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Commutants and hyper-reflexivity of multiplication operators

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Abstract: We characterize the commutants of some multiplication operators on a Banach space of analytic functions defined on a bounded domain in the plane. Under certain conditions on the symbol of a multiplication operator, we show that its commutant is a set of multiplication operators. This partially answers a question of Axler, Cuckovic and Rao. Next, the hyper-reflexivity of these multiplication operators are proved. The paper is concluded by proving the hyper-reflexivity of the multiplication operators with symbols $\varphi(z) = z^k$, $k = 1, 2, \ldots$.

Key words: Bergman space, multiplication operators, hyper-reflexive operator, commutant, hyper-invariant

1. Introduction and preliminaries

Throughout this paper, by an operator we mean a bounded linear operator on a Banach space $\mathcal{X}$, and the set of all operators on $\mathcal{X}$ is denoted by $B(\mathcal{X})$. The weak operator topology (WOT) on $B(\mathcal{X})$ is the topology in which a net $\{T_\alpha\}_\alpha$ converges to T if $T_\alpha x \longrightarrow Tx$ weakly, for all $x \in \mathcal{X}$. The commutant of T, denoted by $\{T\}'$, consists of all operators commuting with T. A closed subspace $\mathcal{M}$ of $\mathcal{X}$ is said to be invariant for T, if $T\mathcal{M} \subseteq \mathcal{M}$, and is called a hyper-invariant subspace for T, if it is invariant for all operators in $\{T\}'$. The lattice of all invariant subspaces of an operator T is denoted by $LatT$.

By a domain we mean a connected open subset of the complex plane $\mathbb{C}$. Let Ω be a bounded domain in the complex plane $\mathbb{C}$. Suppose that $\mathcal{X}$ is a reflexive Banach space of analytic functions on Ω such that $1 \in \mathcal{X}$, for each $\lambda \in \Omega$ the functional $e_\lambda : \mathcal{X} \longrightarrow \mathbb{C}$ of evaluation at λ given by $e_\lambda(f) = f(\lambda)$ is bounded, and if $f \in \mathcal{X}$ then $zf \in \mathcal{X}$. Note that the last condition allows us to define the operator $M_z : \mathcal{X} \longrightarrow \mathcal{X}$ by $M_z f = zf$. As an application of the closed graph theorem, it can be seen that M_z is actually a bounded operator on $\mathcal{X}$.

As usual, $H^\infty(\Omega)$ is the Banach space of all bounded analytic functions on Ω equipped with the supremum norm. An analytic function φ on Ω is said to be a multiplier for $\mathcal{X}$ if $\varphi f \in \mathcal{X}$ for all $f \in \mathcal{X}$. The set of all multipliers of $\mathcal{X}$ is denoted by $\mathcal{M}(\mathcal{X})$. It is well-known that $\mathcal{M}(\mathcal{X}) \subseteq H^\infty(\Omega)$. Every function $\varphi \in \mathcal{M}(\mathcal{X})$ induces a bounded linear operator $M_\varphi : \mathcal{X} \longrightarrow \mathcal{X}$ given by $M_\varphi f = \varphi f$.

Throughout this paper, all multiplication operators are defined on a reflexive Banach space of analytic functions satisfying the previous three conditions; furthermore, we assume that there is a constant $c > 0$ such that for every multiplier φ,

$$\|M_\varphi\| \leq c\|\varphi\|_\infty.$$

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Some spaces such as the Bergman spaces $L^p_a(\Omega)$, and the Hardy spaces $H^p(\Omega)$, $1 < p < \infty$, are examples of these Banach spaces of analytic functions.

The commutant of multiplication operators on spaces of analytic functions (especially on Bergman spaces) are discussed a lot in the literature. Shields and Wallen [13] studied the commutant of the operator M_z on a Hilbert space of analytic functions. Observe that if M_z is in $B(L^p_a(\Omega))$, then $\{M_z\}' = \{M_\varphi : \varphi \in H^\infty(\Omega)\}$. For the proof in the case $p = 2$, see page 73 of [5]; the proof of the general case follows similarly. If φ is continuous on $\overline{\Omega}$, then the commutant of M_φ on the Banach space of analytic functions is discussed in [9, 10]. Cuckovic [6] has investigated the commutant of M_{z^n} on the Bergman space $L^2_a(\mathbb{D})$. In [1], the authors have shown that if two Toeplitz operators on the Bergman space commute, and the symbol of one of them is analytic and nonconstant, then the symbol of the other one is analytic. They have posed three open problems at the end of their paper. The third one has been answered by Cao [2]. In Section 2 of this paper, we discuss the first problem:

"If an operator S in the algebra generated by the Toeplitz operators commutes with a nonconstant analytic Toeplitz operator, then is S itself Toeplitz?"
or equivalently,

"If an operator S in the algebra generated by the Toeplitz operators commutes with a nonconstant multiplication operator, then is S itself a multiplication operator?"

We will give sufficient conditions under which the answer is affirmative for operators not only on Bergman spaces but also on an arbitrary Banach space of analytic functions.

In Section 3, we turn our attention to hyper-reflexivity, which is a strong kind of reflexivity.

Recall that an operator T is said to be reflexive, if the only operators that leave invariant all of its invariant subspaces are contained in the weakly closed algebra generated by T and the identity operator I, denoted by $W(T)$. Furthermore, T is called hyper-reflexive if the only operators that leave every hyper-invariant subspace of T invariant are contained in the commutant of T. In other words, T is a hyper-reflexive operator if

$$\{S : \bigcap_{U \in \{T\}'} \mathrm{Lat}U \subseteq \mathrm{Lat}S\} \subseteq \{T\}'.$$

The reflexivity of multiplication operators on the Bergman spaces $L^p_a(\Omega)$ appeared in a work of Eschmier [7]. A question which arises is which multiplication operators on the Bergman spaces are hyper-reflexive. We will show that all multiplication operators satisfying the conditions of theorems in Section 2 are hyper-reflexive. Also, we will prove that the multiplication operators M_{z^k}, $k = 1, 2, 3, \ldots$ are all hyper-reflexive on special Banach spaces of analytic functions.

2. The commutant of multiplication operators

A sequence $(f_n)_n$ is said to converge to f pointwise boundedly in $\mathcal{X}$, if $(f_n)_n$ is a uniformly bounded sequence that converges pointwise to f.

Theorem 1 *Let φ be a univalent map in $H^\infty(\Omega)$ and $G = \varphi(\Omega)$. If there is a sequence of polynomials converging pointwise boundedly to φ^{-1}, then $\{M_\varphi\}' = \{M_\psi : \psi \in H^\infty(\Omega)\}$.*

Proof Suppose that $(p_n)_n$ is a sequence of polynomials such that $sup\|p_n\|_G \leq M$ for some constant M and $p_n(w) \longrightarrow \varphi^{-1}(w)$, for all $w \in G$. It follows that $p_n(\varphi(z)) \longrightarrow z$ for every z in Ω and $sup\|p_n o \varphi\|_\Omega \leq M$

which implies that $\|M_{p_n o\varphi}\| \leq c\|p_n o\varphi\|_\Omega \leq cM$. Since $\mathcal{X}$ is reflexive, the unit ball of $\mathcal{X}$ is weakly compact. Therefore, the unit ball of $\mathcal{B}(\mathcal{X})$ is WOT-compact. We may assume, by passing to a subsequence if necessary, that $M_{p_n o\varphi} \longrightarrow A$ (WOT) for some operator A. Thus, $M^*_{p_n o\varphi} e_z \longrightarrow A^* e_z$ in the weak star topology. It follows that

$$zf(z) = \lim_n e_z((p_n o\varphi)f) = \lim_n (M^*_{p_n o\varphi} e_z))(f) = (A^* e_z)(f) = e_z(Af) = (Af)(z)$$

for all f in $\mathcal{X}$ and for all z in Ω. But $\{M_\varphi\}' \subseteq \{M_{p_n o\varphi}\}'$ for all n; hence $\{M_\varphi\}' \subseteq \{M_z\}' = \{M_\psi : \psi \in H^\infty(\Omega)\}$. This completes the proof. □

For a bounded domain G in the plane, let G^* be the complement of the closure of the unbounded component of the complement of the closure of G. The set G^* is called the Carathéodory hull of G. Indeed, it can be described as the interior of the outer boundary of G, and, in analytic terms, it can be defined as the interior of the set of all points z_0 in the plane such that $|p(z_0)| \leq \|p\|_G$ for all polynomials p. The components of G^* are simply connected; in fact, it is a simple matter to show that each of these components has a connected complement. The component of G^* containing G is denoted by G^1. If G is bounded and $G = G^1$, then G is called a *Carathéodory region*. To get more information, we refer to Sarason's article [12].

Farrell's Theorem [12] states that if G is a bounded domain in the complex plane, and f is a bounded analytic function on G, then there is a sequence of polynomials converging pointwise boundedly to f if and only if f has a bounded analytic extension to G^1. It is worth noting that this theorem is extended from domains to arbitrary bounded open sets by Rubel and Shields (see page 151 of [8]). With these terminologies, a consequence of Theorem 1 runs as follows.

Corollary 1 *If φ is a univalent map in $H^\infty(\Omega)$ such that $\varphi(\Omega)$ is a Carathéodory region then $\{M_\varphi\}' = \{M_\psi : \psi \in H^\infty(\Omega)\}$.*

Observe that if G is a simply connected region equal to the interior points of $\overline{G}$, and if $\mathbb{C}\backslash\overline{G}$ is connected, then G is a Carathéodory region. Polynomials are in $H^\infty(\mathbb{D})$, and it can be easily seen that their images under $\mathbb{D}$ satisfy the above conditions and so are Carathéodory regions. Thus, we get the following corollary.

Corollary 2 *If $p(z)$ is a univalent polynomial on the unit disk $\mathbb{D}$ then $\{M_p\}' = \{M_\psi : \psi \in H^\infty\}$. In particular, if $n \geq 2$ then $\{M_{(z-a)^n}\}' = \{M_\psi : \psi \in H^\infty\}$ for every $|a| \geq (\sin\frac{\pi}{n})^{-1}$.*

Now, assume that G is a finitely connected domain. It is well known that G is conformally equivalent to a circular domain [3]. Recall that by a circular domain we mean any domain obtained by removing a finite number of disjoint closed subdisks from the open unit disk. So let $G = \mathbb{D}\backslash(\overline{\mathbb{D}}_1 \cup \overline{\mathbb{D}}_2 \cup \cdots \cup \overline{\mathbb{D}}_N)$ where $\overline{\mathbb{D}}_i = \{z : |z - z_i| \leq r_i\}$, $(i = 1, \cdots, N)$ are disjoint closed subdisks of the open unit disk $\mathbb{D}$. We can choose circles $\gamma_i = \{z : |z - z_i| = r_i + \varepsilon_i\}$ $(i = 1, \cdots, N)$, $\gamma_0 = \{z : |z| = 1 - \varepsilon_0\}$, and $\gamma = \{z : |z| = r\}$ in which $r < 1 - \varepsilon_0$, all lying in G and concentric with the boundary circles of G so that they do not meet each other. We denote $G_i = \mathbb{C}\backslash\overline{\mathbb{D}}_i$ $(i = 1, \cdots, N)$.

It is known that

$$H^\infty(G) = H^\infty(\mathbb{D}) + H_0^\infty(G_1) + \cdots + H_0^\infty(G_N), \tag{1}$$

where the subscript zero means that the corresponding functions vanish at ∞ (see [3]).

With the above notations, we prove the following result.

Theorem 2 *Let φ be a univalent map in $H^\infty(\Omega)$ such that $G=\varphi(\Omega)$ is a finitely connected domain. If $\int_{\gamma_0} w^n\varphi^{-1}(w)dw=0$ for every non-negative integer n, then*

$$\{M_\varphi\}^{'}=\{M_\psi:\psi\in H^\infty(\Omega)\}.$$

Proof Considering (1), we observe that $\varphi^{-1}=\phi+\psi$ for some ϕ in $H^\infty(\mathbb{D})$ and ψ in $H_0^\infty(G_1)+\ldots+H_0^\infty(G_N)$. Suppose that the Laurent series of ψ around zero in the annulus $\{z:r<|z|<\infty\}$ is $\psi(z)=\Sigma_{n=-\infty}^{\infty}c_nz^n$, where the coefficients c_n are given by

$$c_n=\frac{1}{2\pi i}\int_{\gamma_0}\frac{\psi(z)}{z^{n+1}}dz\quad(n\in\mathbb{Z}).$$

Let $D(0,1-\varepsilon_0)$ denote the open disk of radius $1-\varepsilon_0$ centered at zero. Define the function f on $D(0,1-\varepsilon_0)\backslash\{0\}$ by $f(z)=\psi(\frac{1}{z})$. Since ψ is a bounded analytic function outside $D(0,1-\varepsilon_0)$, zero is a removable singularity of f, and so $c_n=0$ for $n>1$. Furthermore, $c_0=\lim_{z\longrightarrow 0}f(z)=0$. On the other hand, since ϕ is analytic on $\mathbb{D}$, we see that

$$\int_{\gamma_0}\phi(z)z^ndz=0\quad(n=0,1,2,\cdots),$$

which, in turn, implies that

$$c_n=\frac{1}{2\pi i}\int_{\gamma_0}\varphi^{-1}(z)z^{-n-1}dz=0\quad(n<0).$$

Consequently, $\varphi^{-1}=\phi\in H^\infty(\mathbb{D})$. Now, applying Theorem 1, we conclude that $\{M_\varphi\}^{'}=\{M_\psi:\psi\in H^\infty(\Omega)\}$. □

A consequence of the above theorem runs as follows.

Corollary 3 *Let φ be a univalent map in $H^\infty(\Omega)$ and $G=\varphi(\Omega)$ be a finitely connected domain. Suppose that φ^{-1} is in the closure of polynomials in the Banach space $\mathcal{X}$ of analytic functions defined on G. Then*

$$\{M_\varphi\}^{'}=\{M_\psi:\psi\in H^\infty(\Omega)\}.$$

Proof Suppose that $(q_n)_n$ is a sequence of polynomials converging to φ^{-1} in $\mathcal{X}$ and p is an arbitrary polynomial. Since $\|pq_n-p\varphi^{-1}\|=\|M_p(q_n-\varphi^{-1})\|\leq c\|p\|_\infty\|q_n-\varphi^{-1}\|$, we see that

$$\|pq_n-p\varphi^{-1}\|\longrightarrow 0\quad \text{as } n\longrightarrow\infty. \tag{2}$$

An application of the principle of the uniform boundedness shows that for every compact subset K of G, $\sup_{\lambda\in K}\|e_\lambda\|<\infty$ and so

$$\sup_{\lambda\in K}|p(\lambda)q_n(\lambda)-p(\lambda)\varphi^{-1}(\lambda)|\leq\|pq_n-p\varphi^{-1}\|\sup_{\lambda\in K}\|e_\lambda\|.$$

This fact along with (2) shows that $pq_n\longrightarrow p\varphi_{-1}$ as $n\longrightarrow\infty$ uniformly on compact subsets of G, and especially on γ_0. Consequently,

$$\int_{\gamma_0}p(w)\varphi^{-1}(w)dw=\lim_{n\longrightarrow\infty}\int_{\gamma_0}p(w)q_n(w)dw=0.$$

Thus, the result holds by applying Theorem 2. □

Next, we focus on the commutant of M_{z^n}, $n \geq 1$.

Theorem 3 *Suppose that $n \geq 1$ and $\overline{\Omega}$ lies in an open sector with the angle $\frac{2\pi}{n}$, centered at zero. Then* $\{M_{z^n}\}^{'} = \{M_\psi : \psi \in H^\infty(\Omega)\}$.

Proof Put $\varphi(z) = z^n$. Since φ is univalent on a neighborhood of $\overline{\Omega}$, φ^{-1} is also univalent on a neighborhood of $\overline{\varphi(\Omega)}$. By Runge's Theorem [4] there is a sequence of rational functions $r_k = \frac{p_k}{q_k}$ with poles off $\overline{\varphi(\Omega)}$ converging to φ^{-1} uniformly on $\overline{\varphi(\Omega)}$. It follows that $\sup\{|(r_k \circ \varphi)(z) - z| : z \in \Omega\} \longrightarrow 0$ as $k \longrightarrow \infty$. Thus, $M_{r_k \circ \varphi}$ converges to M_z. But $M_{q_k^{-1} \circ \varphi} = (M_{q_k o \varphi})^{-1}$ and $\{M_\varphi\}^{'} \subseteq \{M_{p \circ \varphi}\}^{'}$ for every polynomial p; consequently, $\{M_\varphi\}^{'} \subseteq \{M_{r_k \circ \varphi}\}^{'}$ which implies that $\{M_\varphi\}^{'} \subseteq \{M_z\}^{'} = \{M_\psi : \psi \in H^\infty(\Omega)\}$. □

Corollary 4 *If $\overline{\Omega}$ does not intersect the real line, then* $\{M_{z^2}\}^{'} = \{M_\psi : \psi \in H^\infty(\Omega)\}$.

3. The hyper-reflexivity of multiplication operators

In this section, we discuss and introduce some hyper-reflexive multiplication operators.

Theorem 4 *If $\varphi \in H^\infty(\Omega)$ and the commutant of $M_\varphi \in B(\mathcal{X})$ is $\{M_\psi : \psi \in H^\infty(\Omega)\}$, then M_φ is a hyper-reflexive operator.*

Proof Suppose that an operator T leaves invariant every hyper-invariant subspace of M_φ on $\mathcal{X}$. For $z \in \Omega$, let e_z be the linear functional of evaluation at z on $\mathcal{X}$. Suppose that $\mathcal{M}$ is the one-dimensional span of $\{e_z\}$. For every $\psi \in H^\infty(\Omega)$, since ${M_\psi}^* e_z = \psi(z) e_z$, we conclude that ${}^\perp\mathcal{M} \in LatM_\psi$ which, in turn, shows that ${}^\perp\mathcal{M}$ is a hyper-invariant subspace of M_φ. Consequently, ${}^\perp\mathcal{M} \in LatT$ and so $\mathcal{M} \in LatT^*$. It follows that there exists a function w so that $T^* e_z = w(z) e_z$ for all $z \in \Omega$. Now, if $f \in \mathcal{X}$ then

$$(Tf)(z) = e_z(Tf) = (T^* e_z)(f) = w(z) f(z),$$

for all $z \in \Omega$. Hence, $T = M_w$ and $w \in H^\infty(\Omega)$. □

In light of the above theorem, we see that if φ satisfies the conditions of Theorems 1, 2, or 3, then M_φ is hyper-reflexive.

In the rest of this paper, let $\Omega = \mathbb{D}$ be the open unit disk. To prove the hyper-reflexivity of M_{z^k}, first we bring a lemma which is needed.

Lemma 1 *Every function f in $\mathcal{X}$ can be represented as $f = f_0 + f_1 + \ldots + f_{k-1}$, $(k \geq 1)$, in which*

$$f_j(z) = \frac{1}{k} \sum_{m=0}^{k-1} e^{\frac{-2jm\pi}{k} i} f(e^{\frac{2m\pi}{k} i} z) \quad (j = 0, 1, \ldots, k-1). \tag{3}$$

Proof Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is in $\mathcal{X}$. It is easy to see that $f(z)$ can be written as

$$f(z) = f_0(z) + f_1(z) + \ldots + f_{k-1}(z),$$

where

$$f_j(z) = \sum_{m=1}^{\infty} a_{km+j} z^{km+j} \quad (j = 0, 1, \ldots, k-1).$$

Thus, a straightforward computation shows that (3) holds. □

Suppose that the set of all analytic polynomials are dense in $\mathcal{X}$; furthermore, if $f(z)$ is a polynomial having a representation as in Lemma 1, then there are constants c_k and d_k so that

$$||f_i|| \le c_k ||f|| \le d_k \max\{||f_j|| : j = 0, 1, \ldots, k-1\}, i = 0, 1, \ldots, k-1.$$

An instance of such spaces is the Bergman space $L^2_a(\mathbb{D})$. Under these additional conditions on $\mathcal{X}$, the main theorem of this section runs as follows:

Theorem 5 *The multiplication operators M_{z^k}, $k = 1, 2, 3, \ldots$ are hyper-reflexive on $\mathcal{X}$.*

Proof Fix a positive integer k. Suppose that T is an operator which leaves invariant every hyper-invariant subspace of M_{z^k}. Take $S \in \{M_{z^k}\}'$. By Lemma 2.2 of [11], for every $f \in \mathcal{X}$,

$$Sf = f_0\varphi_0 + f_1(\frac{\varphi_1}{z} + \varphi_0) + f_2(\frac{\varphi_2}{z^2} + \varphi_0) + \cdots + f_{k-1}(\frac{\varphi_{k-1}}{z^{k-1}} + \varphi_0) \quad (4)$$

where f_j s are as in the preceding lemma, $\varphi_0 = S(1)$, and $\varphi_j = (SM_{z^j} - M_{z^j}S)(1)$ for $j = 1, 2, \ldots, k-1$. Let e_z be the linear functional of evaluation at z on $\mathcal{X}$. For every nonzero scalar z and every S in $\{M_{z^k}\}'$, using (4) we get

$$\begin{aligned}
(S^*e_z)(f) &= (Sf)(z) \\
&= f(z)\varphi_0(z) + \sum_{j=1}^{k-1} \frac{\varphi_j(z)}{z^j} f_j(z) \\
&= f(z)\varphi_0(z) + \sum_{j=1}^{k-1} \frac{\varphi_j(z)}{kz^j} \sum_{m=0}^{k-1} e^{\frac{-2m\pi j}{k}i} f(e^{\frac{2m\pi}{k}i}z) \\
&= e_z(f)\varphi_0(z) + \sum_{m=0}^{k-1} (\sum_{j=1}^{k-1} \frac{\varphi_j(z)}{kz^j} e^{\frac{-2m\pi j}{k}i}) e_{z_m}(f),
\end{aligned}$$

where $z_m = e^{\frac{2m\pi}{k}i}z$, for $m = 0, 1, \ldots, k-1$. Therefore, if $\mathcal{M} = \bigvee\{e_z, e_{z_1}, \ldots, e_{z_{k-1}}\}$, then $S^*e_z \in \mathcal{M}$. Substituting z with $z_1, \ldots, z_{k-1}$ in the above computations, we observe that $S^*e_{z_m} \in \mathcal{M}$, for $m = 0, 1, \ldots, k-1$. This implies that $\mathcal{M} \in LatS^*$ and so $\mathcal{M} \in LatT^*$. Thus,

$$T^*e_z = w_0(z)e_z + w_1(z)e_{z_1} + \cdots + w_{k-1}(z)e_{z_{k-1}},$$

for some scalars $w_0(z),\ w_1(z),\ \ldots,\ w_{k-1}(z)$.

Now, for every $f \in \mathcal{X}$,

$$(Tf)(z) = (T^*e_z)(f) = w_0(z)f(z) + w_1(z)f(z_1) + \cdots + w_{k-1}(z)f(z_{k-1}),$$

and it can be easily verified that $TM_{z^k} = M_{z^k}T$. Hence, M_{z^k} is hyper-reflexive. □

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