{g^N(y)}dy - \int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{v(x,t)}{g^N(x)}dy + f'(u(x,t))v(x,t), \\ v(x,0) = -\varphi(x)H(x) + f(\varphi(x)) + \int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{\varphi(y)}{g^N(y)}dy. \end{cases} \tag{2.3}$$

In fact, there exists a unique solution $v \in C([0,T]; C(\bar{\Omega}))$ of (2.3); analogous to the previous remark, we know that v satisfies

$$\begin{aligned} v(x,t) = & -e^{-H(x)t}\varphi(x)H(x) + e^{-H(x)t}f(\varphi(x)) + e^{-H(x)t}\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{\varphi(y)}{g^N(y)}dy \\ & + \int_0^t e^{-H(x)(t-s)} f'(u(x,s))v(x,s)ds + e^{-H(x)t}\int_0^t\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{v(y,s)}{g^N(y)}dyds. \end{aligned}$$

Define $\omega(x,t) = \varphi(x) + \int_0^t v(x,s)ds$; we only need to show that $\omega(x,t) = u(x,t)$. We use the fact that

$$\begin{aligned} & \int_0^t e^{-H(x)(t-s)} f(\omega(x,s))ds \\ = & f(\varphi(x))\int_0^t e^{-H(x)s}ds + \int_0^t\int_0^s e^{-H(x)(s-z)} f'(\omega(x,z))v(x,z)dzds. \end{aligned}$$

We know

$$\begin{aligned} \omega(x,t) = & e^{-H(x)t}\varphi(x) + \int_0^t e^{-H(x)s}\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{\omega(y,s)}{g^N(y)}dyds \\ & + \int_0^t e^{-H(x)(t-s)} f(\omega(x,s))ds + \int_0^t\int_0^s e^{-H(x)(s-z)} f'(u(x,z))v(x,z)dzds \\ & - \int_0^t\int_0^s e^{-H(x)(s-z)} f'(\omega(x,z))v(x,z)dzds. \end{aligned}$$

Following the above facts, we have for $0 < t < T$, there holds

$$\begin{aligned}
&|\omega(x,t) - u(x,t)| \\
=&\int_0^t e^{-H(x)(t-s)}|f(\omega(x,s)) - f(u(x,s))|ds \\
&+\int_0^t \left| e^{-H(x)s}\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{\omega(y,s)}{g^N(y)}dy - e^{-H(x)(t-s)}\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u(y,s)}{g^N(y)}dy\right| ds \\
&+\int_0^t e^{-H(x)(t-s)}|f'(u(x,s)) - f'(\omega(x,s))||v(x,s)|ds \\
\leq& C\int_0^t |u(x,s) - \omega(x,s)|ds,
\end{aligned}$$

for some constant $C > 0$, by Gronwall's inequality, $u(x,s) = \omega(x,s)$; thus, $u(x,t)$ is differential with respect to time invariable t. □

Lemma 2.2 *If $\varphi(x) \in C(\bar{\Omega})$, $\varphi(x) \geq 0$ and $u \in C(\bar{\Omega} \times [0,T))$ is a solution of* (2.1). *Then $u(x,t) \geq 0$ for any $(x,t) \in \bar{\Omega} \times [0,T)$.*

Proof By the previous remark, we know that u satisfies (2.2). Assume to the contrary that $u(x,t)$ is negative somewhere; since $\varphi(x)$ is non-trivial and $u(x,t)$ is continuous in $\bar{\Omega} \times [0,T)$. Let T_0 be the first quantity such that $T_0 \in (0,T)$ and

$$u(x_0,T_0) = 0 \quad and \quad u(x,t) > 0 \quad in \quad \bar{\Omega} \times [0,T_0],$$

for a certain $x_0 \in \bar{\Omega}$. Then there holds

$$\begin{aligned}
0 =&u(x_0,T_0) \\
=&e^{-H(x_0)T_0}\varphi(x_0) + \int_0^{T_0}\int_\Omega e^{-H(x_0)(T_0-s)}J\left(\frac{x_0-y}{g(y)}\right)\frac{u(y,s)}{g^N(y)}dyds \\
&+\int_0^{T_0} e^{-H(x_0)(T_0-s)}f(u(x_0,s))ds.
\end{aligned} \tag{2.4}$$

As $u(x,t) > 0$ in $\bar{\Omega} \times [0,T_0]$, g is strictly positive and bounded; following the previous remark, we get a contradiction since the right side of (2.4) is strictly positive. □

Remark 2.3 *When the reaction term $f = 0$, then the solution of* (2.1) *is well defined in the whole $t > 0$.*

The comparison principle is a very useful tool in studying evolution equations; before starting the proof of Theorem 1.2, we first prove the following lemma.

Lemma 2.4 *Let* $u,\, v \in C^1(\bar\Omega \times [0,T])$ *satisfying the following inequalities*

$$\begin{aligned} v_t(x,t) - &\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{v(y,t)}{g^N(y)}dy + \int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{v(x,t)}{g^N(x)}dy \\ &\geq u_t(x,t) - \int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g^N(y)}dy + \int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{u(x,t)}{g^N(x)}dy \quad in\ \bar\Omega\times[0,T], \\ v(x,0) \geq &u(x,0), \quad in\ \bar\Omega, \end{aligned} \tag{2.5}$$

Then $u(x,t) \leq v(x,t),\ for\, all\, (x,t) \in \bar\Omega \times [0,T]$.

Proof Indeed, by (2.5) we have

$$(u-v)_t(x,t) \leq \int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)-v(y,t)}{g^N(y)}dy - \int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{u(x,t)-v(x,t)}{g^N(x)}dy.$$

Multiplying the above inequality by $\mathrm{sgn}(u-v)_+(x,t)$ and integrating in $\Omega \times [0,t]$, we get

$$\begin{aligned} &\int_0^t\int_\Omega (u-v)_t(x,t)\mathrm{sgn}(u-v)_+(x,t)dxdt \\ \leq &\int_0^t\int_\Omega\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)-v(y,t)}{g^N(y)}\mathrm{sgn}(u-v)_+(x,t)dydxdt \\ &-\int_0^t\int_\Omega\int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{u(x,t)-v(x,t)}{g^N(x)}\mathrm{sgn}(u-v)_+(x,t)dydxdt, \end{aligned}$$

where

$$\mathrm{sgn}(u-u)_+(x,t) = \begin{cases} 1, & u(x,t) \geq u(x,t), \\ 0, & u(x,t) < u(x,t). \end{cases}$$

By directly computing, we have

$$\begin{aligned} &\int_0^t\int_\Omega (u-v)_t(x,t)\mathrm{sgn}(u-v)_+(x,t)dxdt \\ =&\int_0^t\int_\Omega \frac{\partial}{\partial t}\int_0^{(u-v)_+(x,t)}\mathrm{sgn}(s)dsdxdt \\ =&\int_\Omega (u-v)_+(x,t)dx - \int_\Omega (u-v)_+(x,0)dx, \end{aligned}$$

which implies that

$$\begin{aligned} &\int_\Omega (u-v)_+(x,t)dx \\ \leq &\int_\Omega (u-v)_+(x,0)dx + \int_0^t\int_\Omega\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)-v(y,t)}{g^N(y)}\mathrm{sgn}(u-v)_+(x,t)dydxdt \\ \leq &\int_\Omega (u-v)_+(x,0)dx + \int_0^t\int_\Omega (u-v)_+(y,t)dydt \\ \leq &\int_\Omega (u-v)_+(x,0)dx + \int_0^t\int_\Omega (u-v)_+(x,t)dxdt. \end{aligned}$$

Since $u(x,0) \le v(x,0)$, we know

$$\int_\Omega (u-v)_+(x,t)dx \le \int_0^t \int_\Omega (u-v)_+(x,t)dxdt.$$

Then by Gronwall's inequality, we obtain $(u-v)_+(x,t) = 0$, that is $u(x,t) \le v(x,t)$. $\square$

The proof of Theorem 1.2 directly is a corollary of Lemma 2.4. We conclude this section with the statement of comparison principles. To this end, we introduce the concept of sub–super–solutions.

Definition 2.5 *A function $u \in C^1([0,T]; C(\bar{\Omega}))$ is a super solution of* (2.1) *when $f = 0$, if it satisfies*

$$\begin{cases} u_t(x,t) \ge \int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g^N(y)}dy - \int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{u(x,t)}{g^N(x)}dy, \\ u(x,0) \ge \varphi(x). \end{cases}$$

Sub–solutions are defined similarly by reversing the inequalities.

Analogous to Lemma 2.4, we have the following statement.

Lemma 2.6 *Let $\varphi(x) \in C(\bar{\Omega})$ be nonnegative and $f = 0$, if $\check{\omega}$ and $\hat{\omega}$ are sub- and super–solutions to* (2.1) *respectively. Then we have $\hat{\omega} \ge \check{\omega}$, for every $(x,t) \in \bar{\Omega} \times [0,T]$.*

Lemma 2.7 *Let $\varphi(x)$ and f be as in Lemma 2.6, if $u(x,t)$ is a super-solution of* (2.1)*, then $u(x,t) \ge 0$, for every $(x,t) \in \bar{\Omega} \times [0,T]$.*

3. Global solutions

We devote this section to the proof of Theorem 1.3. We consider problem (2.1) under the increasing conditions (1.5), for $p \le 1$. In this case we will establish that solutions of (2.1) are global.

Proof [Proof of Theorem 1.3] We define the auxiliary function

$$\psi(t) = \int_\Omega u^2(x,t)dx + 1.$$

Using conditions (1.5), we have

$$\begin{aligned} \psi_t(t) =& 2\int_\Omega u(x,t)u_t(x,t)dx \\ \le& 2\int_\Omega\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)u(x,t)}{g^N(y)}dydx - 2\int_\Omega\int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{u^2(x,t)}{g^N(x)}dydx \\ &+ 2a_1\int_\Omega u(x,t)dx + 2a_2\int_\Omega [u(x,t)]^{p+1}dx. \end{aligned} \tag{3.1}$$

It follows by the symmetry of kernel function J that there holds

$$\begin{aligned}
&\int_\Omega\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{(u(y,t)-u(x,t))^2}{g^N(y)}dydx\\
=&\int_\Omega\int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{u^2(x,t)}{g^N(x)}dydx+\int_\Omega\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u^2(x,t)}{g^N(y)}dydx \qquad (3.2)\\
&-2\int_\Omega\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)u(x,t)}{g^N(y)}dydx.
\end{aligned}$$

Substitution of (3.2) in (3.1) results in

$$\begin{aligned}
\psi_t(t)\leq&-\int_\Omega\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{(u(y,t)-u(x,t))^2}{g^N(y)}dydx-\int_\Omega\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u^2(y,t)}{g^N(y)}dydx\\
&+\int_\Omega\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u^2(x,t)}{g^N(y)}dydx+2a_1\int_\Omega u(x,t)dx+2a_1\int_\Omega[u(x,t)]^{p+1}dx\\
\leq&\int_\Omega\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u^2(x,t)}{g^N(y)}dydx+2a_1\int_\Omega u(x,t)dx+2a_2\int_\Omega[u(x,t)]^{p+1}dx\\
\leq&\alpha^{-N}|\Omega|\|J\|_\infty\psi(t)+2a_1|\Omega|^{\frac12}\psi^{\frac12}(t)+2a_2|\Omega|^{\frac{1-p}{2}}[\psi(t)]^{\frac{1+p}{2}}.
\end{aligned}$$

Since $\psi(t)\geq 1$ and $0<p\leq 1$, then

$$\psi_t(t)\leq K\psi(t), \tag{3.3}$$

where $K=\max\left\{\alpha^{-N}|\Omega|\|J\|_\infty, 2a_1|\Omega|^{\frac12}, 2a_2|\Omega|^{\frac{1-p}{2}}\right\}$.

Accordingly

$$\psi(t)\leq e^{Kt}\left(\int_\Omega\varphi^2(x)dx+1\right). \tag{3.4}$$

We conclude from (3.4) that the solution u cannot blow up at any finite time. Thus we end the proof. □

4. Lower bounds for blow-up time

In this section, we consider the problem (2.1) under the conditions (1.5) for $p>1$. Our aim is to derive a lower bound for the blow-up time for solutions of (2.1) when it blows up at finite time T. From Theorem 1.1 and Lemma 2.2, we know if the solution blows up, then it satisfies

$$\lim_{t\to T-}\max_{\bar\Omega}u(x,t)=\infty. \tag{4.1}$$

Proof [Proof of theorem 1.4] Let u be a nonnegative solution of (2.1) and b be a fixed positive constant, and we define

$$\psi(x,t)=bu(x,t)+b.$$

Then by the basic assumption on reaction term f, we have for $t < T$

$$\begin{aligned}
\psi_t(x,t) =& b u_t(x,t)\\
\le& b\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g^N(y)}dy - b\int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{u(x,t)}{g^N(x)}dy + ba_1 + ba_2u^p(x,t)\\
\le& \int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{\psi(y,t)}{g^N(y)}dy - \int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{\psi(x,t)}{g^N(x)}dy\\
&+ b\int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{dy}{g^N(x)} + ba_1 + ba_2u^p(x,t)\\
\le& \int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{\psi(y,t)}{g^N(y)}dy - \int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{\psi(x,t)}{g^N(x)}dy\\
&+ b + ba_1 + b^{1-p}a_2\psi_M^{p-1}(t)\psi(x,t),
\end{aligned} \tag{4.2}$$

where $\psi_M(t) = \max_{\bar\Omega}\psi(x,t)$.
We define the auxiliary function

$$\omega(x,t) = \psi(x,t)\exp\left\{-b(1+a_1)\int_0^t \frac{ds}{\psi_m(s)} - b^{1-p}a_2\int_0^t \psi_M^{p-1}(s)ds\right\},$$

where $\psi_m(t) = \min_{\bar\Omega}\psi(x,t) \ge b > 0$.
By (4.2), we know

$$\begin{cases}
\omega_t(x,t) \le \displaystyle\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{\omega(y,t)}{g^N(y)}dy - \int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{\omega(x,t)}{g^N(x)}dy,\\
\omega(x,0) = b\varphi(x) + b.
\end{cases} \tag{4.3}$$

In order to obtain the lower bounds, we need to find an explicit super–solution. For simplicity, we give a super–solution of (2.1) that only depends on t. It is trivial to see that $v(t) = (b\varphi_M + b)\exp\left\{(\alpha^{-N}|\Omega|\|J\|_\infty - \gamma)t\right\}$ satisfies

$$\begin{aligned}
v_t(t) =& (\alpha^{-N}|\Omega|\|J\|_\infty - \gamma)v(t)\\
\ge& \left\{\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{dy}{g^N(y)} - \int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{dy}{g^N(x)}\right\}v(t)\\
=& \int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{v(t)}{g^N(y)}dy - \int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{v(t)}{g^N(x)}dy,
\end{aligned}$$

where $\varphi_M = \max_{\bar\Omega}\varphi(x)$.
So the function $v(t)$ satisfies

$$\begin{cases}
v_t(t) \ge \displaystyle\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{v(t)}{g^N(y)}dy - \int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{v(t)}{g^N(x)}dy,\\
v(0) = b\varphi_M + b.
\end{cases} \tag{4.4}$$

It is easy to see that $v(t)$ and $\omega(x,t)$ are a pair of super-sub solution of nonlocal inhomogeneous dispersal equation

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g^N(y)}dy - \int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{u(x,t)}{g^N(x)}dy, \quad x\in\bar{\Omega},\ t\in(0,T), \\ u(x,0) = b\varphi(x)+b, \quad x\in\bar{\Omega}. \end{cases} \tag{4.5}$$

By Theorem 1.2, we obtain that for $(x,t)\in\bar{\Omega}\times[0,T)$

$$\omega(x,t)\le v(t).$$

That is

$$\begin{aligned}&\psi(x,t)\exp\left\{-b(1+a_1)\int_0^t\frac{ds}{\psi_m(s)}-b^{1-p}a_2\int_0^t\psi_M^{p-1}(s)ds\right\}\\ &\le(b\varphi_M+b)\exp\left\{(\alpha^{-N}|\Omega|\|J\|_\infty-\gamma)t\right\}.\end{aligned}$$

Taking the maximum about x

$$\begin{aligned}&\psi_M(t)\exp\left\{-b(1+a_1)\int_0^t\frac{ds}{\psi_m(s)}-b^{1-p}a_2\int_0^t\psi_M^{p-1}(s)ds\right\}\\ &\le(b\varphi_M+b)\exp\left\{(\alpha^{-N}|\Omega|\|J\|_\infty-\gamma)t\right\},\end{aligned}$$

and so

$$\begin{aligned}&\psi_M^{p-1}(t)\exp\left\{-b(p-1)(1+a_1)\int_0^t\frac{ds}{\psi_m(s)}-b^{1-p}a_2(p-1)\int_0^t\psi_M^{p-1}(s)ds\right\}\\ &\le(b\varphi_M+b)^{p-1}\exp\left\{(\alpha^{-N}|\Omega|\|J\|_\infty-\gamma)(p-1)t\right\}.\end{aligned}$$

Since $\psi_m(s)\ge b>0$ for $s\in(0,T)$, accordingly

$$\begin{aligned}&\psi_M^{p-1}(t)\exp\left\{-b^{1-p}a_2(p-1)\int_0^t\psi_M^{p-1}(s)ds\right\}\\ &\le(b\varphi_M+b)^{p-1}\exp\left\{(\alpha^{-N}|\Omega|\|J\|_\infty-\gamma)(p-1)t+b(p-1)(1+a_1)\int_0^t\frac{ds}{\psi_m(s)}\right\}\\ &\le(b\varphi_M+b)^{p-1}\exp\left\{(\alpha^{-N}|\Omega|\|J\|_\infty-\gamma)(p-1)t+(p-1)(1+a_1)t\right\}.\end{aligned}$$

Then it follows that

$$\begin{aligned}&\psi_M^{p-1}(t)\exp\left\{-b^{1-p}a_2(p-1)\int_0^t\psi_M^{p-1}(s)ds\right\}\\ &\le(b\varphi_M+b)^{p-1}\exp\left\{(p-1)(\alpha^{-N}|\Omega|\|J\|_\infty+1-\gamma+a_1)t\right\}.\end{aligned}\tag{4.6}$$

Upon integrating about t, from 0 to τ, we have

$$\begin{aligned}&\frac{1-\exp\left\{-b^{1-p}a_2(p-1)\int_0^\tau\psi_M^{p-1}(s)ds\right\}}{b^{1-p}a_2(p-1)}\\ &\le\frac{(b\varphi_M+b)^{p-1}(\exp\left\{(p-1)(\alpha^{-N}|\Omega|\|J\|_\infty+1-\gamma+a_1)\tau\right\}-1)}{(p-1)(\alpha^{-N}|\Omega|\|J\|_\infty+1-\gamma+a_1)},\end{aligned}$$

and on taking the limit as $\tau \to T$

$$T \geq \frac{1}{(p-1)(\alpha^{-N}|\Omega|\|J\|_\infty + 1 - \gamma + a_1)} \ln\left(1 + \frac{\alpha^{-N}|\Omega|\|J\|_\infty + 1 - \gamma + a_1}{a_2(\varphi_M + 1)^{p-1}}\right). \tag{4.7}$$

To end the proof, we may take $C = \alpha^{-N}|\Omega|\|J\|_\infty + 1 - \gamma$. □

Remark 4.1 *From Remark 2.1 we known that*

$$\int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{dy}{g^N(x)} \geq \gamma > 0.$$

Since

$$\int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{dy}{g^N(x)} \leq \alpha^{-N}|\Omega|\|J\|_\infty,$$

we have $\alpha^{-N}|\Omega|\|J\|_\infty \geq \gamma > 0$*, so* $C > 0$ *and the expression in* (4.7) *is well defined for* $a_1 = 0$*.*

5. Criterion for blow-up

In this section we will consider the problem (2.1) under the condition that $f(s) = as^p$. We will establish a criterion for blow-up, and upper bounds for blow-up time are obtained.

Under the assumption $f(s) = as^p$, $p > 1$, $a > 0$, then equation (2.1) becomes

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g^N(y)}dy - \int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{u(x,t)}{g^N(x)}dy + au^p(x,t), \\ u(x,0) = \varphi(x) \geq 0, \end{cases} \tag{5.1}$$

for all $x \in \bar{\Omega}$, $t \in [0,T)$.

Proof [Proof of Theorem 1.5] Since $\varphi(x)$ is non-negative and non-trivial, there exists a constant $c > 0$, such that

$$\int_\Omega \varphi(x)dx \geq c > 0.$$

For our convenience, denote $\beta = (|\Omega|/c)^{p-1}$. We discuss the case that $a > \beta$ first. Let $n > 1$ be a fixed number and

$$\psi(t) = c\int_\Omega u^n(x,t)dx,$$

and compute as in the previous section:

$$\begin{aligned}\psi_t(t) =& cn\int_\Omega [u(x,t)]^{n-1}u_t(x,t)dx\\ =& cn\int_\Omega\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)[u(x,t)]^{n-1}}{g^N(y)}dydx\\ &- cn\int_\Omega\int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{u^n(x,t)}{g^N(x)}dydx\\ &+ acn\int_\Omega [u(x)]^{n+p-1}dx\\ \geq& - cn\int_\Omega\int_\Omega J\left(\frac{x-y}{g(x)}\right)\frac{u^n(x,t)}{g^N(x)}dydx\\ &+ acn\int_\Omega [u(x,t)]^{n+p-1}dx.\end{aligned}$$

It follows by Hölder's inequality:

$$\int_\Omega [u(x,t)]^{n+p-1}dx \geq |\Omega|^{\frac{1-p}{n}}\left(\int_\Omega u^n(x,t)dx\right)^{\frac{n+p-1}{n}}.$$

Since Ω is a bounded domain, as $\int_\Omega J(z)dz \leq 1$, we know

$$\psi_t(t) \geq -n\psi(t) + anc^{\frac{1-p}{n}}|\Omega|^{\frac{1-p}{n}}[\psi(t)]^{\frac{n+p-1}{n}}. \tag{5.2}$$

As $a > \beta$, on integrating (5.2) from 0 to t, we have

$$\begin{aligned} t \leq& \int_{\psi(0)}^{\psi(t)} \frac{ds}{-ns + an|\Omega|^{\frac{1-p}{n}}c^{\frac{1-p}{n}}s^{\frac{n+p-1}{n}}}\\ \leq& \int_{\psi(0)}^{\infty} \frac{ds}{-ns + an|\Omega|^{\frac{1-p}{n}}c^{\frac{1-p}{n}}s^{\frac{n+p-1}{n}}}\\ =& \frac{1}{(1-p)}\ln\left(1 - \frac{1}{a|\Omega|^{\frac{1-p}{n}}\left(\int_\Omega \varphi^n(x)dx\right)^{\frac{p-1}{n}}}\right)\\ =& \frac{1}{(p-1)}\ln\left(1 + \frac{1}{a|\Omega|^{\frac{1-p}{n}}\left(\int_\Omega \varphi^n(x)dx\right)^{\frac{p-1}{n}} - 1}\right).\end{aligned} \tag{5.3}$$

Again by Hölder's inequality

$$\int_\Omega \varphi^n(x)dx \geq c^n|\Omega|^{1-n}. \tag{5.4}$$

Since $\int_\Omega \varphi(x)dx \geq c > 0$, $a > (|\Omega|/c)^{p-1}$ and $n > 1$, substitution of (5.4) in (5.3) results in

$$\begin{aligned} t \leq & \frac{1}{(p-1)} \ln\left(1 + \frac{1}{a|\Omega|^{1-p}c^{p-1} - 1}\right) \\ = & \frac{1}{(p-1)} \ln\left(1 + \frac{1}{\frac{a}{\beta} - 1}\right) \\ = & \frac{1}{(p-1)} \ln\left(1 + \frac{\beta}{a-\beta}\right). \end{aligned} \tag{5.5}$$

Thus, we know that the solution u blows up, since (5.5) cannot hold for all time t, and the blow-up time T is bounded above by

$$T \leq \frac{1}{(p-1)} \ln\left(1 + \frac{\beta}{a-\beta}\right).$$

For the case that $0 < a \leq \beta$, integrating in $x \in \Omega$ on (5.1) and applying Fubini's theorem and the symmetry of J we get

$$\frac{\partial}{\partial t}\int_\Omega u(x,t)dx = a\int_\Omega u^p(x,t)dx \geq a|\Omega|^{1-p}\left(\int_\Omega u(x,t)dx\right)^p. \tag{5.6}$$

Since $p > 1$, we have that $\int_\Omega u(x,t)dx$ cannot be global, and thus u must blow up at some finite time T. Following by (5.6), we obtain

$$T \leq \frac{1}{a(p-1)}\left(\frac{|\Omega|}{\int_\Omega \varphi(x,t)dx}\right)^{p-1}.$$

At last, we give a lower bound for blow-up time T. To this end, we set

$$\begin{aligned} &\psi(x,t) = bu(x,t) + b, \\ &v(t) = (b\varphi_M + b)\exp\{(\alpha^N\|J\|_\infty|\Omega| - \gamma)t\}, \\ &\omega(x,t) = \psi(x,t)\exp\left\{-ab^{1-p}\int_0^t \psi_M^{p-1}ds - \alpha^N\|J\|_\infty|\Omega|b\int_0^t \frac{ds}{\psi_m(s)}\right\}. \end{aligned}$$

By the proof of theorem 1.3, we have

$$T \geq \frac{1}{C(p-1)} \ln\left(1 + \frac{C}{a(\varphi_M + 1)^{p-1}}\right).$$

Thus we complete the proof.

□

6. Applications

In [23], Planos and Rossi analyzed a single biological population model with nonlocal diffusion of the form

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y)(u(y,t) - u(x,t))dy + u^p(x,t), \\ u(x,0) = u_0(x). \end{cases} \tag{6.1}$$

Here J satisfies the same conditions as problem (1.1) and $p>0$. The authors proved that non-negative and non-trivial solutions blow up in finite time if and only if $p>1$; moreover, they estimated the blow-up rate and blow-up set. Also, the solutions is global if $0<p\leq 1$.

In the following, we apply our main results to (6.1).

In the case $0<p\leq 1$, let $v(x,t)=\varphi(x)+2e^{Ct}$; here $C=2\max_{\bar{\Omega}}\varphi(x)+1$, and by the proof of Theorem 1.3, we know that v is a super–solution of (6.1). Then we have

$$u(x,t)\leq \varphi(x)+2e^{Ct}.$$

In the case that $p>1$, the solution of (6.1) blows up, and by the proof of Theorem 1.4, we set

$$\begin{aligned}
\psi(x,t) &= a_2 u(x,t)+1,\\
\omega(x,t) &= \psi(x,t)\exp\left\{-\int_0^t [\psi_M(s)]^{p-1}ds\right\},\\
v(t) &= (a_2\varphi_M+1)e^{a_2 t}.
\end{aligned}$$

and give a lower bound for blow-up time.

We formulate our results as the following theorem.

Theorem 6.1 *Let $u(x,t)$ be a nonnegative solution of the problem*

$$\begin{cases} u_t(x,t)=\displaystyle\int_\Omega J(x-y)(u(y,t)-u(x,t))dy+u^p(x,t),\\ u(x,0)=\varphi(x).\end{cases}$$

Then when $0<p<1$, u is global and there exists a positive constant C, such that

$$u(x,t)\leq \varphi(x)+2e^{Ct}.$$

When $p>1$, $u(x,t)$ blows up at some finite time T. Moreover, we have the following lower bound estimate for blow-up time:

$$T\geq \frac{1}{p-1}\ln\left(\frac{1}{(\varphi_M+1)^{p-1}}+1\right).$$

Remark 6.2 *Our main methods to obtain the blow-up criterion and lower bounds for blow-up time are effective to study general nonlocal problems in material science, phase separation in binary systems, ecology, neurology and genetics (see, e.g., [13, 1]).*

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