

## On the non-propagation theorem and applications

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**Abstract:** We use  $C^*$ -algebras to determine the non-propagation estimates for a certain class of generalized Schrödinger operators acting on the Hilbert space  $\ell^2(\Gamma)$ , where  $\Gamma$  is a tree, and we give some examples for other classes of potentials.

**Key words:**  $C^*$ -algebra, Schrödinger operator, propagation properties, tree

### 1. Introduction and main result

It is well known that the algebraic approach leads to very interesting results on the spectral analysis of self-adjoint operators. As relevant results we quote: the essential spectrum and the Mourre estimate. A good exposition of this formalism and some applications may be found in [1], [3], [4], [10], [11], [8] and [5]. We point out only that the main idea consists in showing that these operators are affiliated to suitable  $C^*$ -algebras which reflect their common properties well. A study of the quotient of these  $C^*$ -algebras by the ideal of compact operators leads to a formula for the essential spectrum of these operators expressed as a union of spectra of some asymptotic operators. The quotient of the same  $C^*$ -algebras by other ideals gives localization results of these operators, which can be interpreted as non-propagation properties of their unitary groups. The last result is discovered and developed in [2], [12], [6] and [13].

Let  $L$  be a self-adjoint operator in a Hilbert space  $\mathcal{H}$  and  $\chi$  a nontrivial multiplication operator (for example the characteristic function of a set having a strictly positive measure). If  $\kappa$  is a continuous function with support intersecting the spectrum of  $L$ , the operator  $\chi \kappa(L)$  has no reason to be small in general. The unique a priori bound would be

$$\|\chi \kappa(L)\| \leq \|\chi\|_\infty \sup_{\lambda \in \sigma(L)} |\kappa(\lambda)|,$$

where  $\sigma(L)$  denotes the spectrum of  $L$ . We are going to correlate  $\chi$  to  $\kappa$  in such a way to make the norm small without asking any of the two factors on the right hand side of the preceding inequality to be small. In order to understand the problem better we recall the following example (see [12] for more detail). In  $\mathcal{H} = L^2(\mathbb{R})$  we consider the Schrödinger operator  $L = L_0 + V$ , where  $L_0$  is the positive Laplace operator and  $V$  is the operator of multiplication with a bounded, uniformly continuous function having a limit at plus infinity:  $V(x) \rightarrow c$  when  $x \rightarrow +\infty$ . Then we have that  $[c, \infty)$  is included in the essential spectrum of  $L$ . The behavior of  $V$  to the left may introduce a spectrum and even an essential spectrum below  $c$ . Now, let  $\kappa : \mathbb{R} \rightarrow \mathbb{R}$  be continuous, with a

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compact support situated below the value  $c$  and  $\{\chi_a, a \in \mathbb{R}\}$  the family of all operators of multiplication with characteristic functions of intervals of the form  $[a, \infty)$ . Then we have that for any  $\epsilon > 0$  there exists  $a \in \mathbb{R}$ , positive and sufficiently large, such that  $\|\chi_a \kappa(L)\| \leq \epsilon$ . As a consequence we get the following non-propagation result:  $\|\chi_a e^{-itL} \kappa(L) f\| \leq \epsilon \|f\|$ , uniformly in  $t \in \mathbb{R}$  and  $f \in \mathcal{H}$ . So, at energies below  $c$ , even when the propagation towards infinity is possible, it does not take place to the right.

The purpose of this paper is to study the non-propagation theorem on trees. Let  $\Gamma$  be a  $\nu$ -fold tree of an origin  $e$  equipped with its canonical metric  $d$ , where  $d(x, y)$  is the shortest path joining  $x$  to  $y$ . We denote by  $x \sim y$  when  $x$  and  $y$  are connected by any edge. For all  $x \in \Gamma$ , we define  $|x| = d(e, x)$ . Then we set  $B(x, r) = \{y \in \Gamma \mid d(x, y) < r\}$  and  $S^n = \{x \in \Gamma \mid |x| = n\}$ .

For each  $x \in \Gamma \setminus \{e\}$  we denote by  $x' = x^{(1)}$  the unique element  $y \sim x$  such that  $|y| = |x| - 1$  and we set  $x^{(p)} = (x^{(p-1)})'$  for all  $1 \leq p \leq |x|$ . Let  $x\Gamma = \{y \in \Gamma \mid |y| \geq |x| \text{ and } y^{(|y|-|x|)} = x\}$ , where the convention  $x^{(0)} = x$  has been used.

We are interested in operators acting on the Hilbert space

$$\ell^2(\Gamma) := \{f : \Gamma \rightarrow \mathbb{C} \mid \sum_{x \in \Gamma} |f(x)|^2 < \infty\}$$

endowed with the inner product  $\langle f, g \rangle := \sum_{x \in \Gamma} \overline{f(x)} g(x)$ . We embed  $\Gamma \subset \ell^2(\Gamma)$  by identifying  $x$  with its characteristic function  $\chi_{\{x\}}$ . Notice that  $\Gamma$  is the canonical basis in  $\ell^2(\Gamma)$ ; indeed, each  $f \in \ell^2(\Gamma)$  can be written as  $f = \sum_{x \in \Gamma} f(x)x$ . We define the bounded operator  $\partial$  given by  $(\partial f)(x) = \sum_{y' = x} f(y)$ . Its adjoint operator is given by  $(\partial^*)f(x) = f(x')$  for all  $x \in \Gamma \setminus \{e\}$  and  $\partial^* f(e) = 0$ . Let  $\mathcal{D}$  be the  $C^*$ -algebra generated by  $\partial$ . In order to define our algebra of potentials, we set  $\widehat{\Gamma} = \Gamma \cup \partial\Gamma$  the compactification of  $\Gamma$ . An element  $x$  of the boundary at infinity  $\partial\Gamma$  is a  $\Gamma$ -valued sequence  $x = (x_n)_{n \in \mathbb{N}}$  such that  $|x_n| = n$  and  $x_{n+1} \sim x_n$  for all  $n \in \mathbb{N}$ . We set  $|x| = \infty$  for  $x \in \partial\Gamma$ . The space  $\widehat{\Gamma}$  is equipped with a natural ultrametric space structure. For  $x \in \partial\Gamma$  and  $(y_n)_{n \in \mathbb{N}}$  a sequence in  $\Gamma$  we have  $\lim_{n \rightarrow \infty} y_n = x$  if for each  $m \in \mathbb{N}$  there is  $N \in \mathbb{N}$  such that for each  $n \geq N$  we have  $y_n \in x_n\Gamma$ . We denote the  $C^*$ -algebra of the complex-valued continuous functions on  $\widehat{\Gamma}$  by  $C(\widehat{\Gamma})$ . Since  $\Gamma$  is dense in  $\widehat{\Gamma}$ , we can view  $C(\widehat{\Gamma})$  as  $C^*$ -subalgebra of  $C_b(\Gamma)$ , the algebra of bounded complex-valued continuous functions on  $\Gamma$ . For  $V \in C(\widehat{\Gamma})$  we denote the operator of multiplication by  $V$  in  $\ell^2(\Gamma)$  by  $V(Q)$ . We denote by  $\mathcal{C}(\widehat{\Gamma})$  the  $C^*$ -algebra generated by  $\mathcal{D}$  and  $C(\widehat{\Gamma})$ . It contains the set of  $K(\ell^2(\Gamma))$  the compact operators of  $\ell^2(\Gamma)$ . Let

$$x\widehat{\Gamma} := \{y \in \widehat{\Gamma} \mid \exists p \in \mathbb{N} \cup \{\infty\}; y^{(p)} = x\}$$

and  $x\partial\Gamma = x\widehat{\Gamma} \cap \partial\Gamma$ . Let  $(x_n)_{n \in \mathbb{N}}$  any sequence converging to  $x$ . Notice that  $\{x_n\widehat{\Gamma}\}_{n \in \mathbb{N}}$  the filter basis adjacent to  $x\widehat{\Gamma}$ . The main result of this paper is the next Theorem.

**Theorem 1.1** *Let  $\nu > 1$  and  $L = \sum_{\alpha, \beta} a_{\alpha, \beta}(Q) \partial^{*\alpha} \partial^\beta + K$ , where  $K$  is a compact operator,  $a_{\alpha, \beta} \in C(\widehat{\Gamma})$ ,  $a_{\alpha, \beta} = 0$  but a finite number of pairs, be a self-adjoint operator. For any  $x$  in  $\widehat{\Gamma}$ , let  $\kappa : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function with*

$$\text{supp}(\kappa) \cap \bigcup_{\gamma \in x\partial\Gamma} \sigma(\sum_{\alpha, \beta} a_{\alpha, \beta}(\gamma) \partial^{*\alpha} \partial^\beta) = \emptyset.$$

Then for each  $\varepsilon > 0$  and for each sequence  $(x_n)_{n \in \mathbb{N}}$  converging to  $x$  there exists  $n_\varepsilon \in \mathbb{N}$  such that

$$\|\chi_{x_n \widehat{\Gamma}} \kappa(L)\| \leq \varepsilon \tag{1.1}$$

for each  $n \geq n_\varepsilon$ .

In particular,

$$\|\chi_{x_n \widehat{\Gamma}} e^{-itL} \kappa(L) f\| \leq \varepsilon \|f\|, \tag{1.2}$$

uniformly in  $t \in \mathbb{R}$  and  $f \in \ell^2(\Gamma)$ .

The Theorem 1.1 remains true even if we consider any other closed subset  $F$  of  $\partial\Gamma$  and any filter basis adjacent to it. Note the obvious fact that one may replace  $\{e^{-itL}\}$  in (1.2) by any bounded family of bounded operators commuting with  $L$ . In this result we consider a small set defined localization at infinity of the operator  $L$  and since the essential spectrum of  $L$  can be written as a union of spectra of the fiber Hamiltonians  $L(\gamma) := \sum_{\alpha, \beta} a_{\alpha, \beta}(\gamma) \partial^{*\alpha} \partial^\beta$ . We can also obtain the non-propagation result in the global version just by taking the continuous function  $\kappa$  with a support disjoint to  $\sigma_{ess}(L) = \bigcup_{\gamma \in \partial\Gamma} \sigma(\sum_{\alpha, \beta} a_{\alpha, \beta}(\gamma) \partial^{*\alpha} \partial^\beta)$ .

The preceding results on trees allow us to treat more general graphs (connected graphs). We recall that a graph is said to be connected if two of its elements can be joined by a sequence of neighbors. Let  $G = \bigcup_{j=1}^m \Gamma_j \cup G_0$  be a finite disjoint union of  $\Gamma_j$ , each  $\Gamma_j$  being a  $\nu_j$ -fold branching tree, with  $\nu_j \geq 1$  and of  $G_0$ , a compact connected graph.

We endow  $G$  with a connected graph structure that respects the graph structure of each  $\Gamma_j$  and the one of  $G_0$ , such that  $\Gamma_k$  is connected to  $\Gamma_j$  ( $k \neq j$ ) only through  $G_0$  and such that  $\Gamma_j$  is connected to  $G_0$  only through  $e_j$ , the origin of  $\Gamma_j$ . The graph  $G$  is hyperbolic and its boundary at infinity  $\partial G$  is the disjoint union  $\bigcup_{j=1}^m \partial\Gamma_j$ .

Let us describe the content of this paper. In section 2, we introduce the framework by recalling some results useful in the spectral theory of self-adjoint operators and explain the construction of the tree  $\Gamma$  and the  $C^*$ -algebra  $C(\widehat{\Gamma})$ . Section 3 is devoted to the proof of the non-propagation result (Theorem 1.1) and the study of some applications on the generalized Schrödinger operators with specific potentials.

## 2. Framework

### 2.1. The $\mathfrak{K}$ -essential spectra

Let  $L$  be a self-adjoint operator in a Hilbert space  $\mathcal{H}$ . By using the spectral theorem we can associate an operator  $\eta(L)$  to a large class of functions  $\eta : \mathbb{R} \rightarrow \mathbb{C}$ . We denote by  $C_0(\mathbb{R})$  the set of all continuous functions  $\eta : \mathbb{R} \rightarrow \mathbb{C}$  that vanish at infinity (i.e. satisfying  $\lim_{x \rightarrow \pm\infty} \eta(x) = 0$ ). Some parts of the spectrum of  $L$  can be easily expressed in terms of these functions: (i) a number  $\lambda \in \mathbb{R}$  belongs to the spectrum  $\sigma(L)$  of  $L$  if  $\eta(L) \neq 0$  whenever  $\eta \in C_0(\mathbb{R})$  and  $\eta(\lambda) \neq 0$ , (ii)  $\lambda$  belongs to the essential spectrum  $\sigma_{ess}(L)$  of  $L$  if  $\eta(L)$  is a non-compact operator whenever  $\eta \in C_0(\mathbb{R})$  and  $\eta(\lambda) \neq 0$ .

Let  $C$  be a  $C^*$ -subalgebra of the  $C^*$ -algebra  $B(\mathcal{H})$  of bounded operators in  $\mathcal{H}$  and  $L$  is a self-adjoint operator  $L$  belonging to  $C$ . We assume that the ideal  $K(\mathcal{H})$  of all compact operators in  $\mathcal{H}$  is contained in  $C$ . The notion of the spectrum has an obvious meaning and it is easy to show that the spectrum of  $\sigma_{K(\mathcal{H})}(L)$  is just the essential spectrum  $\sigma_{ess}(L)$  of the self-adjoint operator  $L$ .

It is obvious that, if  $L$  is a self-adjoint operator belonging to a  $C^*$ -subalgebra  $C$  of  $B(\mathcal{H})$  and if  $\mathfrak{K}_1$  and  $\mathfrak{K}_2$  are two ideals in  $C$  satisfying  $\mathfrak{K}_1 \subset \mathfrak{K}_2$ , then  $\sigma_{\mathfrak{K}_2}(L) \subset \sigma_{\mathfrak{K}_1}(L) \subset \sigma(L)$ .

It is easy to check the next lemma, which will be used in the proof of Theorem 1.1.

**Lemma 2.1** *Let  $\mathfrak{K}$  be an ideal in a  $C^*$ -algebra  $C$  and  $L$  an operator belonging to  $C$ . If  $\eta \in C_0(\mathbb{R})$  is such that  $\eta(\mu) = 0$  for all  $\mu \in \sigma_{\mathfrak{K}}(L)$ , then  $\eta(L) \in \mathfrak{K}$ .*

**2.2.  $C(\widehat{\Gamma})$ - $C^*$ -algebra**

In this subsection we explain the construction of the  $C^*$ -algebra  $C(\widehat{\Gamma})$ . Let any finite set  $A$  consist of  $\nu$  objects. Let  $\Gamma$  be the free monoid over  $A$ ; its elements are *words* and those of  $A$  *letters*. The word  $x$  is a map with values in  $A$  defined on the set of the form  $[1, n] \cap \mathbb{N}$ , with  $n \in \mathbb{N}$ .  $x(i)$  is the  $i^{th}$ -letters of the words  $x$ . The integer  $n$  is the length of the word and will be denoted by  $|x|$ . There is a unique word  $e$  of length 0, its domain is the empty set. This is the neutral element of  $\Gamma$ . From now on we identify  $A$  with the set of the words of length 1.

If  $x \in \Gamma$ , we denote  $x\Gamma$  the *right ideal* generated by  $x$ . We have, on  $\Gamma$ , a canonical order relation defined by:

$$x \leq y \Leftrightarrow y \in x\Gamma.$$

If  $\Gamma$  is an arbitrary ordered set and  $x, y \in \Gamma$ , then one says that  $y$  covers  $x$  if  $x < y$  and if  $x \leq z \leq y \Rightarrow z = x$  or  $z = y$ . For each  $x \in \Gamma$  we denote by:

$$\bar{x} := \{y \in \Gamma \mid y \text{ covers } x\}.$$

In our case,  $y$  covers  $x$  if  $x \leq y$  and  $|y| = |x| + 1$ . Then it is easy to observe that each element  $x$  of  $\Gamma \setminus \{e\}$  covers a unique element  $x'$  and that each element of  $\Gamma \setminus \{e\}$  is covered by  $\nu$  elements. Hence

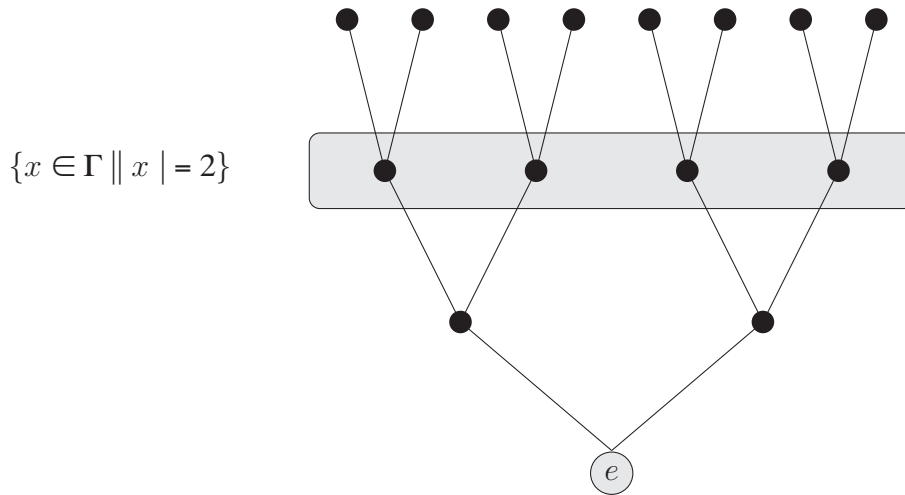
$$y \text{ covers } x \Leftrightarrow y' = x \Leftrightarrow y \in \bar{x}.$$

For  $|x| \geq n$ , we define inductively  $x^{(n)}$  by setting  $x^{(0)} = x$  and  $x^{(m+1)} = (x^{(m)})'$ , for  $m \leq n - 1$ . One may also notice that:  $|x^{(m)}| = |x| - m$ , if  $m \leq |x|$  and for  $m \leq |ab|$ :

$$(ab)^{(m)} = \begin{cases} ab^{(m)}, & \text{if } m \leq |b| \\ a^{(m-|b|)}, & \text{if } m \geq |b|. \end{cases}$$

We recall that a *graph* is a couple  $G = (\mathcal{V}, E)$ , where  $\mathcal{V}$  is the set of *vertices* and  $E$  is a set of pairs of elements of  $\mathcal{V}$  (the *edges*). If  $x$  and  $y$  are joined by an edge, one says that they are *neighbors* and one abbreviates  $x \sim y$ . The graph structure allows one to endow  $\mathcal{V}$  with a canonical metric  $d$ , where  $d(x, y)$  is the length of the shortest path in  $G$  joining  $x$  to  $y$ .

The graph  $G_\Gamma$  associated to the free monoid is defined as follows:  $\mathcal{V} = \Gamma$  and  $x \sim y$  if  $x$  covers  $y$  or  $y$  covers  $x$ . It is usual to identify  $\Gamma$  and  $G_\Gamma$ , the so-called  $\nu$ -fold branching tree. (To understand the construction more, see the following figure.)



**Figure.** A two-fold tree

For all  $x \in \Gamma$ , we have  $|x| = d(e, x)$ . Then we set  $B(x, r) = \{y \in \Gamma | d(x, y) < r\}$  and  $S^n = \{x \in \Gamma | |x| = n\}$ . We define the boundary at the infinity of  $\Gamma$  as the set  $\partial\Gamma = \{x : \mathbb{N}^* \rightarrow \mathbb{A}\}$ . For all  $x \in \partial\Gamma$ , we set  $|x| = \infty$ . Let  $\widehat{\Gamma}$  be  $\Gamma \cup \partial\Gamma$ . For  $x \in \widehat{\Gamma}$  we define the sequence  $(x_n)_{n \in [0, |x|] \cap \mathbb{N}}$  with values in  $\Gamma$  by setting  $x_0 = e$  and  $x_n = x|_{[1, n] \cap \mathbb{N}}$  for  $n \geq 1$ . We observe that the map  $x \mapsto (x_n)_{n \in [0, |x|] \cap \mathbb{N}}$  is injective. There is a natural left action of  $\Gamma$  on  $\widehat{\Gamma}$ . For  $x \in \Gamma$  and  $y \in \widehat{\Gamma}$ ,  $xy$  will be defined on the set  $[1, |x| + |y|] \cap \mathbb{N}$  by  $x(i)$  for  $i \leq |x|$  and by  $y(i - |x|)$  for  $i > |x|$ . Now we will endow  $\widehat{\Gamma}$  with a structure of ultrametric space. We define a kind of valuation  $v$  on  $\widehat{\Gamma} \times \widehat{\Gamma}$  by

$$v(x, y) = \begin{cases} \max\{n | x_n = y_n\}, & \text{if } x \neq y \\ \infty, & \text{if } x = y. \end{cases}$$

And we set  $\widehat{d}(x, y) = e^{-v(x, y)}$ . It is easy to check that  $(\widehat{\Gamma}, \widehat{d})$  is an ultrametric space, i.e. a metric space such that  $\widehat{d}(x, y) \leq \max(\widehat{d}(x, z), \widehat{d}(z, y))$ , for  $x, y, z \in \widehat{\Gamma}$ . For  $r > 0$ , we denote  $\widehat{B}(x, r) = \{y \in \widehat{\Gamma} | \widehat{d}(x, y) < r\}$ . We point out that ultrametric implies that  $\widehat{B}(x, r)$  is closed for all  $x \in \widehat{\Gamma}$  and  $r > 0$ . Notice that the topology induced by  $\widehat{\Gamma}$  on  $\Gamma$  coincides with the initial topology of  $\Gamma$  discrete one. For  $x \in \partial\Gamma$  and  $n \in \mathbb{N}$ ,

$$x_n \widehat{\Gamma} = \{y \in \widehat{\Gamma} | v(x, y) \geq n\} = \widehat{B}(x, e^{-n+1}),$$

which is the closure of  $x_n \Gamma$  in  $\widehat{\Gamma}$ . Hence, for each  $x \in \partial\Gamma$ ,  $\{x_n \widehat{\Gamma}\}_{n \in \mathbb{N}}$  is the neighborhoods basis of  $x$  in  $\widehat{\Gamma}$ . Observe that if  $x \in \Gamma$  then  $x\partial\Gamma = x\widehat{\Gamma} \cap \partial\Gamma$ . It is useful to recall that  $\widehat{\Gamma}$  and  $\partial\Gamma$  are compact spaces and  $\widehat{\Gamma}$  is the compactification of  $\Gamma$  (for more details see [9]).

We denote the  $C^*$ -algebra of continuous complex-valued functions on  $\widehat{\Gamma}$  by  $C(\widehat{\Gamma})$ . The dense embedding  $\Gamma \subset \widehat{\Gamma}$  gives a canonical inclusion  $C(\widehat{\Gamma}) \subset C_b(\Gamma)$ , where  $C_b(\Gamma)$  is the  $C^*$ -algebra of continuous bounded complex-valued functions on  $\Gamma$ . Moreover, we have

$$C_0(\Gamma) = \{f \in C(\widehat{\Gamma}) | f|_{\partial\Gamma} = 0\}.$$

A better understanding of the functions in  $C(\widehat{\Gamma})$  is given in the following result:

**Proposition 2.2** [9] *Let  $E$  a metrizable topological space. A function  $V : \Gamma \rightarrow E$  extends to a continuous function  $\widehat{V} : \widehat{\Gamma} \rightarrow E$  if and only if for each  $x \in \partial\Gamma$  the limit of  $V(y)$  when  $y \in \Gamma$  converges to  $x$  exists.*

### 3. Proof of main result and examples

#### 3.1. Proof of theorem 1.1

(i) Let

$$\mathcal{K}^{x\partial\Gamma} = \{\varphi \in C(\widehat{\Gamma}) \mid \varphi|_{x\partial\Gamma} = 0\}.$$

So if  $\varphi$  belongs to  $\mathcal{K}^{x\partial\Gamma}$ , then for any sequence  $(x_n)_{n \in \mathbb{N}}$  converging to  $x$  and for each  $\delta > 0$  there exists  $n_\varepsilon$  such that

$$|\varphi(y)| \leq \delta \quad \forall y \in x_n \widehat{\Gamma} \quad \forall n \geq n_\varepsilon.$$

(ii) Using Theorem 5.9 in [9] we can deduce that there is a unique morphism  $\Phi : \mathcal{C}(\widehat{\Gamma}) \rightarrow \mathcal{D} \otimes C(x\partial\Gamma)$  such that  $\Phi(D) = D \otimes 1$  for all  $D \in \mathcal{D}$  and  $\Phi(\varphi(Q)) = 1 \otimes (\varphi|_{x\partial\Gamma})$ . Since any function in  $C(x\partial\Gamma)$  is the restriction of a function  $C(\widehat{\Gamma})$  and any element  $D \otimes 1 \in \mathcal{D} \otimes C(x\partial\Gamma)$  has  $D$  as an antecedent through  $\Phi$ , then by taking into account the structure of the  $C^*$ -algebra tensor product we have that any element  $D \otimes \varphi|_{x\partial\Gamma}$  of  $\mathcal{D} \otimes C(x\partial\Gamma)$  can be written as the following product:

$$D \otimes \varphi|_{x\partial\Gamma} = (D \otimes 1)(1 \otimes (\varphi|_{x\partial\Gamma})) = \Phi(D)\Phi(\varphi(Q)) = \Phi(D\varphi(Q)).$$

Thus,  $\Phi$  is surjective with kernel equal to the ideal  $\mathfrak{K}^{x\partial\Gamma} = \langle \mathcal{K}^{x\partial\Gamma}, \mathcal{D} \rangle$ . Then

$$\Phi(L) = \sum_{\alpha, \beta} \partial^{*\alpha} \partial^\beta \otimes (a_{\alpha, \beta})|_{x\partial\Gamma}$$

and as a consequence we obtain

$$\sigma_{\mathfrak{K}^{x\partial\Gamma}}(L) = \bigcup_{\gamma \in x\partial\Gamma} \sigma\left(\sum_{\alpha, \beta} a_{\alpha, \beta}(\gamma) \partial^{*\alpha} \partial^\beta\right).$$

Now using the hypothesis on the support of  $\kappa$  we have  $\kappa(L) \in \mathfrak{K}^{x\partial\Gamma}$  (see Lemma 2.1). So there is a finite number of functions  $\varphi_1, \dots, \varphi_m \in \mathcal{K}^{x\partial\Gamma}$  such that

$$\|\kappa(L) - \sum_{i=1}^m \varphi_i(Q) \partial^{*\alpha_i} \partial^{\beta_i}\| \leq \varepsilon/2.$$

We have

$$\|\chi_{x_n \widehat{\Gamma}}(Q) \kappa(L)\| \leq \sum_{i=1}^m \|\varphi_i\|_{L^\infty(x_n \widehat{\Gamma})} \|\partial^{*\alpha_i} \partial^{\beta_i}\| + \|\kappa(L) - \sum_{i=1}^m \varphi_i(Q) \partial^{*\alpha_i} \partial^{\beta_i}\|.$$

The first term in the r. h. s. of the preceding inequality can be made less than  $\varepsilon/2$  by using the result of (i) with  $\delta = \left[m \sup_{i=1, \dots, m} \alpha_i \beta_i \nu\right]^{-1} \frac{\varepsilon}{2}$ , so the proof is finished.  $\square$

**Remark 3.1** Since  $C_0(\Gamma) \subset \mathcal{K}^{x\partial\Gamma}$  for all  $x \in \partial\Gamma$ , we can easily deduce that the ideal of all compact operators on  $\ell^2(\Gamma)$ ,  $\mathbf{K}(\ell^2(\Gamma)) = \langle C_0(\Gamma), \mathcal{D} \rangle$  is included in  $\mathfrak{K}^{x\partial\Gamma}$ . It follows that for each self-adjoint operator  $L$  belonging to  $\mathcal{C}(\widehat{\Gamma})$ ;  $\sigma_{\mathfrak{K}^{x\partial\Gamma}}(L) \subset \sigma_{ess}(L)$ . Now, with the fact that for all  $x, y \in \widehat{\Gamma}$  with  $x \neq y$ , we have that  $\mathfrak{K}^{x\partial\Gamma}$  is not contained in  $\mathfrak{K}^{y\partial\Gamma}$ , and since  $\bigcap_{x \in \partial\Gamma} \mathfrak{K}^{x\partial\Gamma} = \mathbf{K}(\ell^2(\Gamma))$ , we obtain that

$$\sigma_{ess}(L) = \bigcup_{x \in \partial\Gamma} \sigma_{\mathfrak{K}^{x\partial\Gamma}}(L).$$

Since  $\partial\Gamma$  is a compact space, we do not take the closure of the union of the sets  $(\sigma_{\mathfrak{K}^{x\partial\Gamma}}(L))_{x \in \partial\Gamma}$ . In order to obtain the non-propagation result in the global version just take the continuous function  $\kappa$  with support disjoint to  $\bigcup_{x \in \partial\Gamma} \sigma_{\mathfrak{K}^{x\partial\Gamma}}(L)$  and take any sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\Gamma$  converging to any  $x \in \partial\Gamma$ .

Let  $L$  be a self-adjoint operator in  $\ell^2(\Gamma)$  with a spectral measure  $E_L$  and let  $f \in \ell^2(\Gamma)$  be an arbitrary vector. We call spectral support of  $f$  with respect to  $L$ , we denote  $\text{supp}(f; L)$  for the smallest closed set  $J \subset \mathbb{R}$  such that  $E_L(J)f = f$ . Alternatively one can characterize  $\text{supp}(f; L)$  as follows :

$$\lambda \notin \text{supp}(f; L) \text{ iff } \exists \varepsilon > 0 \text{ such that } E_L(\lambda - \varepsilon, \lambda + \varepsilon)f = 0.$$

Notice that one has  $\text{supp}(f; L) \subset \sigma(L)$ .

**Corollary 3.2** Let  $L$  and  $\Gamma$  be as in Theorem 1.1. Then for each  $\varepsilon > 0$  and for each sequence  $(x_n)_{n \in \mathbb{N}}$  converging to  $x$  there exists  $n_\varepsilon \in \mathbb{N}$  such that

$$\|\chi_{x_n \widehat{\Gamma}} e^{-itL} f\| \leq \varepsilon \|f\|,$$

for all  $n \geq n_\varepsilon$ ,  $t \in \mathbb{R}$  and all  $f \in \ell^2(\Gamma)$  satisfying

$$\text{supp}(f; L) \cap \bigcup_{\gamma \in x\partial\Gamma} \sigma\left(\sum_{\alpha, \beta} a_{\alpha, \beta}(\gamma) \partial^{*\alpha} \partial^\beta\right) = \emptyset.$$

### 3.2. Examples

#### 3.2.1. Non-propagation result for Schrödinger operators with bounded, periodic potentials

We consider the Schrödinger operator  $L = \Delta + V(Q)$  with the potential  $V$  in  $C(\widehat{\Gamma})$ . Here  $\Delta$  is the bounded operator defined on  $\ell^2(\Gamma)$  by  $(\Delta f)(x) = \sum_{x \sim y} (f(y) - f(x))$ . From our definition

$$L = \partial + \partial^* - \nu Id + \chi_{\{e\}} + V(Q),$$

it is clear that  $L \in \mathcal{C}(\widehat{\Gamma})$  and :

$$\sigma_{ess}(L) = \sigma_{ess}(\Delta + V(Q)) = [-\nu - 2\sqrt{\nu}, -\nu + 2\sqrt{\nu}] + V(\partial\Gamma).$$

Then, we obtain the conclusion of Theorem 1.1 for each function  $\kappa$  with support not intersecting the set  $[-\nu - 2\sqrt{\nu}, -\nu + 2\sqrt{\nu}] + V(\partial\Gamma)$ . In particular, if  $V$  is a periodic function i.e.  $V$  satisfying that there is integer  $n_0 \in \mathbb{N}$  such that  $V(x) = V(y)$  if  $|x| = |y| + n_0$ , it is enough to take  $\kappa$  with support disjoint from  $[-\nu - 2\sqrt{\nu}, -\nu + 2\sqrt{\nu}] + V(\{x \in \Gamma \mid |x| \leq n_0\})$ .

**3.2.2. Generalized Schrödinger operators with potentials with asymptotic vanishing oscillation**

The important point in this example is the observation if  $\Gamma$  is a tree then  $\ell^2(\Gamma)$  can be naturally viewed as a Fock space over a finite dimensional Hilbert space. Let  $\mathbf{A}$  be a set consisting of  $\nu$  elements and let

$$\Gamma = \bigcup_{n \geq 0} \mathbf{A}^n \tag{3.3}$$

where  $\mathbf{A}^n$  is the  $n$ -th Cartesian power of  $\mathbf{A}$ . If  $n = 0$  then  $\mathbf{A}^0$  consists of a single element that we denote  $e$ . An element  $x = (a_1, a_2, \dots, a_n) \in \mathbf{A}^n$  is written  $x = a_1 a_2 \dots a_n$  and if  $y = b_1 b_2 \dots b_m \in \mathbf{A}^m$  then  $xy = a_1 a_2 \dots a_n b_1 b_2 \dots b_m \in \mathbf{A}^{n+m}$ , with the convention  $xe = ex = x$ . This provides  $\Gamma$  with a monoid structure. The graph structure on  $\Gamma$  is defined as follows:  $x \leftrightarrow y$  if and only if there is  $a \in \mathbf{A}$  such that  $y = xa$  or  $x = ya$ .

Now, we shall explain how to pass from trees to Fock spaces. We use the following equality (or, rather, canonical isomorphism): if  $\mathbf{A}, \mathbf{B}$  are sets, then  $\ell^2(\mathbf{A} \times \mathbf{B}) = \ell^2(\mathbf{A}) \otimes \ell^2(\mathbf{B})$ . Thus  $\ell^2(\mathbf{A}^n) = \ell^2(\mathbf{A})^{\otimes n}$  if  $n \geq 1$  and clearly  $\ell^2(\mathbf{A}^0) = \mathbb{C}$ . Then, since the union in 3.3 is disjoint, we have  $\mathfrak{H} = \ell^2(\Gamma) = \bigoplus_{n=0}^{\infty} \ell^2(\mathbf{A}^n) = \bigoplus_{n=0}^{\infty} \ell^2(\mathbf{A})^{\otimes n}$  which is the Fock space constructed over the ‘‘one particle’’ Hilbert space  $\mathcal{H} = \ell^2(\mathbf{A})$ . Let  $\mathbf{1}_n$  be the orthogonal projection of  $\mathfrak{H}$  onto  $\mathcal{H}^{\otimes n}$ . Let  $\mathcal{A}$  be a unitary  $C^*$ -subalgebra of  $B(\mathcal{H})$ . We denote by  $\mathbf{1}_{\mathcal{H}}$  its unit element. We are interested in self-adjoint operators  $L = D + V$  where  $D$  is a continuous function of  $\partial$  and  $\partial^*$ ;  $V$  of the form  $\sum_{n \geq 0} V_n \mathbf{1}_n$ , where  $V_n$  are bounded operators in  $\mathcal{H}^{\otimes n}$  and are asymptotically constant in some sense (when  $n \rightarrow \infty$ ). Let  $A_{vo}$  be the set of operators  $V$  such that  $V_n \in \mathbf{A}^{\otimes n}$ ,  $\sup \|V_n\| < \infty$  and  $\|V_n - V_{n-1} \otimes \mathbf{1}_{\mathcal{H}}\| \rightarrow 0$  as  $n \rightarrow \infty$ . We can define the  $C^*$ -algebra  $C_{\infty}$  generated by  $\partial, \partial^*$  and  $A_{vo}$ . Let us denote  $C_0 = C_{vo} \cap K(\mathfrak{H})$ . It is easy to check that  $C_0$  is a closed ideal of  $C_{vo}$ . From [7] we can deduce that  $C_{vo}/C_0$  is isomorphic to  $A_{vo}/C(x\partial\Gamma) \otimes \mathcal{D}$ , then we have  $\Phi(L) = \sum_{\alpha, \beta} \widehat{V_{\alpha, \beta| x \partial \Gamma}} \otimes \partial^{*\alpha} \partial^{\beta}$ . Therefore, if  $L = \sum_k \partial^{*i_k} \partial^{i_k} + V$  is a self-adjoint operator and for each function  $\kappa$  with support not intersecting the set  $[-\sum_k \nu^{i_k}, \sum_k \nu^{i_k}] + \widehat{V}(x\partial\Gamma)$ ; and for any sequence  $(x_n)_{n \in \mathbb{N}}$  converging to  $x$  there is  $n_{\varepsilon} \in \mathbb{N}$  such that

$$\|\chi_{x_n \Gamma} \kappa(L)\| \leq \varepsilon$$

for each  $n \geq n_{\varepsilon}$ .

Observe that the algebras  $\mathcal{A}^{\otimes n}$  are embedded in the infinite tensor product  $C^*$ -algebra  $\mathcal{A}^{\otimes \infty}$ . Thus we may also introduce the  $C^*$ -algebra  $A_{\infty}$  of  $A_{vo}$  consisting of the operators  $V$  such that  $V_{\infty} := \lim_{n \rightarrow \infty} V_n$  exists in norm in  $A_{\infty}$ . For us the algebras of Hamiltonians of interest can now be defined as the  $C^*$ -algebra  $C_{\infty}$  generated by the operators of the form  $L = D + V$  where  $D$  is a polynomial in  $\partial, \partial^*$  and  $V \in A_{\infty}$ . We consider the operator  $L = \sum_k V^k D_k$ . If  $L$  is a self-adjoint operator and  $V^k \in A_{\infty}$ , then

$$\sigma_{\mathfrak{R}^x A^{\infty}}(L) = \bigcup_{a \in x A^{\infty}} \sigma\left(\sum_k V_{\infty}^k(a) D_k\right).$$

It is given that for any sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\Gamma$  converging to  $x$  in  $\partial\Gamma$  and all  $\kappa$  not intersecting the set  $\sigma_{\mathfrak{R}^x A^{\infty}}(L)$  for all  $\varepsilon > 0$  there is  $n_{\varepsilon} \in \mathbb{N}$  such that

$$\|\chi_{x_n \Gamma} \kappa(L)\| \leq \varepsilon$$

for each  $n \geq n_{\varepsilon}$ .



**Remark 3.3** Using the affiliation notion, we can go further by taking an unbounded potential  $V \in C(\widehat{\Gamma}, \overline{\mathbb{R}})$ , where  $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$  i.e.  $V \in C(\widehat{\Gamma}, \overline{\mathbb{R}})$  if and only if for each  $\gamma \in \partial\Gamma$  we have either  $\lim_{x \rightarrow \gamma} V(x) = l$  where  $l \in \mathbb{R}$  or for each  $M \geq 0$  there is  $N \in \mathbb{N}$  such that  $|V(x)| \geq M$  for all  $n \geq N$  and  $x \in \gamma_n\Gamma$ . It is obvious that the domain of  $V(Q)$  is

$$D(V) = \{f \in \ell^2(\Gamma) \mid \|V(Q)f\| < \infty\}.$$

Let  $L_0 \in \mathcal{D}$  be a self-adjoint operator. Since  $L_0$  is bounded, the operator  $L = L_0 + V(Q)$  with the domain  $D(V)$  is self-adjoint and it is affiliated to  $\mathcal{C}(\widehat{\Gamma})$  (i.e. its resolvent belongs to  $\mathcal{C}(\widehat{\Gamma})$ ). In fact, we have  $(V(Q) + z)^{-1} \in C(\widehat{\Gamma})$  for all  $z \in \mathbb{C} \setminus \mathbb{R}$  and for large such  $z$  we have also

$$(L + z)^{-1} = (V(Q) + z)^{-1} \sum_{n \geq 0} (L_0(V(Q) + z)^{-1})^n,$$

where the series is norm convergent. Now, with the same  $z$ , we use Theorem 5.9 in [9] and the fact that  $\mathcal{D} \otimes C(x\partial\Gamma) \simeq C(x\partial\Gamma, \mathcal{D})$ ; we obtain:

$$\Phi_\gamma((L + z)^{-1}) \equiv \Phi((L + z)^{-1})(\gamma) = (V(\gamma) + z)^{-1} \sum_{n \geq 0} (\Phi(L_0)(V(\gamma) + z)^{-1})^n.$$

Note that  $(V(\gamma) + z)^{-1} = 0$  if  $V(\gamma) = \infty$ . By analytic continuation we get

$$\Phi_\gamma((L + z)^{-1}) = (\Phi(L_0) + V(\gamma) + z)^{-1}.$$

We used the convention  $(\Phi(L_0) + V(\gamma) + z)^{-1} = 0$  if  $V(\gamma) = \infty$ . It follows that if  $V(\gamma) = \infty$  then  $\sigma(\Phi_\gamma(L)) = \emptyset$ . Otherwise, one has  $\sigma(\Phi_\gamma(L)) = \sigma(\Phi(L_0)) + V(\gamma)$ . Therefore, we obtain

$$\sigma(\Phi_{x\partial\Gamma}(L)) = \sigma(\Phi(L_0)) + V(x\partial\Gamma_0),$$

where  $\partial\Gamma_0$  is the set of  $\gamma \in \partial\Gamma$  such that  $V(\gamma)$  is a finite value. By using Lemma 1 in [2], we get that  $\kappa : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function with

$$\text{supp}(\kappa) \cap \sigma(\Phi_{x\partial\Gamma}(L)) = \emptyset.$$

Then for any sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\Gamma$  converging to  $x$  and for each  $\varepsilon > 0$  there exists  $n_\varepsilon \in \mathbb{N}$  such that

$$\|\chi_{x_n\widehat{\Gamma}}\kappa(L)\| \leq \varepsilon,$$

for each  $n \geq n_\varepsilon$ .

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