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The cyclic behavior of the constrictive Markov operators

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Abstract: Let S be a Polish space, and let $\mathcal{M}_\Sigma$ be the Banach space of finite signed measures on the Borel Σ-algebra Σ of S. Given a constrictive Markov operator $T : \mathcal{M}_\Sigma \to \mathcal{M}_\Sigma$, we use the asymptotic periodic decomposition of T to determine the set of T-invariant distributions in $\mathcal{M}_\Sigma$ and the set of T-ergodic distributions. We also give the relationship between the asymptotic periodic decomposition and the cycles of the process relative to the operator T.

Key words: Asymptotically periodic, constrictive operator, cyclic decomposition, ergodic decomposition, ergodic measure, Harris decomposition, invariant measure

1. Introduction

A bounded linear operator T on a Banach space $\mathbf{X}$ is called *constrictive* if there is a compact set $F \subset \mathbf{X}$ such that the iterates $T^n x$ tend to F if $\|x\| \leq 1$. If there is a periodic operator $\hat{T}$ such that $\lim_{n\to\infty} \|T^n x - \hat{T}^n x\| = 0$ for all $x \in \mathbf{X}$ we say T is *asymptotically periodic*. There are several results about conditions that imply the equivalence between constrictivity and asymptotic periodicity. Lasota, Li and Yorke [10] showed the equivalence when $\mathbf{X} = L^1(\mu)$ with the strong topology, where μ is a Σ-finite measure. Komorník [6] gave the equivalence for the case in which $L^1(\mu)$ has the weak topology. Komorník and Lasota [8] extended those results for 'quasi-constrictive' (see Section 2 for a definition) instead of 'constrictive' Markov operators. Komorník in [7] collects results concerning asymptotic properties of the iterates of positive contractions or power bounded operators on $L^1(\mu)$; this subject is also studied by Lasota and Mackey in [11]. Bartoszek [1] and Räbiger [15] studied the case in which $\mathbf{X}$ is a real Banach lattice and T is a positive contraction. Bartoszek [2] and Emelýanov [4] studied the case in which $\mathbf{X}$ is an AL Banach space. Bartoszek [3] studied the asymptotic periodicity when $\mathbf{X}$ is ordered F-spaces with the Riesz decomposition property and T is constrictive positive operators.

In all previous cases we have that the operator $\hat{T}$ is of the form $\hat{T}^n x = \sum_{k=1}^{r} \lambda_k(x) x_{\sigma^n(k)}$, where σ is a permutation of $\{1, 2, \ldots, r\}$, each λ_k is a linear functional and $x_1, x_2, \ldots, x_r$ are normalized linearly independent elements of $\mathbf{X}$. We say that such a representation of $\hat{T}$ is an *asymptotic periodic decomposition of* T.

We are interested in the case in which $\mathbf{X}$ is the Banach space $\mathcal{M}_\Sigma$ of finite signed measures with the total variation norm $\|\cdot\|$ on a Σ-algebra Σ of a set S, and T is of the form $\cdot P$, for a Markov transition probability P relative to a time-homogeneous Markov chain $(X_n)_{n=0}^\infty$ whose state space is S. That is to say

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$$P^n(s,A) = \mathrm{P}(X_n \in A | X_0 = s), \quad s \in S \quad \text{and} \quad A \in \Sigma, \tag{1}$$

for all $n \in \mathbb{N} \cup \{0\}$ and

$$T^n\mu = \mu P^n := \int P^n(s,\cdot)\mu(\mathrm{d}\, s), \quad n \in \mathbb{N}. \tag{2}$$

Komorník and Thomas [9] proved an extension of the equivalence between the constrictivity and the asymptotic periodicity. This extension states the existence of the asymptotic periodicity decomposition of constrictive Markov operators on a kind of subset of $\mathcal{M}_\Sigma$. Such extension is reproduced in Section 2. We shall establish a relationship between the periodic asymptotic decomposition and the Harris decomposition of the Markov chain (X_n), and its cyclic decomposition in the case where S is a Polish space. We also shall describe the ergodic and invariant measures.

The Harris decomposition describes a recurrence structure of some Markov chain; Meyn and Tweedie gave [12] several results related with the Harris decomposition in which the state space is composed of a disjoint collection of "recurrent" sets plus a "transient" set.

In Section 3 we enunciate the cyclic decomposition theorem. We use this theorem to prove our main result (Theorem 3.8), which formulates the relation between the asymptotic decomposition and the cycles of the process.

In the next section we give a standard terminology, some of which can be found in Orey [14], and we give necessary previous results to prove the results given in Section 3.

2. Preliminaries

A topological space is said to be a *Polish* space if it is homeomorphic to a complete separable metric space.

Let S be a Borel subset of a Polish space. We denote by Σ the Σ-algebra of Borel subsets of S.

Let us fix the measurable space (S,Σ) and a time-homogeneous Markov chain $(X_n)_{n=0}^{\infty}$ on (S,Σ) whose n-step Markov transition probability is P^n, for $n \in \mathbb{N} \cup \{0\}$.

The *Dirac measure* at x is denoted as δ_x, that is $\delta_x(A) = 1$ if $x \in A$, and $\delta_x(A) = 0$ if $x \notin A$, for $x \in S$ and $A \in \Sigma$.

Note that if we have the linear operator $\cdot P : \mathcal{M}_\Sigma \to \mathcal{M}_\Sigma$ given by (2), then we can recuperate the Markov transition probability given in (1) by means of the formula

$$P^n(x,\cdot) = \delta_x P^n.$$

If μ is a probability measure and $A \in \Sigma$, then $\mu P^n(A)$ represents the probability of $X_n \in A$ given that the probability of $X_0 \in B$ is $\mu(B)$, for all set $B \in \Sigma$.

A set M is a *band in* $\mathcal{M}_\Sigma$ if $M \subset \mathcal{M}_\Sigma$ and it is a Banach lattice such that ($\mu \in M$ and $\nu \ll \mu$) $\Rightarrow \nu \in M$.

Let M be a band in $\mathcal{M}_\Sigma$, and let D_M denote the subset of nonnegative normalized elements of M, called the *distributions* of M. (In other words, D_M is the set of probability measures in M.)

A linear operator $T : M \to M$ is called a *Markov operator* if T maps D_M into itself.

Let M be a band in $\mathcal{M}_\Sigma$. A Markov operator $T : M \to M$ is said to be *quasi-constrictive* if there exists a weakly compact set $F \subset M$ and a nonnegative number $\delta < 1$ such that

$$\limsup_{n\to\infty} d(T^n\mu, F) \le \delta \quad \text{for} \quad \mu \in D_M,$$

where $d(\nu, F) := \inf\{\|\nu - \rho\| : \rho \in F\}$.

We say that $\mu \in M$ is *P-periodic* if there exists a nonnegative integer n such that $\mu P^n = \mu$. The minimum positive integer n such that $\mu = \mu P^n$ is called the *P-period* of μ. If the P-period of μ is 1, we say that μ is *P-invariant.* We say that a P-periodic distribution $\mu \in D_M$ is *minimal* if for any P-periodic measure $\nu \ll \mu$ there exists a scalar t such that $\nu = t\mu$.

An arbitrary set $A \in \Sigma$ is called *absorbing* (or *stochastically closed*) if $P(s, A) = 1$ for $s \in A$. We say that an absorbing set is *indecomposable* if it contains no disjoint pair of absorbing sets.

A P-invariant probability measure μ is said to be *P-ergodic* if for all absorbing set A, we have $\mu(A) = 0$ or $\mu(A) = 1$.

The following theorem is the version of Komorník and Thomas [9] of the asymptotic periodic decomposition.

Theorem 2.1 Asymptotic periodic decomposition theorem (APDT)**.** *Let M be a band in $\mathcal{M}_\Sigma$ and $\cdot P$ be a quasi-constrictive Markov operator on M. Then:*

(a) *There exists*

- *a finite set $F_0 = \{\nu_1, ..., \nu_r\}$ of pairwise orthogonal P-periodic elements of D_M,*
- *a set $\{\lambda_1, ..., \lambda_r\}$ of continuous linear functionals on M, and*
- *a permutation σ of the integers $1, ..., r$ such that*

$$\lim_{n\to\infty} \left\| \mu P^n - \sum_{i=1}^{r} \lambda_i(\mu)\nu_{\sigma^n(i)} \right\| = 0 \quad \text{for each} \quad \mu \in M, \tag{3}$$

and

$$\nu_i P = \nu_{\sigma(i)} \quad \text{for} \quad i \in \{1, ..., r\}. \tag{4}$$

(b) *The functionals λ_i are positive, that is, $\lambda_i(\mu) \geq 0$ if $\mu \geq 0$. Moreover,*

$$\sum_{i=1}^{r} \lambda_i(\nu) = 1 \quad \text{for} \quad \nu \in D_M$$

and

$$|\lambda_i(\mu)| \leq \|\mu\| \quad \text{for} \quad \mu \in M. \tag{5}$$

(c) *The measures ν_i, for $i \in \{1, ..., r\}$, are minimal.*

(d) *The sets $\{\nu_1, ..., \nu_r\}$ and $\{\lambda_1, ..., \lambda_r\}$ satisfying (3) and (4) are unique.*

From APDT we can get the next corollary.

Corollary 2.2 *For a band in $\mathcal{M}_\Sigma$ the concepts of constrictivity and quasi-constrictivity are equivalents.*

We shall denote $D_{\mathcal{M}_\Sigma}$ simply by D; that is, D is the set of probability measures on Σ.

Let $\cdot P : \mathcal{M}_\Sigma \to \mathcal{M}_\Sigma$ be a quasi-constrictive Markov operator. Let us denote the set $\{\mu P^n : \mu \in D\}$ by $[D]P^n$ and let $D_\infty := \bigcap_{n=1}^{\infty} [D]P^n$ be the set of all the limit points of the sequences $(\mu P^n)_{n=1}^{\infty}$, with $\mu \in D$.

By the APDT, $\nu \in D_\infty$ if and only if it is a convex combination of the distributions $\nu_1, ..., \nu_r$. That is, D_∞ is the convex hull of the finite set $F_0 = \{\nu_1, ..., \nu_r\}$ given in the APDT.

We shall now identify the set $D_\infty^I \subset D_\infty$ of P-invariant distributions and the subset $D_\infty^E \subset D_\infty^I$ of all the P-ergodic distributions.

Two integers i and j in $\{1, ..., r\}$ are said to be *equivalent* (denoted by $i \leftrightarrow j$) if $\nu_i P^k = \nu_j$ for some positive integer k. Observe that $\leftrightarrow$ is an equivalence relation, and denote by $O_1, O_2, ..., O_d$ the different equivalence classes of $\{1, ..., r\}$. Let $\bar{O}_l := \{\nu_i : i \in O_l\}$. For $j \in \{1, ..., d\}$, let $\#O_j$ be the number of elements in O_j, and let

$$\tau_j := \frac{1}{\#O_j} \sum_{i \in O_j} \nu_i \tag{6}$$

be the "average" of the elements in $\bar{O}_j$. Observe that $\cdot P : \bar{O}_j \to \bar{O}_j$ is a bijection and $\nu_i P \in \bar{O}_j \Leftrightarrow \nu_i \in \bar{O}_j$. Therefore,

$$\sum_{i \in O_j} \nu_i = \sum_{\nu \in \bar{O}_j} \nu = \sum_{\nu \in \bar{O}_j} \nu P = \sum_{i \in O_j} \nu_i P = \left(\sum_{i \in O_j} \nu_i \right) P,$$

which gives that τ_j is a P-invariant distribution. Also note that $\tau_1, ..., \tau_d$ are mutually singular. The proof of the following theorem proceeds in the same way as in [17] (Theorem 20), in which S was assumed to be a countable set.

Theorem 2.3 Ergodic decomposition theorem. *Let $\cdot P : \mathcal{M}_\Sigma \to \mathcal{M}_\Sigma$ be a quasi-constrictive Markov operator and let $D_\infty^I \subset D_\infty$ be the set of all the P-invariant distributions. Then D_∞^I is a convex set and, in fact, it is the convex hull of $\{\tau_1, ..., \tau_d\}$ with τ_j as in (6), i.e.*

$$D_\infty^I = \left\{ \mu \in D : \mu = \sum_{j=1}^{d} \alpha_j \tau_j \quad \textit{with} \quad \alpha_j \geq 0 \quad \textit{and} \quad \sum_{j=1}^{d} \alpha_j = 1 \right\}. \tag{7}$$

Hence, $D_\infty^E = \{\tau_1, ..., \tau_d\}$ is the collection of all the P-ergodic distributions.

We denote by $Q(s, A)$ the probability that $X_n \in A$ for infinitely many n, given $X_0 = s$, and we denote by $L(s, A)$ the probability that $X_n \in A$ for some $n \in \mathbb{N}$, given $X_0 = s$, that is to say

$$Q(s, A) := P\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} [X_n \in A] | X_0 = s \right)$$

and

$$L(s, A) := P\left(\bigcup_{n=1}^{\infty} [X_n \in A] | X_0 = s \right),$$

for $s \in S$ and $A \in \Sigma$. An absorbing set A is called *Harris* if there exists a Σ-finite measure φ on Σ with $\varphi(A) > 0$ such that $\varphi(B) > 0 \Rightarrow Q(s,B) = 1$ for all $s \in A$. If we want to be specific, we say the set A is φ*-Harris*. The Harris sets are indecomposable. A set A is said to be *uniformly transient* if there exists $M < +\infty$ such that

$$\sum_{n=1}^{\infty} P^n(s,A) \leq M \quad \text{for } \ s \in S. \tag{8}$$

The sum given in (8) is the expectation of the number of times that the chain visits the set A given $X_0 = s$. A set is Σ*-transient* if it can be represented as a countable union of uniformly transient sets.

A set $A \in \Sigma$ said to be *strongly transient* if

$$\sum_{n=1}^{\infty} P^n(s,A) < +\infty \quad \text{for } \ s \in S. \tag{9}$$

It is easy to prove the next lemma.

Lemma 2.4 *If a set is strongly transient, then it is* Σ*-transient.*

We say that the Markov chain (X_n) admits a *Harris decomposition* for S if there exists a countable disjoint family $\{H_n\}$ of Harris sets and a Σ-transient set E such that

$$S = \left(\bigcup_n H_n\right) \cup E. \tag{10}$$

We shall prove that if the Markov chain (X_n) admits a Harris decomposition, then each ergodic measure μ is concentrated in one of the Harris sets H_n given in (10). We say that a measure μ on Σ is *concentrated* in a set $A \in \Sigma$, if $\mu(S \setminus A) = 0$.

Lemma 2.5 *If* μ *is a finite invariant measure and* E *is a* Σ*-transient set, then* $\mu(E) = 0$*.*

Proof Note that it is sufficient to prove the lemma when E is uniformly transient. Suppose that $\mu(E) > 0$. As μ is invariant, we have

$$\mu(E) = \mu P^j(E) = \int P^j(z,E)\mu(\mathrm{d}\, z) \quad \text{for } \ j \in \mathbb{N}.$$

On the other hand, as E is uniformly transient, there exists $M < +\infty$ such that

$$\sum_{j=1}^{\infty} P^j(s,E) \leq M, \quad \text{for } \ s \in S,$$

hence, as $\mu(E) > 0$, we have

$$\begin{aligned} +\infty = \sum_{j=1}^{\infty} \mu(E) &= \sum_{j=1}^{\infty} \int P^j(z,E)\mu(\mathrm{d}\, z) \\ &= \int \left(\sum_{j=1}^{\infty} P^j(z,E)\right) \mu(\mathrm{d}\, z) \leq \int M\mu(\mathrm{d}\, z) = M\mu(S), \end{aligned}$$

which is a contradiction. Therefore $\mu(E)=0$. $\square$

Theorem 2.6 *If ν is a finite periodic measure and E is a Σ-transient set, then $\nu(E)=0$.*

Proof Suppose that ν is periodic with period n and E is a Σ-transient set. Let $\mu=\nu+\nu P+\cdots+\nu P^{n-1}$ and observe μ is invariant. As $\nu \ll \mu$, from Lemma 2.5 it follows $\nu(E)=0$. $\square$

Theorem 2.7 *If the Markov chain (X_n) admits a Harris decomposition*

$$S=\left(\bigcup_n H_n\right)\cup E,$$

then:

(a) *each P-invariant probability measure is concentrated in $\bigcup_n H_n$,*

(b) *each P-ergodic measure is concentrated in some H_n.*

Proof Part (a) follows directly from Theorem 2.6. If μ is an ergodic measure, then we have from (a) that $\mu(\bigcup_n H_n)=1$. Hence, $\mu(H_n)>0$ for some n. Now, since H_n is absorbing and μ is ergodic, $\mu(H_n)=1$. $\square$

From Theorems 2.3 and 2.7 and Equation (6), we have the next corollary.

Corollary 2.8 *If the hypothesis of Theorem 2.7 is fulfilled and the operator $\cdot P$ is quasi-constrictive on $\mathcal{M}_\Sigma$, then:*

(a) *each distribution $\mu\in D_\infty$ (see Theorem 2.3) is concentrated in $\bigcup_n H_n$,*

(b) *each distribution ν_i given in APDT is concentrated in some H_n.*

3. The cyclic decomposition

We begin this section defining the concept of cycle.

A finite sequence $(C_1,C_2,\ldots,C_q)$ of q disjoint sets is a *cycle (of length q)* if each $C_j\in\Sigma$ and

$$P(s,C_{j+1})=1 \quad \text{for} \quad s\in C_j \quad \text{and} \quad 1\le j\le q-1, \quad \text{and}$$
$$P(s,C_1)=1 \quad \text{for} \quad s\in C_q.$$

In order to establish the cyclic decomposition we need some terminology.

Let φ bea Σ-finite measure on Σ. If $\varphi(A)>0\Rightarrow L(s,A)>0$, for all $A\in\Sigma$ and $s\in S$, we say that the Markov chain (X_n) is *φ-irreducible.* If $\varphi(A)>0\Rightarrow L(s,A)=1$, for all $A\in\Sigma$ and $s\in S$, we say that the Markov chain (X_n) is *φ-recurrent.*

An equivalent way to define a φ-recurrent Markov chain is saying that $\varphi(A)>0\Rightarrow Q(s,A)=1$, for all $A\in\Sigma$ and $s\in S$. We can use both versions according to our convenience.

A set $A\in\Sigma$ is said to be *inessential* if $Q(s,A)=0$ for all $s\in S$.

A Σ-algebra is *separable* if it is generated by a countable collection of sets.

Remark 3.1 In our context S has structure of separable metric space (S is Polish), so its topology is generated by a denumerable collection of sets. Thus, the Borel Σ-algebra Σ is separable.

Keeping in mind Remark 3.1, we have the hypothesis of Theorem 3.1, Ch. 1 in [14], Theorem 1 in [5] or Theorem 1.3 of Ch. 5 in [16], whose results are given in the following theorem.

Theorem 3.2 Cyclic decomposition theorem. *If the Markov chain on (S, Σ) is φ-irreducible, then there is a cycle $(C_1, C_2, \dots, C_q)$, such that the following conditions hold:*

(a) *The measure φ is concentrated in $\bigcup_{j=1}^{q} C_j$ and the set $S \setminus \bigcup_{j=1}^{q} C_j$ is a countable union of inessential sets.*

(b) *If $(C'_1, C'_2, \dots, C'_{q'})$ is a cycle, then q' divides q, and each C'_i differs from a union of q/q' members of $\{C_1, C_2, \dots, C_q\}$ only by a φ-null set.*

Corollary 3.3 *If the Markov chain on (S, Σ) is φ-irreducible, then there is a cycle $(C_1, C_2, \dots, C_q)$, such that the following conditions hold:*

(a) *The measure φ is concentrated in $\bigcup_{j=1}^{q} C_j$ and the set $S \setminus \bigcup_{j=1}^{q} C_j$ is Σ-transient.*

(b) *If $(C'_1, C'_2, \dots, C'_{q'})$ is a cycle, then q' divides q, and each C'_i differs from a union of q/q' members of $\{C_1, C_2, \dots, C_q\}$ only by a φ-null set.*

Proof Theorem 6 in [12] claims that any inessential set is a countable union of strongly transient sets. So, any countable union of inessential sets is a countable union of strongly transient sets, and by Lemma 2.4, it is Σ-transient. □

If a cycle $(C_1, C_2, \dots, C_q)$ holds the condition (b) of the cyclic decomposition theorem we say that it is a φ-*maximal* cycle.

Notation 3.4 In the context of Equation (6), let us denote $q_j := \#O_j$ and let $\nu_{j,1}, \nu_{j,2}, \dots, \nu_{j,q_j}$ probability measures such that $\{\nu_{j,1}, \nu_{j,2}, \dots, \nu_{j,q_j}\} = O_j$, $\nu_{j,k}P = \nu_{j,k+1}$ for $k \in \{1, 2, \dots, q_j - 1\}$, and $\nu_{j,q_j}P = \nu_{j,1}$, where $j \in \{1, 2, \dots, d\}$.

We denote the *Cesàro sums* by

$$P^{(n)} := \frac{1}{n} \sum_{k=0}^{n-1} P^k.$$

If we see P^k as a transition probability, we denote

$$P^{(n)}(s, \cdot) := \frac{1}{n} \sum_{k=0}^{n-1} P^k(s, \cdot)$$

and it is called the n-step *expected occupation measure* with initial state s.

If $\nu \in O_j$ then $\nu P^{q_j} = \nu P^{r!} = \nu$ and $\nu P^{(q_j)} = \nu P^{(r!)} = \tau_j$, where r is the number given in the APDT (Theorem 2.1). Furthermore, there are positive continuous linear functionals $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_d$ on $\mathcal{M}_\Sigma$ such that if $\mu \in \mathcal{M}_\Sigma$, then

$$\lim_{k\to\infty} \left\| \mu P^{(k)} - \sum_{k=1}^{d} \hat{\lambda}_k(\mu)\tau_k \right\| = 0, \quad \text{where} \quad \sum_{k=1}^{d} \hat{\lambda}_k(\mu) = 1. \tag{11}$$

We shall see that any constrictive operator admits a Harris decomposition.

Lemma 3.5 *The set* $S_j := \left\{ s \in S : \lim_{k\to\infty} \left\| \delta_s P^{(k)} - \tau_j \right\| = 0 \right\}$ *is* τ_j*-Harris.*

Proof Let us probe first that S_j is nonempty. Note that if $s \notin S_j$, then there is an $i \neq j$ such that $\hat{\lambda}_i(\delta_s) > 0$. We have

$$\tau_j = \tau_j P^{(n)} = \int P^{(n)}(s,\cdot)\tau_j(\mathrm{d}\, s) = \int \delta_s P^{(n)}(\cdot)\tau_j(\mathrm{d}\, s), \tag{12}$$

for all $n \in \mathbb{N}$, so from (11) and (12) we get

$$\tau_j = \int \sum_{k=1}^{d} \hat{\lambda}_k(\delta_s)\tau_k(\cdot)\tau_j(\mathrm{d}\, s) = \sum_{k=1}^{d} \left(\int \hat{\lambda}_k(\delta_s)\tau_j(\mathrm{d}\, s) \right) \tau_k(\cdot). \tag{13}$$

Now, if S_j were empty the coefficient of some $\tau_k \neq \tau_j$ in the right part of (13) would be different than 0, in contradiction with the fact that the measures $\tau_1, \tau_2, \ldots, \tau_d$ are mutually singular, therefore $S_j \neq \varnothing$.

Let us probe now that S_j is absorbing. Let $s_0 \in S_j$ and $p_0 := P(s_0, S_j)$. If $p_0 < 1$, then

$$\mu_0 := \frac{P(s_0, (S \setminus S_j) \cap \cdot)}{1 - p_0}$$

is a probability measure concentrated in $S \setminus S_j$, but

$$P^{n+1}(s_0, \cdot) = P^{n+1}(s_0, S_j \cap \cdot) + (1 - p_0)\mu_0 P^n,$$

so

$$P^{(n+1)}(s_0, \cdot) = P^{(n+1)}(s_0, S_j \cap \cdot) + (1 - p_0)\mu_0 P^{(n)}. \tag{14}$$

Using the definition of S_j and making n to tend to ∞ in (14), we have

$$\tau_j = \tau_j(S_j \cap \cdot) + (1 - p_0) \sum_{k=1}^{d} \hat{\lambda}_k(\mu_0)\tau_k(\cdot). \tag{15}$$

Since μ_0 is concentrated in $S \setminus S_j$ we get that $\hat{\lambda}_i(\mu_0) > 0$ for some $i \neq j$, and therefore Equation (15) and the fact that $\tau_1, \tau_2, \ldots, \tau_d$ are mutually singular are in conflict with the assumption that $p_0 < 1$; thus, S_j is absorbing.

We turn now to show that S_j is τ_j-Harris. From Equation (13) we can conclude that τ_j is concentrated in S_j. Let $A \subset S_j$ be a set such that $\tau_j(A) > 0$ and let $s_0 \in S_j$. Since there is an $N \in \mathbb{N}$ such that $\|P^{(n)}(s_0, \cdot) - \tau_j\| < \frac{\tau_j(A)}{4}$ for all integers, $n \geq N$, we have that for all integers, $N' \geq N$ there is an $n \geq N'$

such that $P^n(s_0, A) > \frac{\tau_j(A)}{2}$, which implies $L(s_0, A) = 1$. □

From Lemma 3.5 and from Equation (11) we get the next lemma.

Lemma 3.6 *If $S_1, S_2, \ldots, S_d$ are as in Lemma 3.5, then the set $E = S \setminus \bigcup_{i=1}^{d} S_i$ is inessential.*

From Lemmas 3.5 and 3.6 we get the next theorem.

Theorem 3.7 *If the Markov chain (X_n) admits a spectral decomposition as in the APDT, then there is a Harris decomposition of the form*

$$S = \left(\bigcup_{i=1}^{d} S_i \right) \cup E, \tag{16}$$

where each S_i is τ_i-Harris and E is inessential.

Watching one absorbing set S_j as if it were all the space S and remembering that it is τ_j-Harris, we shall use Notation 3.4 and the cyclic decomposition theorem to show that there is a cycle $(C_{j,1}, C_{j,2}, \ldots, C_{j,q_j})$ with the properties of such a theorem (replacing S by S_j, C_i by $C_{j,i}$, q by q_j and φ by τ_j), and moreover each $\nu_{j,k}$ is concentrated in $C_{j,k}$.

If we take the process (X_n) restricted to the set S_j, we shall denote the corresponding Markov operator by P_j.

The spectral decomposition of the Markov operator P_j is given by the formula

$$\lim_{n\to\infty} \left\| \mu P_j^n - \sum_{i=1}^{q_j} \lambda_{j,i}(\mu) \nu_{j,\sigma_j^n(i)} \right\| = 0, \tag{17}$$

for each finite measure μ concentrated in S_j, where $\sigma_j(k) = k+1$ for $k \in \{1, 2, \ldots, q_j - 1\}$, $\sigma_j(q_j) = 1$ and $\sum_{i=1}^{q_j} \lambda_{j,i}(\mu) = 1$. So, the spectral decomposition of the operator $P_j^{q_j}$ is of the form

$$\lim_{n\to\infty} \left\| \mu P_j^{nq_j} - \sum_{i=1}^{q_j} \lambda_{j,i}(\mu) \nu_{j,i} \right\| = 0,$$

thus the ergodic measures of the operator $P_j^{q_j}$ are $\nu_{j,1}, \nu_{j,2}, \ldots, \nu_{j,q_j}$. Therefore, in a similar way to the construction of the Harris decomposition given in Equation (16), we take

$$C_{j,i} := \left\{ s \in S_j : \lim_{k\to\infty} \|\delta_s P_j^{nq_j} - \nu_{j,i}\| = 0 \right\} \tag{18}$$

and we have the Harris decomposition of S_j relative to the operator P^{q_j} in the form

$$S_j = \left(\bigcup_{i=1}^{q_j} C_{j,i} \right) \cup E_j, \tag{19}$$

where each $\nu_{j,i}$ is concentrated in $C_{j,i}$ and E_j is inessential. We claim that the finite sequence $(C_{j,1}, C_{j,2}, \ldots, C_{j,q_j})$ of sets given in (18) is a cycle of the process relative to the operator P_j. Effectively, if $s \in C_{j,i}$ then, by formulae (17) and (18), we have $\lim_{n\to\infty} \left\| \delta_s P_j^{nq_j+1} - \nu_{j,\sigma_j(i)} \right\| = 0$, but $\delta_s P_j^{nq_j+1} = P_j(s,\cdot) P_j^{nq_j}$, therefore $P_j(s,\cdot)$ is concentrated in $C_{j,\sigma_j(i)}$, that is $P_j(s, C_{j,\sigma_j(i)}) = 1$ so $(C_{j,1}, C_{j,2}, \ldots, C_{j,q_j})$ is a cycle of length q_j and τ_j is concentrated in $\bigcup_{i=1}^{q_j} C_{j,i}$.

Let us show that $(C_{j,1}, C_{j,2}, \ldots, C_{j,q_j})$ is a τ_j-maximal cycle. If we had another cycle $(C_1^*, C_2^*, \ldots, C_{q^*}^*)$ of length $q^* > q_j$ such that q_j divides q^* and each $C_{j,i}$ differs from a union of q^*/q_j members of $\{C_1^*, C_2^*, \ldots, C_{q^*}^*\}$ only by a τ_j-null set, then we would have q^*/q_j different points $s_1, s_2, \ldots, s_{q^*/q_j} \in C_{j,1}$ with the property that $s_1, s_2, \ldots, s_{q^*/q_j}$ are in q^*/q_j different sets of $\{C_1^*, C_2^*, \ldots, C_{q^*}^*\}$ and, for all positive integers, k, the probability measures $\delta_{s_1} P_j^{kq_j}, \delta_{s_2} P_j^{kq_j}, \ldots, \delta_{s_{q^*/q_j}} P_j^{kq_j}$ would be mutually singular, in contradiction with (18). Hence, from the cycle decomposition theorem, $(C_{j,1}, C_{j,2}, \ldots, C_{j,q_j})$ is a τ_j-maximal cycle.

In view of (19) and the above discussion, we have proved the next theorem.

Theorem 3.8 *Let $\cdot P$ be a constrictive operator on $\mathcal{M}_\Sigma$ and let us use the notation of this article. We have a decomposition of the space S as a finite disjoint union given by*

$$S = E \cup \left(\bigcup_{j=1}^{d} \left(E_j \cup \bigcup_{i=1}^{q_j} C_{j,i} \right) \right) = \left(E \cup \bigcup_{j=1}^{d} E_j \right) \cup \left(\bigcup_{j=1}^{d} \bigcup_{i=1}^{q_j} C_{j,i} \right),$$

where

(a) *The set $E \cup \bigcup_{j=1}^{d} E_j$ is inessential.*

(b) *We have each $S_j = E_j \cup \bigcup_{i=1}^{q_j} C_{j,i}$ and each S_j is τ_j-Harris.*

(c) *Each set $\bigcup_{i=1}^{q_j} C_{j,i}$ is also τ_j-Harris.*

(d) *Each finite sequence $(C_{j,1}, C_{j,2}, \ldots, C_{j,q_j})$ is a τ_j-maximal cycle.*

(e) *Each probability measure of the form $\nu_{j,i}$ is concentrated in $C_{j,i}$.*

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References

[1] Bartoszek, W.: Asymptotic periodicity of the iterates of positive contractions on Banach lattices. *Studia Math.* **91**(3), 179–188 (1988).

[2] Bartoszek, W.: Asymptotic properties of the iterates of stochastic operators on (AL) Banach lattices. *Ann. Polon. Math.* **52**(2), 165–173 (1990).

[3] Bartoszek, W.: Riesz decomposition property implies asymptotic periodicity of positive and constrictive operators. *Proc. Amer. Math. Soc.* **116**(1), 101–111 (1992).

[4] Emelýanov, È.Y.: Regularity conditions for Markov semigroups on abstract L^1-spaces (translation of *Mat. Tr.* **7**(1), 50–82 (2004)). *Siberian Adv. Math.* **14**(4), 1–29 (2004).

[5] Jain, N. and Jaimson, B.: Contributions to Doeblin's theory of Markov processes. *Z. Wahrscheinlichkeitstheorie verw. Geb.* **8**, 19–40 (1967).

[6] Komorník, J.: Asymptotic periodicity of the iterates of weakly constrictive Markov operators. *Tohoku Math. J.* (2) **38**(1), 15–27 (1986).

[7] Komorník, J.: Asymptotic periodicity of Markov and related operators. In: *Dynamics Reported* (Eds.: C.K.R.T. Jones et al.) 31–68, Expositions in Dynamical Systems. New series **2**. Berlin. Springer-Verlag 1993.

[8] Komorník, J. and Lasota, A.: Asymptotic decomposition of Markov operators. *Bull. Pol. Acad. Sci., Math.* **35**(5–6), 321–327 (1987).

[9] Komorník, J. and Thomas, E.G.F.: Asymptotic periodicity of Markov operators on signed measures. *Math. Bohem.* **116**(2), 174–180 (1991).

[10] Lasota, A., Li, T.-Y. and Yorke, J.A.: Asymptotic periodicity of the iterates of Markov operators. *Trans. Amer. Math. Soc.* **286**(2), 751–764 (1984).

[11] Lasota, A. and Mackey, M.C.: *Chaos, Fractals and Noise*, 2nd ed. New York. Springer-Verlag 1994.

[12] Meyn, S.P. and Tweedie, R.L.: The Doeblin decomposition. In: *Doeblin and Modern Probability* (Ed.: S. Grigorescu) 211–225, Contemp. Math. **149** (1993).

[13] Meyn, S.P. and Tweedie, R.L.: *Markov Chains and Stochastic Stability*. Berlin. Springer-Verlag 1993.

[14] Orey, S.: *Limit Theorems for Markov Chains Transition Probabilities*. London. Van Nostrand Reinhold 1971.

[15] Räbiger, F.: Attractors and asymptotic periodicity of positive operators on Banach lattices. *Forum Math.* **7**(6), 665–683 (1995).

[16] Revuz, D.: *Markov Chains*. Amsterdam. North-Holand Publishing Company 1975.

[17] Villarreal, C.E.: Ergodic decomposition of Markov chains. *Linear Algebra Appl.* **283**(1), 61–73 (1998).