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Threshold complexes and connections to number theory

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Abstract: In this paper we study quota complexes (or equivalently in the case of scalar weights, threshold complexes) and how the topology of these quota complexes changes as the quota is changed. This problem is a simple "linear" version of the general question in Morse Theory of how the topology of a space varies with a parameter. We give examples of natural and basic quota complexes where this problem frames questions about the distribution of primes, squares and divisors in number theory and as an example provide natural topological formulations of the prime number theorem, the twin prime conjecture, Goldbach's conjecture, Lehmer's Conjecture, the Riemann Hypothesis and the existence of odd perfect numbers among other things. This builds on the original work of A. Björner who had studied similar topological formulations for the Riemann Hypothesis and prime number theorem.

We also consider random quota complexes associated to sequences of independent random variables and show that various formulas for expected topological quantities give L-series and Euler product analogs of interest.

Key words: Quota system, threshold complex, persistent homology, Goldbach conjecture, Riemann Hypothesis, random complexes

1. Introduction

In the area of voting theory in political science, monotone yes/no voting systems are studied. If V is a set of voters, then a subset of voters, $C \subset V$, is called a losing coalition if the voters in C are not sufficient to force an initiative to pass. One can form a simplicial complex (see [7] or [12] for background on simplicial complexes) X with vertex set V to encode the voting system by declaring $F = [v_0, \dots, v_n]$ to be a face of X if and only if $\{v_0, \dots, v_n\}$ is a losing coalition. The monotone property of the voting system then guarantees that any face of a face in X is also a face in X and so X is indeed a simplicial complex. In voting theory, it is shown that every voting system can be weighted so that it is a quota system. In this paper we study the topological behaviour of various quota complexes as the quota is changed. The issues involved are essentially those involved in Morse theory or persistant homology where the change in topology of a set is studied as a parameter is increased.

For example in [6] and [9], the topology of a finite union of balls of radius r in $\mathbb{R}^n$, is studied as a function of r for fixed centers. This is used both in generating random complexes and in studying the persistant shapes/homology of random data sets.

In this paper we first discuss some basic topological properties of quota sets. We then illustrate the theory with examples in arithmetic involving the distribution of primes, squares, cubes and divisors of a fixed number.

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In all these cases, the quota complexes encode significant distributional information and our aim is to illustrate some of this.

Ian Leary and other anonymous readers have pointed out that scalar weighted quota complexes are equivalent to threshold complexes, which have been an object of interest in graph theory and areas of computer science. These arise for example in work of Kahn, Saks, and Sturtevant [10] where they solve the "Karp Conjecture" in computer science. They also arise in the pioneering work of A. Björner [3] where he uses variants of them to provide topological formulations of the Riemann Hypothesis and prime number theorem. (This was not known to us at the time of original submission and we thank them for bringing it to our attention.)

Let $\mathbb{R}_+$ be the set of positive real numbers.

Definition 1.1 *Let V be a vertex set. A scalar-valued quota system on V is given by a weight function $w: V \to \mathbb{R}_+$ and quota $q > 0$. The quota complex $X[w:q]$ is the simplicial complex on vertex set V such that a face $F = [v_0, \dots, v_n]$ is in $X[w:q]$ if and only if $w(F) = \sum_{i=0}^{n} w(v_i) < q$.*

For example, if we have 3 vertices $\{a, b, c\}$ with weights $\{2, 3, 5\}$ respectively then at quota $q = 9$, the corresponding quota complex would be the boundary of a triangle with corners $\{a, b, c\}$. The interior would not be included as the face $[a, b, c]$ has weight $2 + 3 + 5 = 10 > 9$. If the quota is raised to 11, this face would be included to give a solid triangle, whereas if the quota were lowered to 8, the edge $[b, c]$ would be excluded from the complex as it has weight $3 + 5 = 8$.

Definition 1.2 *Let V be a vertex set. A vector-valued quota system on V is given by a weight function $\hat{w}: V \to \mathbb{R}^s_+$ and quota $\hat{q} \in \mathbb{R}^s_+$. The quota complex $X[\hat{w}:\hat{q}]$ is the simplicial complex on vertex set V such that a face $F = [v_0, \dots, v_n]$ is in $X[\hat{w}:\hat{q}]$ if and only if $\sum_{i=0}^{n} w_j(v_i) < q_j$ for* **some** *$1 \leq j \leq s$.*

It is easy to see $X[\hat{w}:\hat{q}] = \bigcup_{j=1}^{s} X[w_j : q_j]$. s is refered to as the weight dimension of the quota system.

In the appendix, it is shown that every finite simplicial complex is isomorphic to a vector-valued quota complex $X[\hat{w}:\hat{q}]$ where the weights and quota can be taken to be vectors with positive integer entries. It is a question of interest in voting theory, what the minimum weight dimension is for a given simplicial complex.

In section 2, the basic topology of quota complexes is studied, especially scalar-valued quota complexes. One of the key results is (all the relevant topological definitions can be found in that section):

Theorem 1.3 *Let $X = X[w:q]$ be a scalar valued quota complex, then X is homotopy equivalent to a bouquet of spheres. Let v_0 be a vertex of minimal weight, then there is one sphere of dimension s in the bouquet for every face F of dimension s in X, not containing v_0, such that $q - w(v_0) \leq w(F) < q$.*

If X is a vector valued quota complex of weight dimension N then X can be covered by N scalar valued quota complexes. As long as X has no "shell vertices" we have $Cat(X) \leq 2N - 1$ where $Cat(X)$ denotes the category of the space X. Thus $\frac{Cat(X)+1}{2}$ provides a homotopy invariant lower bound on the weight dimension of any quota complex for X with no shell vertices.

These quota systems are then applied to examples in number theory (for the basic background needed in number theory see for example [1] or [17]) :

The prime complex is the full simplicial complex with vertex set equal to the set of primes $P = \{2, 3, 5, 7, 11, \dots\}$. For a fixed integer $q \geq 3$, $Prime(q)$ is the quota complex on vertex set P and quota

q (thus the actual vertex set will be all primes less than q and the faces will be collections of such primes whose sum is less than q.)

In section 3 we study this basic prime complex and show

Proposition 1.4 *Let $q = 2k$ be an even integer ≥ 4, then $[q-2, q) \cap \mathbb{Z} = \{O, E\}$ where $O = 2k-1$, $E = 2k-2$ are the unique odd (respectively even) integer in the interval $[q-2, q)$. Then the prime complex $Prime(q)$ has the homotopy type of a bouquet of spheres where there is one j-sphere in the bouquet for every way of writing an element in $\{O, E\}$ as a sum of $j+1$ distinct odd primes.*

The dimension of the reduced jth homology of $Prime(q)$ is equal to the number of ways of writing an element in $\{O, E\}$ as a sum of $j+1$ distinct odd primes.

For $q \geq 6$, $Prime(q)$ is not connected if and only if $O = q-1$ is a prime number.

For $q \geq 6$, $Prime(q)$ has a non simply-connected component if and only if $E = q-2$ is a sum of two distinct odd primes.

Thus the Twin Prime Conjecture is equivalent to the statement that $Prime(q)$ and $Prime(q+2)$ are both disconnected complexes for infinitely many values of q.

The Goldbach Conjecture is equivalent to the statement that $Prime(q)$ has a non-simply connected component for all $q \geq 6$, with q not equal to twice an odd prime. (The Goldbach conjecture trivially holds for $q = 2p$, p an odd prime anyway.)

Thus while the $Prime(q)$ complex is defined as a natural quota complex with vertices the set of primes less than q and faces given by collections of such primes whose sum is less than q, the topology of the complex $Prime(q)$ is carried by the sums whose sum lies in the shell $[q-2, q)$. Hence one can view the statement above as a topological form of the sieve method important in number theory.

In section 3 we also include data that shows how $H_i(Prime(q))$ varies as a function of q for fixed i. The behaviour observed is similar to that found in the study of random simplicial complexes (see [9]) and to behaviour observed in the birth-death process in the theory of continuous Markov chains (see [11]).

In section 5 we study how another topological quantity, namely the Euler characteristic of $Prime(q)$, varies as the quota q is changed. We find

$$\chi(Prime(q)) = -\sum_{n=2}^{\infty} \mu(n) L_q(n)$$

where μ is the Möbius function and L_q is a certain characteristic function which converges point-wise to the characteristic function of the square-free integers as $q \to \infty$. In order to clarify the issue, we introduce the LogPrime quota complex, which is the simplicial complex on the set of primes as vertices but with the weight of a vertex p being $ln(p)$. We then study how the Euler characteristic of this quota complex changes with quota and obtain:

Theorem 1.5 *Let $LogPrime(q)$ be the LogPrime complex with quota $q > 2$. Then*

$$\chi(LogPrime(q)) = -\sum_{2 \leq n < e^q} \mu(n)$$

where μ is the Möbius function. The Riemann hypothesis, that the nontrivial zeros of the Riemann zeta function lie on the critical line, is equivalent to the growth condition $|\chi(LogPrime(q))| = O(e^{q(0.5+\epsilon)})$ for all $\epsilon > 0$.

The substance of this theorem is really due to the work of Titchmarsh (see [16]) who gave an equivalent statement to the Riemann Hypothesis based on the rate of growth of the Mertens function. A. Björner recently studied a complex equivalent to $LogPrime(p)$ in [3], where he obtained this topological formulation of the Riemann hypothesis and also studied the asymptotics of its Betti numbers. We still include this theorem here, stated in the language of this paper, as it is a useful comparison for other similar results stated in this paper and illustrates where this complex fits into the general scheme.

In section 4 we give data for similar complexes encoding the distribution of integer squares and cubes. For example one can let $Square(q)$ be the simplicial complex on vertices the positive integer squares less than q and with the faces consisting of collections of positive integer squares whose sum is less than q. Similarly one can define a complex $Cube(q)$ where cubes replace squares.

One then gets similar flavor results:

Proposition 1.6 *If $Square(q)$ is the square complex, then $Square(q)$ is homotopy equivalent to a bouquet of spheres and $\dim(\bar{H}_j(Square(q)))$ is equal to the number of ways to write $q-1$ as a sum of $j+1$ distinct positive integer squares >1. Thus $Square(q)$, $q \geq 3$, is connected if and only if $q-1$ is not a positive integer square. It is simply connected if and only if $q-1$ is not a positive integer square or the sum of two distinct positive integer squares >1.*

If $Cube(q)$ is the cube complex, then $\dim(\bar{H}_j(Cube(q)))$ is equal to the number of ways to write $q-1$ as a sum of $j+1$ distinct positive integer cubes >1.

Thus the change in the homology of these complexes as quota is varied encodes various Waring type problems.

Data describing the growth of H_i for these two complexes as a function of q is also presented in that section.

As a final arithmetical application, in section 6 we define for any integer $n \geq 2$ the divisor complex $Div(n)$ whose vertices consist of the proper positive integer divisors of n and whose faces consist of collections of such divisors whose sum is less than n. (Hence we are using quota n.) This complex encodes the distribution of the divisors of n, keeping track of which collections of divisors have sums less than n or greater or equal to n.

Recall an integer $n \geq 2$ is called deficient if the sum of its proper divisors is less than n, perfect if the sum of its proper divisors is equal to n, and abundant if the the sum of its proper divisors is greater than n. We obtain the following result:

Proposition 1.7 *For any $n \geq 2$, $Div(n)$ is homotopy equivalent to a bouquet of spheres where there is one j-sphere for every collection of $j+1$ proper divisors of n, not including 1 which sum to $n-1$.*

Thus if n is deficient, $Div(n)$ is contractible.

Thus n is perfect if and only if $Div(n)$ is homotopic to a sphere of dimension $\tau(n)-3$ where $\tau(n)$ is the number of positive integer divisors of n.

The complex $Div(n)$ in the case that n is abundant can be relatively complicated. Data is presented in section 6 which shows an example of an odd number whose complex $Div(n)$ is spherical and of dimension close to $\tau(n)-3$ but not equal. This odd number is “close” to being an odd perfect number in a topological sense; of course the existence of an actual odd perfect number is still open. Even perfect numbers are in bijective correspondence with Mersenne primes and it is an open question whether there are infinitely many of these.

In section 7, a generating function for Euler characteristics of quota complexes is found and a quota-complex formulation of Lehmer's conjecture is provided among other things.

In the final section, we consider a finite set of independent continuous random variables $X_1, \dots X_N$ with continuous density functions f_i with compact support in $[m, \infty)$ where $m > 0$. We examine the random quota complex associated to these random variables with fixed minimal weight $X_0 = m$ and quota $q > m > 0$. We derive various formulas for the expected topology of this complex (see the section for the relevant definitions and [2] for basic background.)

Theorem 1.8 *Let $X_0 = m > 0$. Let $X_1, \dots, X_N$ be independent, continuous random variables with density functions $f_1, \dots, f_N$ which are continuous with compact support in $[m, \infty)$ and let $\mathbb{X}[q]$ be the random scalar quota complex determined by this collection and quota $q > m > 0$.*

Then for $j \geq 1$,

$$E[\dim(\bar{H}_{j-1}(\mathbb{X}[q], \mathbb{Q}))] = \sum_{\mathfrak{J}, |\mathfrak{J}|=j} (f_{\mathfrak{J}} \star \mathbb{I}_m)(q)$$

is a continuous function of q with compact support, where $\star$ denotes convolution.

Furthermore we have

$$1 - E[\chi(\mathbb{X}[q])] = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{2\pi i \alpha x} \left(\prod_{j=0}^{N} (1 - \hat{f}_j(\alpha)) - (1 - \hat{f}_0(\alpha)) \right) \frac{d\alpha}{\alpha},$$

which is a continuous function of q with compact support, where $\hat{f}$ denotes the Fourier transform of f.

We then give an example where the final equality in the last theorem is:

$$\sum_{1 \leq n < e^q} \mu(n) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{sx} \left(\prod_{p \in P} (1 - \frac{1}{p^s}) - (1 - \frac{1}{2^s}) \right) \frac{ds}{s}$$

where P is the set of primes less than e^q. Note for $Re(s) > 1$, $\prod_{p \in P}(1 - \frac{1}{p^s}) \to \frac{1}{\zeta(s)}$ as $q \to \infty$ where $\zeta(s)$ is the Riemann zeta function.

2. Topology of quota complexes

In this section we discuss the topological results for scalar quota complexes. The theorem on the topological type of scalar quota complexes follows from work done on shifted complexes, see for example [4] however, we also provide an elementary, self-contained proof in appendix A. The definition of a closed star can also be found there.

Theorem 2.1 (Scalar quota complexes are homotopy equivalent to bouquets of spheres) *Let X be a finite scalar quota complex and let $A = \bar{St}(v_{min})$ be the closed star of a vertex v_{min} of minimal weight. Then the quotient map $\pi : X \to X/A$ is a homotopy equivalence and X/A is a bouquet of spheres where there is one i-sphere for each i-face $F = [v_0, \dots, v_i]$ in X not containing v_{min} and such that $q - w(v_{min}) \leq w(F) < q$.*

Thus the reduced integer homology groups of X, $\bar{H}_i(X)$ are free abelian groups of finite rank equal to the number of i-faces F of X, not containing v_{min} with $q - w(v_{min}) \leq w(F) < q$.

Proof The fact that the collapse map $\pi : X \to X/A$ is a homotopy equivalence follows as A is contractible.

The complex is a shifted complex in the sense of [4]. To see this choose a linear order $v_1 < \cdots < v_k$ on the vertices of X where $w(v_i) \leq w(v_j)$ whenever $i < j$. It follows that if $i < j$ and F is a face of the complex with $v_i \notin F$ and $v_j \in F$ then $(F - \{v_j\}) \cup \{v_i\}$ is also a face of the complex as we have just swapped a vertex for a vertex of lower weight. This theorem then follows as a consequence of the theorem on shifted complexes in [4].

□

Note Theorem 2.1 is a sort of topological sieve in the sense that it says the interesting topology of a scalar quota complex is carried by the "shell faces", i.e. the faces F with $q - w(v_{min}) \leq w(F) < q$, so that their weight is concentrated in a narrow shell near the quota q.

Using work of Taylor and Zwicker [15], one can provide a combinatorial characterization of finite simplicial complexes which are scalar weighted quota complexes. Such a characterization is provided in Appendix B.

As we will not use vector-weighted quota complexes in the applications of the rest of the paper, the theorems and proofs about the vector-weighted case are deferred to Appendix C. The reader is also referred to Appendix D, for the proof that every finite simplicial complex is isomorphic to a (vector weighted) quota complex. Thus, for example, every closed manifold is homeomorphic to some vector weighted quota complex. The picture for vector weighted quota complexes is much less complete than that for scalar weighted quota complexes. In the applications in the rest of the paper, scalar weighted quota complexes are used predominantly because their topological structure is completely determined up to homotopy by the results of this section.

3. Application: the prime complex and the Goldbach conjecture

In the current and following sections we examine three examples of scalar quota complexes that arise from sequences of increasing positive integers $V = \{v_n\}_{n=1}^{\infty}$. In general, for integers $q > v_1$ we take $V(q)$ to be the quota complex on the vertex set V with quota q. As described in section 2 the topology of $V(q)$ is entirely determined by the number of ways to add integers $v_n \in V$, where $v_1 \neq v_n < q$, so that the sum falls in the interval $[q - v_1, q)$. So $V(q)$ topologically encodes the number of ways to express integers in $[q - v_1, q)$ as sums of distinct elements in $V - \{v_1\}$. Our main interest is in describing the behavior of $V(q)$ as q is increased for the cases where V is the set of primes, squares and cubes.

Our description of $V(q)$ will be data-based and will center on the functions $h_i(q)$ and $s_i(q)$, where $h_i(q) = \dim(\bar{H}_i(V(q)), \mathbb{Q})$ and $s_i(q)$ is the number of i-simplexes in $V(q)$ not containing v_1. So $s_i(q) = \left|\left\{ U \subset V - \{v_1\} \,:\, |U| = i+1, \sum_{v \in U} v < q \right\}\right|$ and $h_i(q) = s_i(q) - s_i(q - v_1)$ by Theorem 2.1. Furthermore, note that for $i > 0$, $s_i(q) + s_{i-1}(q - v_1)$ is the number of i-simplexes in $V(q)$ and of course $s_0(q) + 1$ is the number of zero simplexes. In order to capture the relative growth of these values we will examine the ratios:

$$S_i(q) = \frac{s_i(q)}{\sum_j s_j(q)} \quad \text{and} \quad H_i(q) = \frac{h_i(q)}{\sum_j h_j(q)}.$$

Note of course that $0 \leq S_i(q), H_i(q) \leq 1$ count the fraction of the cells (respectively homology) of the quota complex $V(q)$ that are concentrated in dimension i.

Before describing the data we first develop a simple theoretical context. Let $\kappa > 0$ be an integer and suppose there is a monotonically increasing, differentiable function $f : [\kappa, \infty) \to \mathbb{R}$, with $\lim_{x\to\infty} f(x) = \infty$ and such that $s_0(q) \sim f(q)$, by which we mean that $\lim_{q\to\infty} s_0(q)/f(q) = 1$. Such a function f will be called an interpolating function for the vertex count function $s_0(q)$.

Set $\widehat{s}_i(x) = \binom{f(x)}{i+1}$, where we are thinking of $\binom{x}{k}$ as the degree k polynomial $x(x-1)(x-2)\cdots(x-k+1)/k!$. So certainly $\widehat{s}_i(x) \sim f(x)^{i+1}/(i+1)!$. We make the approximation

$$\widehat{s}_i(q/(i+1)) \sim \binom{s_0(\lceil q/(i+1)\rceil)}{i+1} \leq s_i(q) \leq \binom{s_0(q)}{i+1} \sim \widehat{s}_i(q).$$

This comes from noting that $\binom{s_0(q)}{i+1}$ is the number of potential i-simplices in the complex that one can make out of the $s_0(q)$ vertices other than v_1 and hence certainly bounds $s_i(q)$ from above. On the other hand, $\binom{s_0(\lceil q/(i+1)\rceil)}{i+1}$ is the number of i-simplices possible that one can make out of vertices other than v_1 but with weight below $\frac{q}{i+1}$; such simplices definitely are below quota and hence definitely count towards $s_i(q)$.

In this setup, various general expectations can be derived for the functions $S_i(q)$ and $H_i(q)$. The expectations and their proofs are collected in proposition E.1 in Appendix E. It is shown there that $\{S_i : \mathbb{Z} \to [0,1] \,|\, i = 0,1,2,\ldots\}$ should be a family of unimodal functions of q, each with a single local maximum which moves to the right as i increases and such that the heights of the maxima decrease towards 0 as $i \to \infty$. All of these expectations are realized in the data.

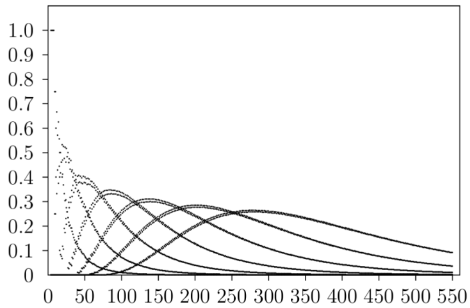
In the primes quota complex ($v_1 = 2$) the data indicates that the family of functions $H_i(q)$ has the same global behavior as $S_i(q)$, whereas in the squares and cubes cases ($v_1 = 1$), $H_i(q)$ has no discernible shape outside of appearing to tend to zero as $q \to \infty$.

We now consider the case $V = P = \{2, 3, 5, \cdots\}$, the set of primes, and its quota complex $Prime(q)$. For this vertex set we will denote the corresponding simplex and homology functions discussed above with a superscript P, so for example $s_i^P(q)$ is the number of i-simplexes in $Prime(q)$ not containing 2. Note that in this case $s_0^P(q) = \pi(q-1) - 1$, where π is the prime number counting function.

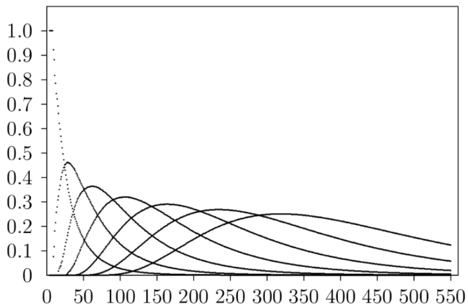
Recall that the prime number theorem says $\pi(q) \sim f_P(q)$ where $f_P(x) = x/\ln x$. Certainly $f_P : [3, \infty) \to \mathbb{R}$ has the required properties to be an interpolating function, so we expect $S_i^P(q)$ to have the behavior described above, at least for $i \geq 3$. Our data was generated using the first 100 odd primes (i.e. the primes 3 through 547), producing complete information in the range $0 < q \leq 550$ and up to homology dimension 16. Figure 1a shows $S_i^P(q)$ versus q for $0 < q \leq 550$ and cell-dimensions $0 \leq i \leq 6$; here our global expectations are closely exhibited: $S_i^P(q)$ is unimodal and as i increases the maximum height decreases and the position of the maximum moves to the right. The "double-line" effect evident in Figure 1a occurs because we are summing odd numbers, so depending on the length of the sum only odd or even sums occur in each dimension. Figure 1b is a smoothing of $S_i^P(q)$, showing the quantity $S_{i,ave}^P(q) = \frac{1}{q}\sum_{k=1}^{q} S_i^P(k)$.

As described in the introduction, the homology data of $Prime(q)$ encodes several classical arithmetic conjectures in number theory. In particular if $h_1^P(q) > 0$ for all $q > 8$ then the Goldbach conjecture holds; in fact, in this case, a stronger refinement of Goldbach holds, namely that every even number greater than or equal to 8 is the sum of two distinct odd primes. Our data is consistent with the stronger refinement of Goldbach and thus we conjecture:

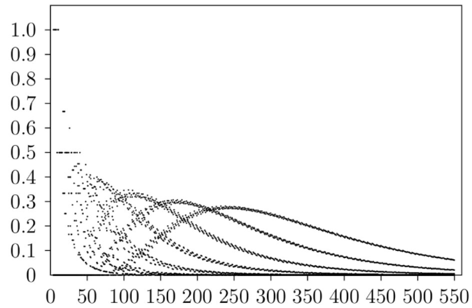
Conjecture 3.1 $\dim(H_1(Prime(q), \mathbb{Q})) \neq 0$ *for all* $q > 8$.



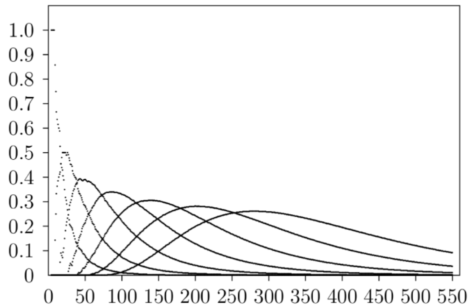
(A) $S_i^P(q)$, $0 < q \leq 550$, $0 \leq i \leq 6$



(B) $S_{i,ave}^P(q)$, $0 < q \leq 550$, $0 \leq i \leq 6$



(C) $H_i^P(q)$, $0 < q \leq 550$, $0 \leq i \leq 6$



(D) $H_{i,ave}^P(q)$, $0 < q \leq 550$, $0 \leq i \leq 6$

Figure 1. *Prime*(q).

Equivalently, every even number $q \geq 8$ *is a sum of exactly two distinct odd primes.*

The classical Goldbach conjecture is equivalent to the above conjecture holding for all even numbers greater than 3 and not equal to twice a prime. (The Goldbach conjecture is trivial for even numbers equal to twice a prime anyway.)

Figure 1c shows $H_i^P(q)$ for $0 < q \leq 550$ and $0 \leq i \leq 6$. Although appearing fairly scattered below $q \approx 100$ in these dimensions, the same global behavior as $S_i^P(q)$ manifests fairly rapidly as q increases. As before, we also provide a smoothing of the data in Figure 1d.

In general, the quantities $h_i^P(q)$ count the ways to sum to even or odd positive integers using $i+1$ distinct odd primes. Specifically if $h_i^P(q) > 0$ for all sufficiently large q and $i+1$ is even/odd, then all sufficiently large positive even/odd integers are expressible as a sum of distinct odd primes. At $i = 2$ this is a refinement of the weak Goldbach conjecture that all odd numbers greater than 7 are a sum of three odd primes. The weak Goldbach conjecture has been shown to hold for almost all odd numbers by Vinogradav.

In our data $h_2^P(q) > 0$ for $20 \leq q \leq 550$ and so it is reasonable to conjecture:

Conjecture 3.2 $\dim(H_2(Prime(q), \mathbb{Q})) \neq 0$ *for all* $q \geq 20$.
Equivalently, every odd number $q \geq 19$ *is a sum of exactly three distinct odd primes.*

Perhaps the most striking feature of the data is displayed in Figure 2. Here, in a naive attempt to capture the expected asymptotic growth of $s_i^P(q)$, we graph $\left(s_i^P(q)\right)^{1/(i+1)} ln(q)$ versus q for $0 < q \leq 550$ and

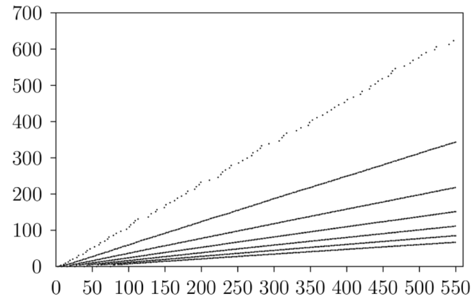


Figure 2. $\left(s_i^P(q)\right)^{1/(i+1)} ln(q)$, $0 < q \leq 550$, $0 \leq i \leq 6$.

$0 \leq i \leq 6$. In this range the data displays strong linearity. The data line with slope nearly one is the line for the zero-simplexes, the data line with the next greatest slope is the one-simplexes, and so on. Note that since $\widehat{s}_i(q) \sim f_P(q)^{i+1}/(i+1)! = \frac{q^{i+1}}{ln(q)^{i+1}(i+1)!}$ by the prime number theorem and $\widehat{s}_i(\frac{q}{i+1}) \leq s_i(q) \leq \widehat{s}_i(q)$, heuristically one might expect lines when plotting $\left(s_i^P(q)\right)^{1/(i+1)} \ln q$ versus q with slopes roughly somewhere between $\frac{((i+1)!)^{-1/(i+1)}}{i+1}$ and $((i+1)!)^{-1/(i+1)}$. Running a least squares approximation on the data gives slopes of 0.632374 when $i = 1$; 0.404613 when $i = 2$; 0.284124 when $i = 3$; 0.211868 when $i = 4$; 0.164796 when $i = 5$; and 0.132366 when $i = 6$. These values are consistent with the heuristics. Thus it is reasonable to conjecture:

Conjecture 3.3 *Let $s_i^P(q)$ denote the number of sets of $i+1$ distinct odd primes whose sum is below q. Then*

$$s_i^P(q) \sim \frac{C_i^{i+1} q^{i+1}}{ln(q)^{i+1}}$$

where $1 = C_0 > C_1 > C_2 > \cdots > 0$.

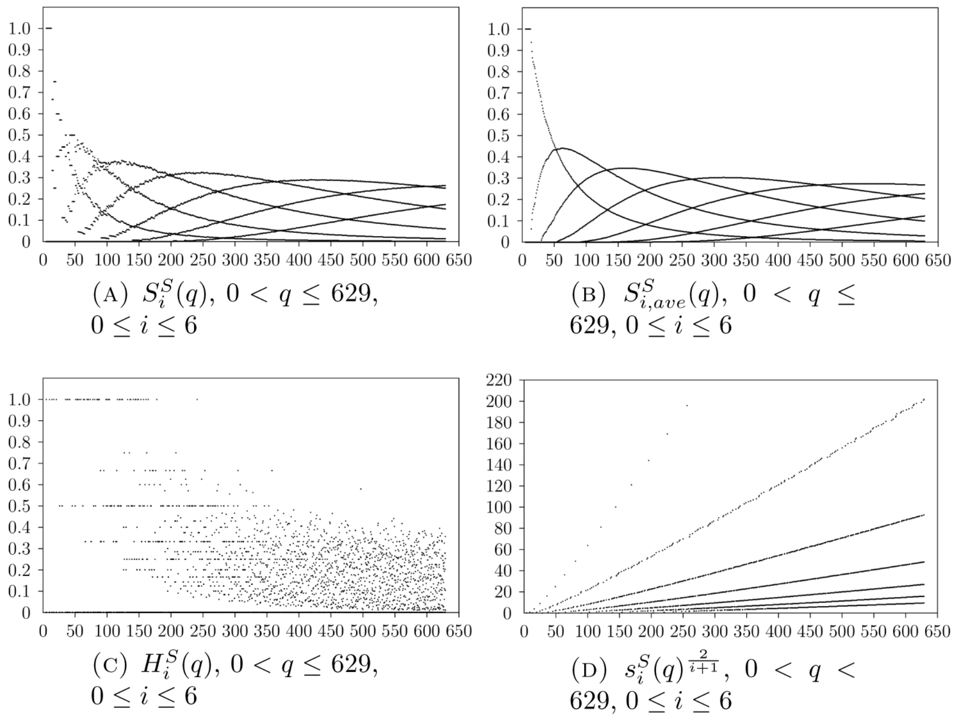
Note the case $i = 0$ is the prime number theorem which is of course known to be true. Approximate values of the constants $C_1, \ldots, C_6$ are listed in the paragraph before the conjecture.

4. Application: sum of squares and cubes complexes

In this section we present data for the quota complexes $V(q)$ with $V = S = \{1, 4, 9, \ldots\}$ the set of squares, and $V = C = \{1, 8, 27, \ldots\}$ the set of cubes, which we denote $Square(q)$ and $Cube(q)$, respectively. We indicate the various functions of interest for these quota complexes with a superscript S or C as we did in section 3. Note that since $s_0^S(q) = \lfloor \sqrt{q-1} \rfloor - 1$ and $s_0^C(q) = \lfloor \sqrt[3]{q-1} \rfloor - 1$, our expectations for the global behavior of the families S_i^S and S_i^C are once again informed by proposition E.1.

Data for $Square(q)$ was generated using the first 25 squares, producing complete information for $0 < q \leq 629$ and up to homology dimension 9. Figure 3a displays $S_i^S(q)$ for $0 < q \leq 629$ and cell-dimensions $0 \leq i \leq 6$. Here our expectations are strongly evident and are made all the more so in Figure 3b, which shows $S_{i,ave}^S(q) = \frac{1}{q}\sum_{k=1}^{q} S_i^S(k)$. The homology data $H_i^S(q)$ is notable for its lack of shape and is displayed in Figure 3c. Finally, the heuristic growth of $s_i^S(q)$ as $q^{\frac{i+1}{2}}$ is verified in Figure 3d as a graph of $s_i^S(q)^{\frac{2}{i+1}}$ versus q.

Data for $Cube(q)$ was generated using the first 25 cubes, producing complete information in the range $0 < q \leq 15633$ and up to homology dimension 13. Once again our expectations are closely realized in the data:



(A) $S_i^S(q)$, $0 < q \leq 629$, $0 \leq i \leq 6$

(B) $S_{i,ave}^S(q)$, $0 < q \leq 629$, $0 \leq i \leq 6$

(C) $H_i^S(q)$, $0 < q \leq 629$, $0 \leq i \leq 6$

(D) $s_i^S(q)^{\frac{2}{i+1}}$, $0 < q < 629$, $0 \leq i \leq 6$

Figure 3. *Squares*(q).

Figure 4a shows $S_i^C(q)$ for $0 \leq i \leq 6$ and Figure 4b displays the growth of $s_i^C(q)$ as $q^{\frac{i+1}{3}}$. A graph of H_i^C is not included as this data is scattered and apparently formless.

In the case of squares, this data fits with well-known asymptotics. For example $s_i^S(q)$ is the number of sets of $i+1$ distinct integer squares (each greater than 1) whose sum is below q. Equivalently $s_i^S(q)$ is $\frac{1}{2^{i+1}}$ times the number of integer lattice points in an open ball of radius $\sqrt{q}$ in $\mathbb{R}^{i+1}$ where the hyperplanes where two coordinates are equal or where one coordinate is 0 or 1 have been removed. There are $\binom{i+1}{2} + 2(i+1)$ such hyperplanes. Without the removal of the hyperplanes, this count is asymptotic to $C_{i+1}(\sqrt{q})^{i+1}$ where C_{i+1} is $\frac{1}{2^{i+1}}$ times the volume of the unit ball in R^{i+1} with an error no larger than the radial surface area, i.e. $(i+1)C_{i+1}(\sqrt{q})^i$. Now when one removes the hyperplanes, one is removing balls of codimension 1 and so one can see that $s_i^S(q) = C_{i+1}q^{\frac{(i+1)}{2}} + O(q^{\frac{i}{2}})$ where the implied constant only depends on i but not on q.

The case of cubes can be treated similarly. Let $X_{q,d} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d | (x_1)^3 + \cdots + (x_d)^3 < q, x_i \geq 0\}$. Let D_d be the volume of $X_{1,d}$. Then as $X_{q,d} = q^{\frac{1}{3}} \cdot X_{1,d}$ we find $Vol(X_{q,d}) = D_d q^{\frac{d}{3}}$. Thus $s_i^C(q)$, the number of sets of $i+1$ distinct integer cubes (each greater than 1) whose sum is below q can be viewed as the number of integer lattice points in $X_{q,i+1}$ with the hyperplanes where two coordinates are equal or where one coordinate is 0 or 1 removed. This is asymptotic to $Vol(X_{q,i+1}) = D_{i+1}q^{\frac{i+1}{3}}$ with an error of $O(q^{\frac{i}{3}})$ (as the boundary surface of $X_{1,i+1}$ is i-dimensional) where the implied constant only depends on i. Thus we have shown:

Proposition 4.1 *Let $s_i^S(q)$ be the number of sets of $i+1$ distinct integer squares greater than one whose sum is below q. Similarly let $s_i^C(q)$ be the number of sets of $i+1$ distinct positive integer cubes greater than one* *whose sum is below q; then we have seen that*

$$s_i^S(q) \sim (A_i q)^{\frac{i+1}{2}}$$

and

$$s_i^C(q) \sim (B_i q)^{\frac{i+1}{3}}.$$

We conjecture that the constants satisfy $1 = A_0 > A_1 > \cdots > 0$ and $1 = B_0 > B_1 > \cdots > 0$.

Now as the data suggests, the asymptotics of the Betti numbers on the other hand are irregular as the number of lattice points in a shell with inner radius $\sqrt{q-1}$ and outer radius $\sqrt{q}$ cannot be approached as simply with volume arguments as the errors that occur are comparable to the final estimate. In fact the case of counting lattice points in disks of the plane shows that shell counts can in fact even be 0 as not every integer is a sum of two squares. Thus, though one can bound the ith Betti number from above as $O(q^{\frac{i}{2}})$ for the squares complex and $O(q^{\frac{i}{3}})$ for the cubes complex, lower bounds are harder to come by. In fact finding expressions for the number of ways to write an integer as a sum of $i+1$ squares is an active and interesting area which uses the theory of modular forms extensively. For example, Jacobi's four square theorem shows that the number of ways to write a positive integer as a sum of four (not necessarily distinct or nonzero) squares is 8 times the sum of its divisors not divisible by four.

Thus, while asymptotics for individual Betti numbers are very difficult, asymptotics for their averages do follow easily from Proposition 4.1:

Proposition 4.2 *Let $\beta_i^S(q)$ denote the ith Betti number of the square complex with quota q and $\beta_i^C(q)$ denote the ith Betti number of the cube complex with quota q. Define their averages*

$$\beta_{i,ave}^S(q) = \frac{1}{q}\sum_{n=1}^{q} \beta_i^S(n) = \frac{1}{q} s_i^S(q)$$

and

$$\beta_{i,ave}^C(q) = \frac{1}{q}\sum_{n=1}^{q} \beta_i^C(n) = \frac{1}{q} s_i^C(q)$$

Then we have

$$\beta_{i,ave}^S(q) \sim C_{i+1} q^{\frac{i-1}{2}}$$

and

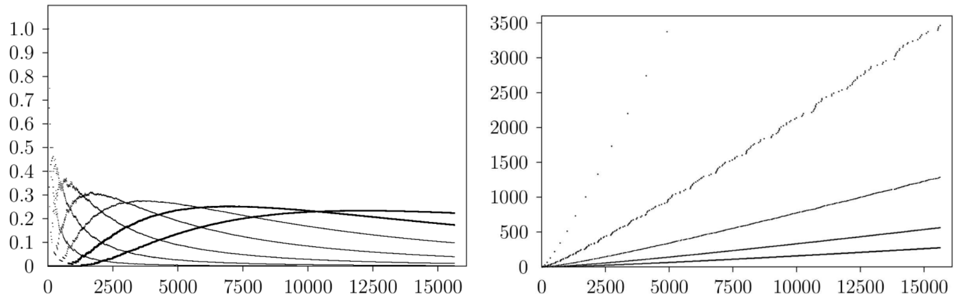
$$\beta_{i,ave}^C(q) \sim D_{i+1} q^{\frac{i-2}{3}}$$

where C_i and D_i are constants defined in the paragraph prior to Proposition 4.1.

Proof Follows immediately from Proposition 4.1 and the fact that $\beta_i^S(q)$ is the number of ways to write $q-1$ as a sum of distinct integer squares strictly bigger than one. Similar comment holds for $\beta_i^C(q)$. □

5. Application: Euler characteristics, the Möbius function and the Riemann hypothesis

In Section 3 we looked at the topology of the Prime complex $Prime(q)$ as the quota q was varied. In this section we go back to that complex and consider the variance of another topological quantity, the Euler characteristic, of this complex as the quota q changes.



(A) $S_i^C(q)$, $0 < q \leq 15633$, $0 \leq i \leq 6$

(B) $s_i^C(q)^{\frac{3}{i+1}}$, $0 < q < 15633$, $0 \leq i \leq 4$

Figure 4. *Cubes* (q).

Recall for a finite cell complex X, the Euler characteristic of X, $\chi(X)$, can be defined as an alternating sum of the number of cells of various dimensions. Namely, $\chi(X) = \sum_{j=0}^{\infty}(-1)^j c_j \in \mathbb{Z}$ where c_j is the number of j-cells in the complex X. Recall that the Euler characteristic is a homotopy invariant, i.e. homotopy equivalent spaces have the same Euler characteristic.

Let us first consider the unrestricted $Prime$ complex where $q = \infty$. This is an infinite dimensional simplex, but let us write down an (infinite) expression for its Euler characteristic. When restricted to the finite quota complexes $Prime(q)$, we will then get finite sum expressions for their Euler characteristic.

Recall the j-dimensional faces of the prime complex are in bijective correspondence with finite sets of $j+1$ distinct primes. Such a finite set of primes can be uniquely associated to its product by the unique factorization of integers. Thus the faces of the prime complex are in bijective correspondence with the square-free integers > 1. (Recall an integer is square-free if no prime is repeated in its factorization).

Recall the Möbius function $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is given by

$$\mu(n) = \begin{cases} (-1)^{\text{ number of prime factors of } n} \text{ if } n \text{ is square-free} \\ 0 \text{ if } n \text{ is not square-free} \end{cases}$$

Now note that for a j-cell of the prime complex, the corresponding square-free number n will have $j+1$ distinct prime factors and so $\mu(n) = (-1)^{j+1} = -(-1)^j$. It is now easy to see that formally one has:

$$\chi(Prime) = \sum_{j=0}^{\infty}(-1)^j c_j = -\sum_{n=2}^{\infty}\mu(n)$$

To get the corresponding finite sum expression for $\chi(Prime(q))$ we need to include the quota restriction which will allow only a finite number of the cells to be in $Prime(q)$. Notice a j-face $[p_0, p_1, \ldots, p_j]$ lies in $Prime(q)$ if and only if $\sum_{i=0}^{j} p_i < q$. Let us define $L_q : \mathbb{N} \to \{0, 1\}$ by $L_q(n) = 1$ if n is square-free and the sum of its prime divisors is less than q, and $L_q(n) = 0$ otherwise. Note that $L_q(n) = 0$ for sufficiently large n as there are only finitely many primes less than q and finitely many square-free numbers with only those prime factors.

Then we have proven:

Proposition 5.1 (Euler characteristic of the Prime complex) *Let $q > 2$ and $Prime(q)$ be the prime complex with quota q. Then*

$$\chi(Prime(q)) = -\sum_{n=2}^{\infty} \mu(n) L_q(n)$$

and this sum is finite.

Note that as $q \to \infty$, L_q converges pointwise to the characteristic function of the set of square-free integers.

In the data the Euler characteristic of $Prime(q)$ behaves very regularly as q increases. Figure 5 displays the Euler characteristic for quotas $0 < q \leq 550$. We will see in section 7 (example 3) that the growth of $\chi(Prime(q))$ with q is subexponential.

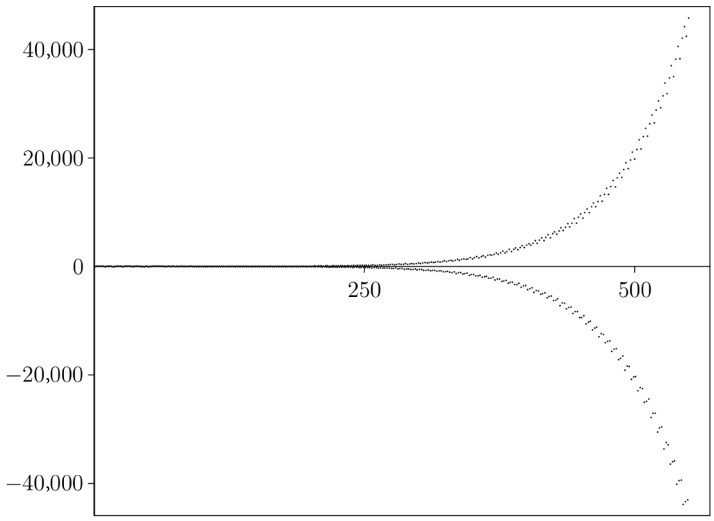


Figure 5. $\chi(Prime(q))$, $0 < q \leq 550$.

In analytic number theory, one defines Merten's function:

$$M(N) = \sum_{n=1}^{N} \mu(n).$$

By the work of Titchmarsh it is known that the Riemann hypothesis that the nontrivial zeros of the Riemann zeta function lie on the critical line is equivalent to the statement that $M(N) = O(N^{0.5+\epsilon})$ for all $\epsilon > 0$. In other words, for every $\epsilon > 0$, there exists a constant $C = C(\epsilon) > 0$ such that

$$|M(N)| \leq CN^{0.5+\epsilon}$$

for all N large enough.

With this in mind we introduce the *LogPrime* complex, which is the full quota complex ($q = \infty$) with vertices the set of primes where the weight of the vertex p is $ln(p)$. Thus as a simplicial complex $LogPrime = Prime$ but $LogPrime(q) \neq Prime(q)$ for most finite quotas q.

The j-dimensional faces of $LogPrime$ are still in bijective correspondence with the square-free integers with $j+1$ prime factors. We need only consider the effect of imposing quota q with the new weights. Note $[p_0, \ldots, p_j]$ is a face of $LogPrime(q)$ if and only if $\sum_{i=0}^{j} ln(p_i) < q$. If n is the square-free number corresponding to the face then this is equivalent to $ln(n) < q$ or $n < e^q$.

From this one easily deduces:

Theorem 5.2 (Euler characteristic of the LogPrime complex) *Let $LogPrime(q)$ be the LogPrime complex with quota $q > 2$. Then*

$$\chi(LogPrime(q)) = -\sum_{2 \leq n < e^q} \mu(n).$$

If we set $q = ln(N+1)$ then

$$\chi(LogPrime(q)) = -\sum_{n=2}^{N} \mu(n) = 1 - M(N)$$

and so

$$|\chi(LogPrime(q))| = O(N^{0.5+\epsilon}) = O(e^{q(0.5+\epsilon)})$$

for all $\epsilon > 0$ if and only if the Riemann Hypothesis is true. Equivalently, for any $\epsilon > 0$, there is a constant $C = C(\epsilon) > 0$ such that

$$ln(|\chi(LogPrime(q))|) \leq (0.5+\epsilon)q + ln(C(\epsilon))$$

for all q large enough.

Figure 6 shows the natural logarithm of the magnitude of the Euler characteristic of 6276 of the $LogPrime(q)$ complexes in the range $7 \leq q \leq 16.5554$. This data, which was generated using the first 10^6 odd primes, clearly displays the behavior expected by the Riemann Hypothesis, namely most data points occur below a line of slope roughly $1/2$. Note also that by comparing this data with that for the $Prime$ complex it is clear that $L_q(n)$ has a strongly regularizing effect.

From the last theorem, we see that a thorough understanding of how the Euler characteristic of the LogPrime complex varies with quota q is equivalent to the Riemann hypothesis.

It would be nice to have a similar characterization of the Riemann Hypothesis using the Prime complex itself. As the prime number theorem, twin prime conjecture and Goldbach conjecture have topological characterizations using the Prime complex, it seems likely the Riemann Hypothesis does also, but we do not know of any such clean statement currently. However, Section 7 addresses this further.

The $LogPrime(q)$ complexes are equivalent to complexes studied by A. Björner in [3] - in that paper Betti number asymptotics are studied for this complex and the reader is referred to it for details. These asymptotics count the number of odd squarefree numbers below $N = e^q$ with a given number of prime factors, as N grows large.

6. Application: the divisor complex, perfect, deficient and abundant numbers

Recall that a positive integer is perfect/deficient/abundant if the sum of its proper positive divisors is equal/less/more than itself.

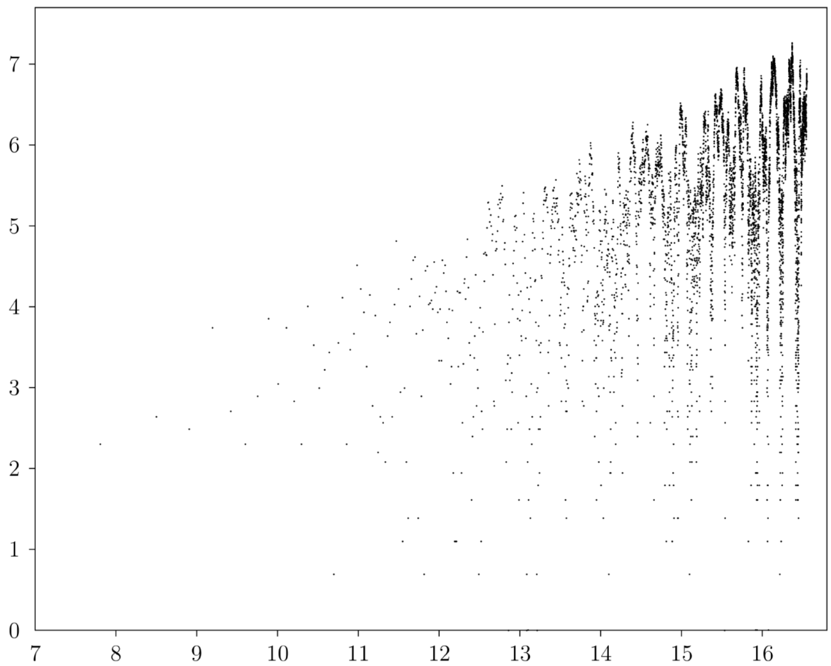


Figure 6. $\ln(|\chi(LogPrime(q))|)$, $7 \leq q \leq 16.5554$.

Consider the divisor complex $Div(n)$ for $n \geq 2$. This is the scalar quota complex with vertex set the set of proper positive integer divisors of n and quota n. So, since the minimal vertex has weight 1, $Div(n)$ is homotopic to a bouquet of spheres, consisting of one sphere of dimension i for each way $n-1$ can be written as a sum of $i+1$ distinct non-unit divisors of n. In particular, as stated in the introduction, n is perfect if and only if $Div(n)$ is homotopic to a sphere of dimension $\tau(n)-3$, where $\tau(n)$ is the number of positive integer divisors of n. Note that $Div(n)$ does not fit into the simple framework developed in section 3 since as the quota n increases the vertex set of $Div(n)$ changes completely. By the Euclid-Euler theorem, there is a bijective correspondence between even perfect numbers and Mersenne primes, i.e. primes of the form $2^p - 1$. More specifically a Mersenne prime $2^p - 1$ is associated to even perfect number $2^{p-1}(2^p - 1)$. It is unknown if there are infinitely many Mersenne primes (and hence infinitely many even perfect numbers). As of 2010, there were 47 known Mersenne primes. It is also unknown if there are any odd perfect numbers.

Figure 7 displays $Div(n)$ for $2 \leq n < 12384$. The horizontal axis measures n and the vertical axis measures topological dimension. Non-contractible complexes are represented as solid vertical lines with height equal to the maximal dimension of a sphere in the bouquet; the spheres themselves are plotted as white squares along the vertical line at height equal to their dimension with spheres of the same dimension being listed horizontally. Also, a data point at $(n, \tau(n)-3)$ is included for each n with $Div(n)$ non-contractible, together with a dashed line connecting it to a highest dimensional sphere in $Div(n)$. Thus, perfect numbers can be identified as those vertical lines with no dashed component. In this range there are four perfect numbers, 6, 28, 496 and 8128, though due to the scale of the graph only 8128 is clearly visible.

One may think of a number n with $Div(n)$ non-contractible as being topologically perfect; from this perspective the length of the dashed lines in Figure 7 measure how close a topologically perfect number is to being perfect in the classical sense. The last non-contractible divisor complex plotted in Figure 7 is particularly

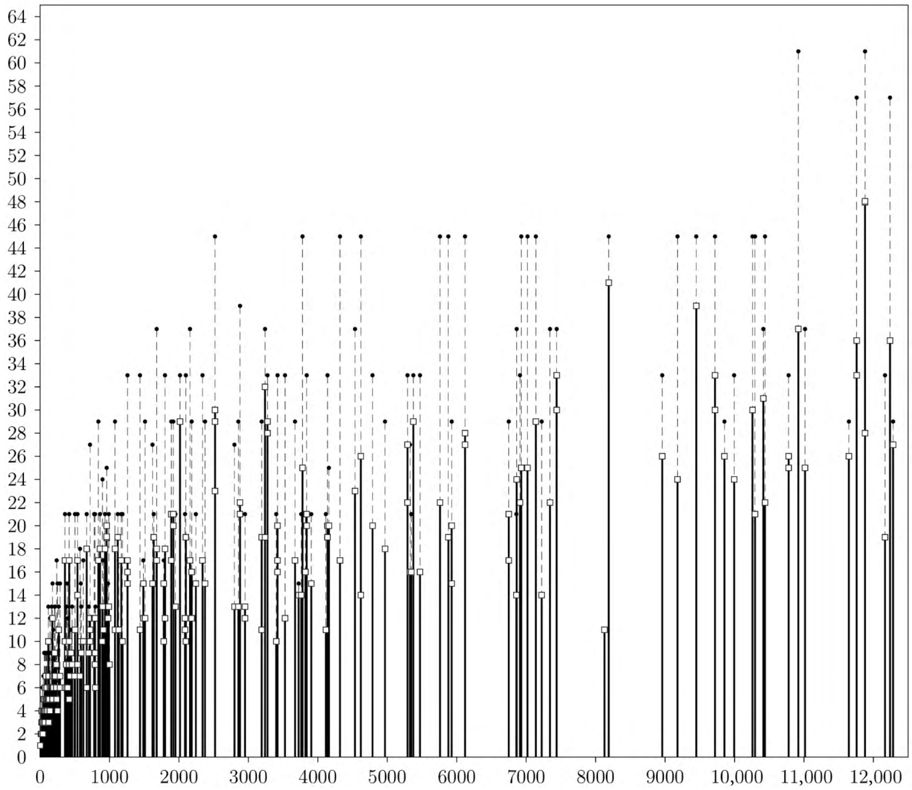


Figure 7. $Div(n)$.

interesting in this light as it occurs at the odd number $n = 12,285$ and is only a distance of two divisors away from being perfect. A simple check with the data-generating algorithm showed that $n = 12,285$ is the only topologically perfect odd number in the range $2 \leq n \leq 10^6$. In general, of course, it is not known if odd perfect numbers exist.

7. Combinatorics of Euler characteristics and Lehmer's conjecture

In this section, we use the simple combinatorial nature of the Euler characteristic to explicitly connect the Euler characteristics of various basic quota systems with well-known combinatorics. We will work with formal power series but state radii of convergence when reasonable without proof (the proofs are elementary).

For the general setup let $1 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_i \leq \ldots$ be a sequence of nondecreasing positive integers with the property that only finitely many of them lie below any given $N > 0$.

It is easy to check that we have an identity of formal power series:

$$\prod_{i=2}^{\infty}(1 - x^{\nu_i}) = 1 + \sum_{j=\nu_2}^{\infty} C_j x^j$$

where

$$C_j = \sum_{n=1}^{\infty}(-1)^n D_{n,j} = -\sum_{n=0}^{\infty}(-1)^n D_{n+1,j}$$

and $D_{n,j}$ is the number of n-fold sums of ν_i, $i \neq 1$ with distinct indices, which sum to j. (We define $C_j = 0$ for $j < \nu_2$ also.) If $X(q)$ denotes the corresponding scalar quota complex with quota $q > \nu_1$ and $\chi[q]$ its Euler chacteristic then by Theorem 2.1, we have $\chi[q] = 1 - C_{q-1} - \cdots - C_{q-\nu_1}$ or equivalently $1 - \chi[q] = \sum_{q-\nu_1 \leq i < q} C_i$. Thus

$$(1 + x + \cdots + x^{\nu_1 - 1}) \prod_{i=2}^{\infty} (1 - x^{\nu_i})$$

$$= (1 + x + \cdots + x^{\nu_1 - 1}) + \sum_{j=\nu_2}^{\infty} C_j x^j (1 + x + \cdots + x^{\nu_1 - 1})$$

$$= (1 + x + \cdots + x^{\nu_1 - 1}) + \sum_{j=\nu_2}^{\infty} (C_j + C_{j-1} + \cdots + C_{j-\nu_j+1}) x^j$$

$$= (1 + x + \cdots + x^{\nu_1 - 1}) + \sum_{j=\nu_2}^{\infty} (1 - \chi[j+1]) x^j.$$

Multiplying by $1 - x$ we get:

$$\prod_{i=1}^{\infty} (1 - x^{\nu_i}) = 1 - x^{\nu_1} + (1 - x) \sum_{j=\nu_1}^{\infty} (1 - \chi[j+1]) x^j$$

where we used $(1 - \chi[j+1]) = 0$ for $\nu_1 \leq j < \nu_2$. Using the geometric series formula we can simplify to get:

$$\prod_{i=1}^{\infty} (1 - x^{\nu_i}) = 1 - (1 - x) \sum_{j=\nu_1}^{\infty} \chi[j+1] x^j.$$

We summarize our results:

Theorem 7.1 (Generating function for Euler characteristics) *Let $1 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_i \leq \ldots$ be a sequence of nondecreasing positive integers such that only finitely many terms of the sequence are below any given $N > 0$. Let $\chi[q]$ denote the Euler characteristic of the corresponding scalar quota complex with quota q. Then $\prod_{i=1}^{\infty} (1 - x^{\nu_i})$ defines a well-defined formal power series and*

$$\prod_{i=1}^{\infty} (1 - x^{\nu_i}) = 1 - (1 - x) \sum_{j=\nu_1}^{\infty} \chi[j+1] x^j$$

or equivalently

$$\sum_{j=\nu_1}^{\infty} \chi[j+1] x^j = \frac{1 - \prod_{i=1}^{\infty} (1 - x^{\nu_i})}{1 - x}.$$

Theorem 7.1 shows that in theory, one can recover the complete quota system from the sequence of Euler characteristics $\chi[q]$. This is because on expanding the product on the left hand side of the first equation, the first nonzero term determines both the lowest weight and its multiplicity by comparison with the right hand side of the equation. Once this is known, one can divide through by the corresponding factor(s) and determine the 2nd lowest weight with its multiplicity and, proceeding in this manner, recursively recover the full weight system.

Example 1: *Counting complex.* Let $\nu_i = i$ for all $i \geq 1$. Then the corresponding quota complex will be denoted $Count(q)$. We then have

$$1 - (1-x)\sum_{j=1}^{\infty} \chi(Count(j+1))x^j = \prod_{n=1}^{\infty}(1-x^n) = \phi(x)$$

where $\phi(x)$ is called Euler's function (not to be confused with the Euler totient function). In this case the fomal series and product converge for complex numbers x with $|x| < 1$. In fact $\frac{1}{\phi(x)} = \sum_{n=1}^{\infty} p(n)x^n$ where $p(n)$ is the number of partitions of n. Furthermore, writing $x = e^{2\pi i z}$ makes $\phi(z)$ an analytic function on the upper half of the complex plane which turns out to be a modular form. In fact $\phi(x) = x^{\frac{-1}{24}}\eta(z)$ where η is the Dedekind eta function.

Example 2: $Count^{(24)}$ *and Lehmer's conjecture.* Let us take the union of 24 copies of the previous example; thus, the weights will be the positive integers but there will be 24 vertices for every given weight. We will denote the corresponding quota complex by $Count^{(24)}(q)$. Then

$$1 - (1-x)\sum_{j=1}^{\infty} \chi(Count^{(24)}(j+1))x^j = \prod_{n=1}^{\infty}(1-x^n)^{24} = \sum_{n=0}^{\infty} \tau(n+1)x^n$$

where $\tau(n)$ is Ramanujan's Tau function. From this it is easy to see that Lehmer's conjecture that $\tau(n) \neq 0$ for all $n \geq 1$ is equivalent to

$$\chi(Count^{(24)}(m)) \neq \chi(Count^{(24)}(m+1))$$

for all $m \geq 2$.

Example 3: *Prime complex.* Let ν_i be the ith prime. The corresponding quota complex is just $Prime(q)$, the prime complex considered in earlier sections. We have

$$\sum_{q=2}^{\infty} \chi(Prime(q+1))x^q = \frac{1 - \prod_{p \text{ prime}}(1-x^p)}{1-x}.$$

The formal power series and product converge for complex numbers x with $|x| < 1$ but the sum diverges at $x = 1$ and the product diverges at $x = -1$ for example. They define analytic functions in the open unit disk and hence on the upper half plane. The fact that the radius of convergence is 1 shows that the growth of $\chi(Prime(q))$ with q is subexponential, i.e. $\chi(Prime(q)) = o(A^q)$ for any $A > 1$.

8. Random quota complexes

In this section we consider the topology of a random quota complex. For the basic facts about probability used in this section see [2] and for the basic facts about Fourier transforms see [8].

Fix $X_0 = m > 0$ a (nonrandom) value. Let $X_1, \ldots, X_N$ be independent random variables. Furthermore assume X_i are continuous random variables with continuous density function f_i with **compact** support in

$[m, \infty)$ for all $i \geq 1$. Fix a quota $q > m$. Since we assume the densities are continuous with compact support, $f_i \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for $1 \leq i \leq N$ and we will hence be able to use convolutions, Fourier transforms and the inverse Fourier transform without technical difficulties. We will do so without further mention. Note one can consider the density for $X_0 = m$ as the delta measure centered at m but while this has a well-defined Fourier transform, the inverse transform formula is singular and we will worry about this when necessary.

Definition 8.1 (Random scalar quota complex) *Let $\mathbb{X} = \{X_0 = m, X_1, \ldots, X_N\}$ be chosen as in the previous paragraph and $q > m > 0$. $\mathbb{X}[q]$ is the quota complex on vertices $\{0, 1, 2, \ldots, N\}$ with weights $w(i) = X_i$ and quota q.*

$\mathbb{X}[q]$ is called a random scalar quota complex. On any run of the experiment, the weights will take on specific values and $\mathbb{X}[q]$ will determine a specific scalar quota complex. Thus $\mathbb{X}[q]$ can be viewed as a random variable on the sample space of the underlying experiment which takes values in the set of finite abstract simplicial complexes.

By Theorem 2.1, we see immediately that $\dim(\bar{H}_j(\mathbb{X}[q], \mathbb{Q}))$ is an integer valued random variable whose value is the number of $(j+1)$-fold sums of the variables $X_1, \ldots, X_N$ (repeats not allowed) that lie in the interval $[q - m, q)$. The number of j-dimensional faces of $\mathbb{X}[q]$ is another integer random variable whose value is the number of $(j+1)$-fold sums of the variables $X_0, \ldots, X_N$ (repeats not allowed) that are below quota q. Note that as each random variable takes values greater than or equal to $m > 0$, there will be no j-faces when $q \leq (j+1)m$ and so the dimension of $\mathbb{X}[q]$ is bounded by $\frac{q}{m} - 1$.

Let $\mathfrak{J} \subseteq \{1, \ldots, N\}$ have $|\mathfrak{J}| = j$ and let $X_{\mathfrak{J}} = \sum_{i \in \mathfrak{J}} X_i$ be the corresponding j-fold sum.

Let us form a Bernoulli indicator random variable:

$$B_{\mathfrak{J}} = \begin{cases} 1 \text{ if } X_{\mathfrak{J}} \in [q - m, q) \\ 0 \text{ otherwise} \end{cases}.$$

Then calculating expected values, we see:

$$E[B_{\mathfrak{J}}] = Pr(X_{\mathfrak{J}} \in [q - m, q))$$

and so to figure out this expectation, we need to determine the distribution for the corresponding sum $X_{\mathfrak{J}}$. This is standard but we include a quick exposition here:

Definition 8.2 *Let $f, g \in L^1(\mathbb{R})$, then we define the convolution*

$$(f \star g)(\alpha) = \int_{-\infty}^{\infty} f(\alpha - x) g(x) dx$$

for all $\alpha \in \mathbb{R}$. It is well known that $f \star g \in L^1(\mathbb{R})$ and indeed $||f \star g||_1 \leq ||f||_1 ||g||_1$. $(L^1(\mathbb{R}), \star)$ forms a commutative Banach algebra.

Definition 8.3 *Let f be a continuous density function for a real random variable X. The cumulative density function F is defined as*

$$F(\alpha) = Pr(X \leq \alpha) = \int_{-\infty}^{\alpha} f(x) dx$$

and so we have $F'(\alpha) = f(\alpha)$ for all $\alpha \in \mathbb{R}$.

The relevant proposition is:

Proposition 8.4 *Let X and Y be* **independent** *continuous real valued random variables with continuous density functions f_X, f_Y respectively and cumulative density functions F_X, F_Y respectively. Then $F_{X+Y} = F_X \star f_Y$ and $f_{X+Y} = f_X \star f_Y$.*

Proof Computing we find:

$$\begin{aligned} F_{X+Y}(\alpha) &= \int\int_{x+y\leq\alpha} f_X(x)f_Y(y)dxdy \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\alpha-y} f_X(x)dx\right)f_Y(y)dy \\ &= \int_{-\infty}^{\infty} F_X(\alpha-y)f_Y(y) \\ &= (F_X \star f_Y)(\alpha) \end{aligned}$$

The corresponding identity for $f_{X+Y}(\alpha) = F'_{X+Y}(\alpha)$ is obtained by differentiating the 2nd row of equations above with respect to α. □

Corollary 8.5 *Let $X_1, \ldots X_N$ be independent real random variables with continuous densities $f_1, \ldots f_N$. Then $f_{X_1+\cdots+X_N} = f_1 \star f_2 \star \cdots \star f_N$*

Proof Follows by induction on Proposition 8.4 by setting $X = X_1$ and $Y = X_2 + \cdots + X_N$. □

Returning to the calculation of $E[B_{\mathfrak{J}}] = Pr(X_{\mathfrak{J}} \in [q-m,q))$ let us define $f_{\mathfrak{J}}$ to be the convolution of f_i for $i \in \mathfrak{J}$. Thus $f_{\mathfrak{J}}$ is a $|\mathfrak{J}|$-fold convolution and $E[B_{\mathfrak{J}}] = \int_{q-m}^{q} f_{\mathfrak{J}} dx$.

Since

$$\dim(\bar{H}_{j-1}(\mathbb{X}[q])) = \sum_{\mathfrak{J},|\mathfrak{J}|=j} B_{\mathfrak{J}}$$

we find $E[\dim(\bar{H}_{j-1}(\mathbb{X}[q],\mathbb{Q}))] = \sum_{\mathfrak{J},|\mathfrak{J}|=j} \int_{q-m}^{q} f_{\mathfrak{J}} dx$.

Let

$$\mathbb{I}_m(x) = \begin{cases} 1 \text{ if } x \in [0,m] \\ 0 \text{ otherwise} \end{cases}.$$

Then it is easy to check that $(f \star \mathbb{I}_m)(q) = \int_{q-m}^{q} f(x)dx$. Thus we conclude

$$E[\dim(\bar{H}_{j-1}(\mathbb{X}[q],\mathbb{Q}))] = \sum_{\mathfrak{J},|\mathfrak{J}|=j} (f_{\mathfrak{J}} \star \mathbb{I}_m)(q).$$

Since the convolution of continuous functions with compact support is continuous with compact support, we see that $E[\dim(\bar{H}_{j-1}(\mathbb{X}[q],\mathbb{Q}))]$ is a continuous function of q with compact support. (Though $\mathbb{I}_m$ is not continuous, one can go back to the integral expression a couple of lines back to check the continuity of the expected dimension of homology.)

Since the Euler characteristic can be expressed as the alternating sum of these homology groups, we have

$$E[\chi(\mathbb{X}[q])] = 1 + \sum_{\mathfrak{J} \subseteq \{1,\ldots,N\}} (-1)^{|\mathfrak{J}|-1} (f_{\mathfrak{J}} \star \mathbb{I}_m)(q)$$

is a continuous function of q with compact support.

Note we do not include the empty set $\mathfrak{J}$ in the sum and the extra 1 in front on the right hand side is due to $\dim(H_0) = \dim(\bar{H}_0) + 1$, or equivalently to account for the vertex of minimal weight m.

Rewriting a bit we get:

$$1 - E[\chi(\mathbb{X}[q])] = \sum_{\mathfrak{J} \subseteq \{1,\ldots,N\}} (-1)^{|\mathfrak{J}|} (f_{\mathfrak{J}} \star \mathbb{I}_m)(q).$$

Note that though the left hand side is defined only for $q > m$, the right hand side is defined for all q and so can be viewed as a continuation of the left hand side to the whole real line. To shed light we will take the Fourier transform of the function

$$G(q) = \sum_{\mathfrak{J} \subseteq \{1,\ldots,N\}} (-1)^{|\mathfrak{J}|} (f_{\mathfrak{J}} \star \mathbb{I}_m)(q)$$

appearing on the right hand side. (Note that Laplace transforms would work just as well for our purposes.)

As the Fourier transform of a convolution is the product of the Fourier transforms, we have

$$\hat{G}(\alpha) = \sum_{\mathfrak{J} \subseteq \{1,\ldots,N\}} (-1)^{|\mathfrak{J}|} \hat{f}_{\mathfrak{J}}(\alpha) \hat{\mathbb{I}}_m(\alpha).$$

However, now we have $\hat{f}_{\mathfrak{J}}(\alpha) = \prod_{i \in \mathfrak{J}} \hat{f}_i$. Using this it is easy to see that

$$\hat{G}(\alpha) = \left[\prod_{j=1}^{N} (1 - \hat{f}_j) - 1 \right] \hat{\mathbb{I}}_m(\alpha)$$

where $\hat{\mathbb{I}}_m(\alpha) = \frac{1 - e^{-2\pi i \alpha m}}{2\pi i \alpha}$ is found by direct computation. To make the formula more uniform, we note that $X_0 = m$ can be thought of as having density function given by the delta measure centered at m, i.e. $f_0 = \delta_m$. The Fourier transform of this measure, $\hat{f}_0(\alpha)$, is given by $\int_{-\infty}^{\infty} \delta_m(x) e^{-2\pi i \alpha x} dx = e^{-2\pi i \alpha m}$. Thus we can write $\hat{I}_m = \frac{1-\hat{f}_0}{2\pi i \alpha}$ and so

$$\hat{G}(\alpha) = \frac{1}{2\pi i \alpha} \left[\prod_{j=0}^{N} (1 - \hat{f}_j(\alpha)) - (1 - \hat{f}_0(\alpha)) \right].$$

(The reader is warned that though $\hat{G}(\alpha)$ has a continuous Fourier inverse with compact support, $(1 - \hat{f}_0)$ by itself has badly behaved inverse Fourier transform.) By the Fourier inversion formula we get:

$$1 - E[\chi(\mathbb{X}[q])] = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{2\pi i \alpha x} \left(\prod_{j=0}^{N} (1 - \hat{f}_j(\alpha)) - (1 - \hat{f}_0(\alpha)) \right) \frac{d\alpha}{\alpha}.$$

We will see in an example that this is an example of an Euler product decomposition in analytic number theory but first let us record the probability results we have obtained.

Theorem 8.6 (Expected topology of random scalar quota complexes) *Let $X_0 = m > 0$. Let $X_1, \ldots, X_N$ be independent, continuous random variables with density functions $f_1, \ldots, f_N$ which are continuous with compact support in $[m, \infty)$ and let $\mathbb{X}[q]$ be the random scalar quota complex determined by this collection and quota $q > m > 0$.*

Then for $j \geq 1$,

$$E[\dim(\bar{H}_{j-1}(\mathbb{X}[q], \mathbb{Q}))] = \sum_{\mathfrak{J}, |\mathfrak{J}|=j} (f_{\mathfrak{J}} \star \mathbb{I}_m)(q)$$

is a continuous function of q with compact support.

Furthermore we have

$$1 - E[\chi(\mathbb{X}[q])] = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{2\pi i \alpha x} \left(\prod_{j=0}^{N} (1 - \hat{f}_j(\alpha)) - (1 - \hat{f}_0(\alpha)) \right) \frac{d\alpha}{\alpha}$$

is a continuous function of q with compact support.

We now present a degenerate example to illustrate that in some sense in Theorem 8.6, the last equality has right hand side a form of Euler product.

First recall (see [1]) the definition of the Riemann zeta function and its reciprocal:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges for all complex numbers s with $Re(s) > 1$. This is the L-series associated to the constant function with value 1 (as the numerators are all 1s). It is well known and easy to check that due to the unique factorization of positive integers into primes, we have an Euler product for the zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in Prime} \frac{1}{1 - \frac{1}{p^s}}$$

which also holds for $Re(s) > 1$. Similarly, the reciprocal of the Riemann zeta function is the L-series associated to the Möbius function and so

$$\psi(s) = \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_{p \in Prime} \left(1 - \frac{1}{p^s}\right)$$

for $Re(s) > 1$. It is common to look at restricted Euler products to various sets of primes and we will consider the simple case where P is a finite set of primes. Let us write

$$\psi_P(s) = \prod_{p \in P} \left(1 - \frac{1}{p^s}\right).$$

Note ψ_P is just a finite product of entire functions and so is entire with zeros only lying on the imaginary axis of the form $\frac{2\pi i k}{ln(p)}$ where $p \in P$ and k an integer.

Consider the case where P is the set of primes $\leq N$ for some $N > 3$ and set quota $q = ln(N+1)$. Consider "random" variables $X_0 = ln(2), X_1 = ln(3), X_2 = ln(5), \ldots$ which are just constant at the log-prime values. The corresponding density functions are delta measures centered at the corresponding log-primes and the corresponding "random" complex is just the LogPrime complex. We will apply Theorem 8.6 to this scenario. If the reader is worried about the the fact that the density functions f_i are not continuous with compact support, just replace the delta measures with very small continuous bumps of mass 1 and compact support localized around these log-primes. Since there are no serious benefits of the more rigorous approach we will just use the delta measures for this example.

By Theorem 5.2, we have $1-\chi(LogPrime(q)) = \sum_{1 \leq n < e^q} \mu(n)$. Note the Fourier transform of the delta measure $\delta_{ln(p)}$ centered at $ln(p)$ is $e^{-2\pi i ln(p)\alpha} = p^{-2\pi i\alpha}$ and so the formula from Theorem 8.6 becomes

$$\sum_{1 \leq n < e^q} \mu(n) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{2\pi i\alpha x}\left(\prod_{p \in P}(1-p^{-2\pi i\alpha}) - (1-2^{-2\pi i\alpha})\right)\frac{d\alpha}{\alpha}.$$

Doing a change of variable $s = 2\pi i\alpha$ one finds:

$$\sum_{1 \leq n < e^q} \mu(n) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{sx}\left(\psi_P(s) - \left(1-\frac{1}{2^s}\right)\right)\frac{ds}{s}.$$

9. Concluding remarks

We have seen the standard Morse theoretic problem of studying the change of topology of a space as a parameter is varied can be applied to the study of quota complexes as the quota is varied. In this case the quota determines the complex through a set of linear inequalities. A similar though more difficult nonlinear version of this problem is encountered in the study of random data sets through the method of persistant homology.

In this context a finite set of data points in $\mathbb{R}^n$ is studied by considering the space $X[r]$ consisting of the union of open balls of radius r around these points. The topology of $X[r]$ changes from being discrete when r is very small to a single contractible blob when r is very large. Usually r is varied in a range corresponding to reasonable error estimates for the experiment and the persistant features of the topology of $X[r]$ as r ranges over this error interval are looked for and attributed as features of the data set.

In this context one often forms the associated Vietoris-Rips complex instead of the actual space $X[r]$ for computational simplicity. This is an abstract simplicial complex where there is one vertex for each of the original data points and where $[v_0, \ldots, v_n]$ is a face if and only if the radius r balls around the v_i have pairwise nonempty intersection. This is equivalent to $D(v_0, \ldots, v_n) = sup_{i,j}|v_i - v_j| < 2r$. Thus, instead of the linear inequalities present in scalar quota complexes, these inequalities are nonlinear ones saying the diameters of various sets should be below quota $q = 2r$. In these cases, one has to do more work to study the change of topology as, for example, Theorem 2.1 does not apply.

Please see [9] for more details. We will not pursue this here.

Appendix

A. Topology of scalar quota complexes

In this section we provide a self-contained, elementary proof of the theorem about the topology of scalar quota complexes or equivalently threshold complexes. For the basic background needed in this section see [7] or [12].

First we recall some elementary definitions for simplicial complexes.

Definition A.1 *Let X be a simplicial complex and let v be a vertex in X. The closed star of v, $\bar{St}(v)$, is the union of all faces in X that contain v. The open star of v, $St(v)$, is the union of the interior of all faces in X that contain v. The link of v is defined as $Lk(v) = \bar{St}(v) - St(v)$. Thus $Lk(v)$ consists of all faces $F = [v_0, \ldots, v_k]$ in X not containing v such that $[v_0, \ldots, v_k, v]$ is also a face of X.*

The open star $St(v)$ is an open neighborhood of v in X, the closed star $\bar{St}(v)$ is its closure and $Lk(v)$ is its boundary in X.

The topological structure of a scalar valued quota complex is controlled to a large degree by the part of the complex outside the closed star of a vertex of minimal weight as the next theorem illustrates:

Proposition A.2 *Let X be a scalar weighted finite simplicial complex. Let v_{min} be a vertex of minimal weight. If F is a face of X that is not in $\bar{St}(v_{min})$ then the boundary of F is completely contained in $Lk(v_{min}) \subset \bar{St}(v_{min})$.*

Furthermore, F is such a face if and only if F does not contain v_{min} and $q - w(v_{min}) \leq w(F) < q$.

Proof If V is the vertex set of X, $w : V \to \mathbb{R}_+$ the weight function and $q > 0$ the quota then $[v_0, \ldots, v_k]$ is a k-face of X if and only if $\sum_{i=0}^{k} w(v_i) < q$. Let v_{min} be a vertex of minimal weight which exists as V is finite.

Note that if $F = [v_0, \ldots, v_k]$ is a face of X not contained in the closed star of v_{min} then $v_{min} \notin F$ and $w(F) < q$ as F is a face of X. Furthermore, $[v_0, \ldots, v_k, v_{min}]$ cannot be a face in X as if not it would be in $\bar{St}(v_{min})$ and hence F being a face of $[v_0, \ldots, v_k, v_{min}]$ would also be in $\bar{St}(v_{min})$. Thus $w(F) + w(v_{min}) \geq q$. Putting the inequalities together gives $q - w(v_{min}) \leq w(F) < q$. Conversely, reversing the argument, it is easily checked that any face F of X not containing v_{min} and satisfying these final inequalities is not in the closed star of v_{min}.

Let $F = [v_0, \ldots v_k]$ and let $\sigma = [v_0, \ldots, \hat{v}_i, \ldots, v_k]$ be a face in the boundary of F obtained by removing the ith vertex. Note that $w(\sigma) = w(F) - w(v_i) \leq w(F) - w(v_{min})$ and so $w([\sigma, v_{\mathrm{min}}]) = w(\sigma) + w(v_{min}) \leq w(F) < q$ and so $[\sigma, v_{min}]$ is a face in X and so $\sigma \subseteq Lk(v_{min}) \subset \bar{St}(v_{min})$ as desired.

Thus the boundary of F is completely contained in $Lk(v_{min}) \subset \bar{St}(v_{min})$ as desired.

□

For the next theorem, we recall the basic fact that if A is a contractible subcomplex of a CW or simplicial complex X then the quotient map $\pi : X \to X/A$, which collapses A to a point, is a homotopy equivalence. However, even if A and X are simplicial complexes, the collapsed space X/A is only a CW-complex in general.

We will apply this fact to the case of a finite scalar quota complex X where A will be the closed star of a minimal weight vertex. Closed and open stars of a vertex v are always contractible as they are star-convex with respect to the vertex v.

In this context, it is important to note that if $F = [v_0, \ldots, v_k]$ is a k-face of X then the space $F/\partial F$ obtained by collapsing the boundary of F to a point is homeomorphic to a k-sphere S^k, i.e. the space of unit vectors in $\mathbb{R}^{k+1}$. Recall that a bouquet of spheres is the wedge product of a collection of spheres (not necessarily of the same dimension and the 0-sphere is allowed). Intuitively this is a collection of spheres attached at a common point.

Theorem A.3 (Scalar quota complexes are homotopy equivalent to bouquets of spheres) *Let X be a finite scalar quota complex and let $A = \bar{S}t(v_{min})$ be the closed star of a vertex of minimal weight. Then the quotient map $\pi : X \to X/A$ is a homotopy equivalence and X/A is a bouquet of spheres where there is one i-sphere for each i-face $F = [v_0, \dots, v_i]$ in X not containing v_{min} and such that $q - w(v_{min}) \leq w(F) < q$.*

Thus the reduced integer homology groups of X, $\bar{H}_i(X)$ are free abelian groups of finite rank equal to the number of i-faces F of X, not containing v_{min} with $q - w(v_{min}) \leq w(F) < q$.

Proof The fact that the collapse map $\pi : X \to X/A$ is a homotopy equivalence was explained in the paragraphs before the statement of the theorem. Any face of X inside $A = \bar{S}t(v_{min})$ maps to the collapse basepoint A/A in X/A.

By proposition A.2, the i-faces F of X not contained in A are exactly the i-faces F not containing v_{min} with $q - w(v_{min}) \leq w(F) < q$ and such a face maps to an i-sphere $F/\partial F = S^i$ in X/A. From this it is easy to see that X/A is a bouquet of spheres as claimed.

The comment on reduced integer homology follows immediately from basic facts about homology. □

B. Combinatorial characterization of scalar quota complexes

For the basic language of voting theory used in this section, please consult [14]. Let V be a finite set of people which will be considered as a set of vertices. A monotone yes/no voting system on V is determined (up to power equivalence) by its set of losing coalitions which determines a simplicial complex with vertex set V. It is easy to see that this gives us a bijection between the set of monotone yes/no voting systems (up to equivalence) on V and the set of simplicial complexes with vertex set V.

It is the purpose of this section to rephrase an old result of Taylor and Zwicker in [15], in the context of simplicial complexes. Precisely, we will use their characterization of when a voting system is scalar weighted to give a characterization of when a finite simplicial complex is given by a scalar quota system.

The key concept is that of trade robustness. A trade between two coalitions L_1 and L_2 is a choice of subsets $S \subset L_1 - L_2$ and $T \subset L_2 - L_1$ to form new coalitions $L_1' = (L_1 - S) \cup T$ and $L_2' = (L_2 - T) \cup S$. A trade between two sets of vertices F_1 and F_2 in a simplicial complex is defined similarly as a choice of vertex subsets $S \subset F_1 - F_2$ and $T \subset F_2 - F_1$ to form new vertex sets $F_1' = (F_1 - S) \cup T$ and $F_2' = (F_2 - T) \cup S$. Note these vertex subsets are not required to be faces of the simplicial complex.

Definition B.1 *For a fixed integer $k \geq 1$, a voting system is k-trade robust if when given a sequence $L_1, \dots, L_k$ of losing coalitions (not necessarily distinct), it is impossible to transform them into a sequence of winning coalitions $W_1, \dots, W_k$ through a series of trades between pairs of coalitions.*

A voting system is trade robust if it is k-trade robust for all $k \geq 1$.

For a fixed integer $k \geq 1$, a finite simplicial complex is k-trade robust if when given a sequence of faces $F_1, \dots, F_k$, it is impossible to transform them into a sequence of nonfaces $W_1, \dots, W_k$ through a series of trades between pairs of vertex sets.

A finite simplicial complex is trade robust if it is k-trade robust for all $k \geq 1$.

It is clear that the concept of trade robustness is preserved under the bijection between voting systems and finite simplicial complexes mentioned earlier.

We now recall the main result of [15]:

Theorem B.2 (Taylor-Zwicker) *A voting system on a finite set of people is scalar weighted if and only if it is trade robust.*

This has the immediate corollary:

Corollary B.3 (Combinatorial characterization of scalar valued quota complexes) *A finite simplicial complex is isomorphic to a scalar valued quota complex if and only if it is trade robust, i.e. it is impossible to transform a finite sequence of faces into a sequence of nonfaces using a series of pairwise trades.*

C. Vector-weighted quota complexes

In order to state topological structure results about vector weighted quota complexes, we need to recall the definition of the Lusternik-Schnirelmann category of a space.

Definition C.1 *Let X be a topological space. Let n be a nonnegative integer. We say that $Cat(X) \leq n$ if X can be covered by $n+1$ open sets $U_0, \dots, U_n$ such that each U_i is contractible in X, i.e. the inclusion map $j_i : U_i \to X$ is homotopically trivial.*

The category of a space is hence either a nonnegative integer or infinite. Note $Cat(X) = 0$ if and only if X is contractible. $Cat(X)$ is a homotopy invariant of X; i.e. homotopy equivalent spaces have the same category.

The notion of category was introduced to give a lower bound on the number of critical points of a Morse function on a smooth manifold. It has also proved fruitful in proving bounds on degrees of nilpotence of various algebraic structures associated to a space X. (See [13] for example.)

Note any closed n-manifold can be covered by a finite number of open sets homeomorphic to $\mathbb{R}^n$ and hence contractible. Thus all closed manifolds have finite category. A sphere S^n that has $Cat(S^n) = 1$ as a sphere can always be covered by two contractible open sets (an upper and lower hemisphere) but is itself not contractible. If $\mathbb{C}P^n$ is n-dimensional complex projective space then $\mathbb{C}P^n$ can be covered by $n+1$ charts which are contractible pieces so $Cat(\mathbb{C}P^n) \leq n$. However, cup product arguments can be used to show that $Cat(\mathbb{C}P^n) > n-1$ and so $Cat(\mathbb{C}P^n) = n$ for all integers $n \geq 0$.

A bouquet of positive dimensional spheres X has $Cat(X) = 0$ if it is an empty bouquet (just a point) or $Cat(X) = 1$ otherwise. To see this let U_0 be an open thickening of the attaching point of the bouquet and U_1 be X minus the attaching point. U_0 is contractible and though U_1 is not in general contractible, it is always contractible in X, i.e. the inclusion map $U_1 \to X$ is homotopically trivial. This is because each component of U_1 is contractible and X is path connected as the bouquet does not involve 0-spheres.

A bouquet with 0-spheres involved can have category larger than 1 as the bouquet will not be path connected. Each 0 sphere gives an extra point component which increases the category by 1. To address the issue of 0-spheres in quota complexes we define the concept of a "shell vertex" in a quota complex.

Definition C.2 (Shell vertices) *Let X be a finite scalar quota complex with vertex of minimal weight v_{min}. A vertex s of X is called a shell vertex if $q - w(v_{min}) \leq w(s) < q$.*

Note by Theorem 2.1, X is homotopy equivalent to a bouquet of spheres where there is one 0-sphere for each shell vertex not equal to v_{min}. Thus $Cat(X) =$ (Number of shell vertices not equal to v_{min}) $+ \epsilon$ where $\epsilon = 1$ if there are any positive dimensional spheres in the bouquet and $\epsilon = 0$ if not.

For a vector valued quota complex, for the i*th coordinate of the weights and quota we can find a vertex* $v_{min,i}$ *of minimal weight for that coordinate. Then a vertex* s *is called a shell vertex if* $q_i - w_i(v_{min,i}) < w_i(s) < q_i$ *for* **some** *coordinate* i.

Theorem C.3 (Category of vector weighted quota systems) *Let* X *be a finite vector weighted quota system with weight dimension* N, *weight function* $\hat{w}$ *and quota* $\hat{q}$. *So* $\hat{w} : V \to \mathbb{R}_+^N$ *and quota* $\hat{q} \in \mathbb{R}_+^N$.

X *is then the union of* N *scalar quota complexes.*

Furthermore if X *has no shell vertices, we have* $Cat(X) \leq 2N - 1$. *Thus* $\frac{Cat(X)+1}{2}$ *provides a homotopy invariant lower bound for the weight dimension of a quota complex with no shell vertices.*

Proof Let X_i be the complex determined by the ith component of the weight-quota system for $1 \leq i \leq N$. Then by definition $X = \cup_{i=1}^N X_i$. By Theorem 2.1, each X_i is homotopy equivalent to a bouquet of spheres. If X has no shell vertices then no 0-spheres occur in these bouquets and hence each X_i can be covered by two open sets which are contractible in X_i and hence in X. Thickening up these open sets so that they are open in X, we see that X can then be covered by $2N$ open sets, each contractible in X. Thus $Cat(X) \leq 2N - 1$ by definition. The rest of the theorem then follows readily. □

D. Every finite simplicial complex is a quota complex

In this appendix we provide a proof that every finite simplicial complex is a quota complex. This fact already appeared in unpublished work of Manuel Alves who was working on independent undergraduate research on voting theory under the guidance of the first author. Given a monotone voting system on a set V of voters, the voting complex associated to the system is an abstract simplicial complex based on V whose faces consist of the losing coalitions of the system.

The proof of the following theorem then parallels the fact that every monotone voting system is a (vector valued) quota system. Let $|F|$ denote the number of vertices in a face F.

Theorem D.1 *If* X *is a finite simplicial complex on vertex set* V *then* X *is a quota complex, i.e. there exists a weight function* $\hat{w} : V \to \mathbb{R}_+^n$ *for some* n *and quota* $\hat{q} \in \mathbb{R}_+^n$ *such that* X *is the simplicial complex associated to quota system* $[\hat{w} : \hat{q}]$.

Furthermore the weights and the quota can be chosen to have positive integer coordinates and these coordinates can be assumed to all be distinct.

Proof Let $\mathfrak{L}$ denote the set of maximal faces (facets) of X. Order these faces $\mathfrak{L} = \{F_1, \ldots, F_s\}$. Define a scalar valued quota system $w_i : V \to \mathbb{R}_+$ by setting

$$w_i(v) = \begin{cases} 1 \text{ if } v \in F_i \\ |F_i| + 1 \text{ if } v \notin F_i \end{cases}.$$

Set quota $q_i = |F_i| + 1$. It is simple to check that the subcomplex of the simplex on V determined by the quota system $[w_i : q_i]$ is exactly the simplex F_i and its faces. Letting $\hat{w} : V \to \mathbb{R}_+^s$ be the function with coordinates the w_i and $\hat{q} \in \mathbb{R}_+^s$ be the vector with coordinates the q_i, then $[\hat{w} : \hat{q}]$ is subcomplex of the simplex on V given by $\bigcup_{i=1}^s F_i$. Hence $X = \bigcup_{i=1}^s F_i$ is the quota complex $[\hat{w} : \hat{q}]$.

Define as usual the weight of a face to be the sum of the weights of the vertices in the face. Since X has a finite number of faces, only a finite number of vectors occur as weights of the faces of X. It is clear then that we may change the coordinates of the quota by subtracting sufficiently small positive numbers from the coordinates so that the quota determines the same complex as the original quota system and has distinct positive rational entries and that these quota coordinates are not acheived as the weight of any face in the simplex on V. It is then clear that we can perturb the weights of the vertices by small amounts so that the resulting quota complex still determines the same complex and so that the weights of the vertices consist of vectors of distinct positive rational numbers and such that no weight vectors share any coordinates with each other or with the quota vector.

Finally, scaling all weight vectors and the quota vector by the least common multiple of the denominators of all rational numbers involved, we get a quota system determining the same complex as the original, where weights and quota have positive integer coordinates and all these coordinates are distinct. □

E. Expectation justification

We adopt the setup and notation from the beginning of section 3 in this section of the appendix.

Furthermore set $S_i^{high}(x) = \widehat{s}_i(x)/2^{f(x/(i+1))}$ and $S_i^{low}(x) = \widehat{s}_i(x/(i+1))/2^{f(x)}$. We have that

$$S_i^{low}(q) \approx \frac{\binom{s_0(\lceil q/(i+1)\rceil)}{i+1}}{\sum_j \binom{s_0(q)}{j}} \leq S_i(q) \leq \frac{\binom{s_0(q)}{i+1}}{\sum_j \binom{s_0(\lceil q/(i+1)\rceil)}{j}} \approx S_i^{high}(q).$$

So if we take $\widehat{S}_i(x) = \binom{f(x)}{i+1}/2^{f(x)}$ then for x with $f(x/(i+1)) \geq i$, $S_i^{low}(x) < \widehat{S}_i(x) < S_i^{high}(x)$. The global behavior of the family of functions $\widehat{S}_i(x)$ is easy to ascertain; we collect some of the features in the following proposition.

Proposition E.1 *Let k' be the smallest positive integer such that $f(\kappa) \leq k'$, and for $j \geq k'$ let $x_j \in [\kappa, \infty)$ such that $f(x_j) = j$. Then for all $i \geq k'$, $\widehat{S}_i(x_i) = 0$, $\lim_{x\to\infty} \widehat{S}_i(x) = 0$ and $\widehat{S}_i(x)$ has exactly one critical value in $[x_i, \infty)$ which is a maximum. Moreover, if m_i is the critical point of $\widehat{S}_i(x)$ in $[x_i, \infty)$, then $x_{2i+1} < m_i < x_{2i+2}$ and $\lim_{i\to\infty} \widehat{S}_i(m_i) = 0$.*

Proof The first assertions are an exercise in undergraduate calculus: we may compute

$$\widehat{S}_i'(x) = f'(x)\widehat{S}_i(x)\left[\sum_{j=0}^{i} \frac{1}{f(x)-j} - \ln 2\right],$$

from which it follows that $\widehat{S}_i(x)$ has exactly one critical value in $[x_i, \infty)$ which is a maximum.

In order to establish the bounds on m_i we use the following sequence (see [5] pg. 10):

$$l_i = \sum_{j=2^i}^{2^{i+1}-1} \frac{1}{j} \to \ln 2 \quad \text{as} \quad i \to \infty.$$

Namely, consider the sequences

$$a_i = \sum_{j=0}^{i} \frac{1}{f(x_{2i+1})-j} = \sum_{j=i+1}^{2i+1} \frac{1}{j} \quad \text{and} \quad b_i = \sum_{j=0}^{i} \frac{1}{f(x_{2i+2})-j} = \sum_{j=i+2}^{2i+2} \frac{1}{j}.$$

Then, a_i is a bounded decreasing sequence and b_i is a bounded increasing sequence, so they converge. Furthermore $a_{2^i-1} = l_i$ and $b_{2^i-2} - l_i \to 0$ as $i \to \infty$, so $a_i, b_i \to \ln 2$ as $i \to \infty$. Hence, since a_i is decreasing we must have that $x_{2i+1} < m_i$, and since b_i is increasing we must have that $m_i < x_{2i+2}$. So $\widehat{S}_i(m_i) < \binom{2i+2}{i+1}/2^{2i+1}$ and since, by Stirling's formula, $\binom{2n}{n} \sim 2^{2n}/\sqrt{\pi n}$ it follows that $\lim_{i\to\infty} \widehat{S}_i(m_i) = 0$. $\square$

Note that while it is reasonable to make the the approximation

$$H_i(q) = \frac{s_i(q) - s_i(q-v_1)}{\sum_j s_j(q) - \sum_j s_j(q-v_1)} \approx \frac{\binom{f(q)}{i+1} - \binom{f(q-v_1)}{i+1}}{2^{f(q)} - 2^{f(q-v_1)}},$$

the function on the right seems to be less amenable to a simple general analysis.

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