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# The total graph of a finite commutative ring

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**Abstract:** Let $R$ be a commutative ring with $Z(R)$, its set of zero-divisors and $\mathrm{Reg}(R)$, its set of regular elements. Total graph of $R$, denoted by $T(\Gamma(R))$, is the graph with all elements of $R$ as vertices, and two distinct vertices $x, y \in R$, are adjacent in $T(\Gamma(R))$ if and only if $x+y \in Z(R)$. In this paper, some properties of $T(\Gamma(R))$ have been investigated, where $R$ is a finite commutative ring and a new upper bound for vertex-connectivity has been obtained in this case. Also, we have proved that the edge-connectivity of $T(\Gamma(R))$ coincides with the minimum degree if and only if $R$ is a finite commutative ring such that $Z(R)$ is not an ideal in $R$.

**Key words:** Commutative rings, total graph, regular elements, zero-divisors

## 1. Introduction

We assume throughout that all rings are commutative with $1 \neq 0$. We denote the set of zero-divisor elements and the set of regular elements of $R$ by $Z(R)$ and $\mathrm{Reg}(R)$, respectively ($\mathrm{Reg}(R) = R \backslash Z(R)$). Let $Z(\Gamma(R))$ and $\mathrm{Reg}(\Gamma(R))$ be the (induced) subgraphs of $T(\Gamma(R))$ with vertices $Z(R)$ and $\mathrm{Reg}(R)$, respectively. In [2] Anderson and Badawi introduced the total graph of $R$, denoted by $T(\Gamma(R))$, as the graph with all elements of $R$ as vertices, and two distinct vertices $x, y \in R$ are adjacent in $T(\Gamma(R))$ if and only if $x+y \in Z(R)$. The subgraph $Z(\Gamma(R))$ of $T(\Gamma(R))$ is always connected; see Theorems 2.1 and 3.1 in [2]. Also, $Z(\Gamma(R))$ is complete if and only if $Z(R)$ is an ideal of $R$.

Additionally, if $Z(R)$ is an ideal of $R$, then $Z(\Gamma(R))$ and $\mathrm{Reg}(\Gamma(R))$ are disjoint subgraphs of $T(\Gamma(R))$, and $\mathrm{Reg}(\Gamma(R))$ is the union of disjoint subgraphs, each of which is either a complete graph or a complete bipartite graph; see Theorem 2.2 in [2]. However, if $Z(R)$ is not an ideal of $R$, then the subgraphs $Z(\Gamma(R))$ and $\mathrm{Reg}(\Gamma(R))$ of $T(\Gamma(R))$ are never disjoint. In Theorems 3.3 in [2], it was proved that for every commutative ring $R$, if $Z(R)$ is not an ideal of $R$, then $T(\Gamma(R))$ is connected if and only if the ideal generated by $Z(R)$ is $R$ (i.e $R = (z_1, z_2, \cdots, z_n)$ for some $z_1, z_2, \cdots, z_n \in Z(R)$). In particular, if $R$ is a finite commutative ring and $Z(R)$ is not an ideal of $R$, then $T(\Gamma(R))$ is connected. Also in Theorem 3 in [1], it was proved that for every finite commutative ring $R$, if $Z(R)$ is not an ideal, then $T(\Gamma(R))$ is a Hamiltonian graph.

In this paper we show that the subgraph $\mathrm{Reg}(\Gamma(R))$ with vertices $\mathrm{Reg}(R)$ is connected and its vertices have the same degree and its diameter is at most 2. Moreover for each $x \in Z(R)$ which is not nilpotent, there is $p \in \mathbb{N}$ such that $x^p - 1 \in Z(R)$ and $x \pm x^p - 1 \in \mathrm{Reg}(R)$. At the end, since the total graph of a finite commutative ring has a finite number of vertices and edges, we are able to investigate edge-connectivity and

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vertex-connectivity in terms of $T(\Gamma(R))$ and show that in finite commutative ring $R$, vertex-connectivity of $T(\Gamma(R))$ is less than $\alpha - |\mathrm{Nil}(R)|$, where $\alpha = |Z(R)|$ and $Nil(R)$ is the set of nilpotent elements of $R$. Also if $R$ is a finite commutative ring such that $Z(R)$ is not an ideal, then edge-connectivity of $T(\Gamma(R))$ is always equal to $(\alpha - 1)$.

## 2. Preliminaries

At first, we recall various conventions and definitions from graph theory. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ is the set of vertices of $G$ and $E(G)$ is the set of edges of $G$. We say that $G$ is connected if there is a path between any two distinct vertices of $G$. For any $x, y \in V(G)$, we define $d(x, y)$ to be the length of the shortest path from $x$ to $y$ ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The diameter of $G$ is defined: $\mathrm{diam}(G) = \sup\{d(x, y) \mid x, y \in V(G)\}$.

We say that two distinct subgraph $G_1$ , $G_2$ of $G$ are disjoint if they have no common vertices and no vertex of $G_1$ (respectively, $G_2$) is adjacent (in $G$) to any vertex of $G_2$ (respectively, $G_1$). If vertex $v$ is an end point of edge $e$, then $v$ and $e$ are incident. The degree of a vertex $v$ in a graph $G$, written $\deg_G(v)$ is the number of edges incident to $v$. The minimum degree in a graph $G$ is denoted by $\delta(G)$. Graph $G$ is called $k$-regular if degree of each vertex of $G$ be $k$. If $S \subseteq V(G)$ is any subset, we denote by $G - S$, the graph whose vertex set is $V(G) - S$ and whose edge set is $E(G) - \{\{x, y\} | \{x, y\} \cap S \neq \emptyset\}$. A vertex cut of $G$ is a subset $S \subseteq V(G)$ such that $G - S$ is disconnected. If $T \subseteq E(G)$ is any subset, we denote by $G - T$, the graph whose vertex set is $V(G)$ and whose edge set is $E(G) - T$. An edge cut of $G$ is a subset $T \subseteq E(G)$, such that the graph $G - T$ is disconnected. The vertex-connectivity of $G$ is defined by

$$\kappa(G) = \min\{n \geq 0 \mid \text{there exists a vertex cut } S \subseteq V(G) \text{ such that } |S| = n\},$$

if $G$ has a finite vertex cut, and $\kappa(G) = \infty$ otherwise. Similarly, the edge-connectivity of $G$ is defined by:

$$\lambda(G) = \min\{n \geq 0 \mid \text{there exist a edge cut } T \subseteq E(G) \text{ such that } |T| = n\},$$

if $G$ has a finite edge cut, and $\lambda(G) = \infty$ otherwise. One can refer to [8] for further information.

The following lemma is a direct consequence of the definitions of $\kappa(G), \lambda(G), \delta(G)$.

**Lemma 2.1** *For any graph $G$, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

We refer to [7] for the following lemma.

**Lemma 2.2** *For any graph $G$ with $diam(G) = 1$ or $2$; $\lambda(G) = \delta(G)$.*

The group of units of a commutative ring $R$ will be denoted by $U(R)$ and the nonzero elements of $A \subseteq R$ will be denoted by $A^*$. Also we say that $R$ is reduced if $\mathrm{Nil}(R) = \{0\}$; one can refer to [3],[4] and [6] for further information on ring theory. Recall that for each natural number $n$, the function $\varphi(n)$ is the number of integers $t$, $1 \leq t \leq n$, such that $\gcd(t, n) = 1$.

If $R$ is a finite commutative ring such that $Z(R)$ is an ideal of $R$, then $R$ is local with $Z(R) = \mathrm{Nil}(R)$ its unique maximal ideal.

Lemma 2.3 below follows directly from Theorem 1 in [5].

**Lemma 2.3** *$Z(R)$ is finite if and only if either $R$ is finite or an integral domain.*

**Corollary 2.4** *Let $R$ be a commutative ring such that $Z(R)$ is not an ideal and $|Z(R)| < \infty$, then $T(\Gamma(R))$ is finite.*

We refer to Exercise 2.26 in [4] for the following lemma.

**Lemma 2.5** *Let $R$ be a commutative ring. There are non-trivial rings $R_1$ and $R_2$ such that $R \simeq R_1 \times R_2$ if and only if there exists a non-trivial idempotent $e \in R$. In this case one can choose $R_1 = Re$ and $R_2 = R(1-e)$.*

According to the previous lemma, if we consider a finite commutative ring containing idempotent elements, we can write it as the product of two rings. We continue the process until we reach an idempotent-free ring. The process will always stop, because $R$ is finite.

## 3. Main results

The basic properties of $T(\Gamma(R))$ and $\mathrm{Reg}(\Gamma(R))$ are given below, independent of whether or not $Z(R)$ is an ideal of $R$.

Let $|Z(R)| = \alpha$ (we allow $\alpha$ to be infinite cardinals); then according to Theorem 2.3, if $2 \leq \alpha < \infty$ then the commutative ring $R$ is finite.

**Theorem 3.1** *Let $R$ be a commutative ring with $|Z(R)| = \alpha$, then*

$$deg_{T(\Gamma(R))}(x) = (\alpha - 1) \qquad \forall x \in Z(R).$$

**Proof** Each $x \in Z(R)^*$ is adjacent to $0$. Now, assume that $z_1 \neq 0$ and $2z_1 = z_\alpha, z_1, z_2, \cdots z_{\alpha-1} \in Z(R)$, then

$$z_1 + (z_i - z_1) \in Z(R) \text{ for each integer } i; 1 \leq i \leq \alpha - 1,$$

and proof is completed. □

**Theorem 3.2** *Let $R$ be a commutative ring with $|Z(R)| = \alpha$, then*

*1) If $2 \in Z(R)$ then $deg_{T(\Gamma(R))}(x) = (\alpha - 1)$ for each $x \in Reg(R)$.*

*2) If $2 \notin Z(R)$ then $deg_{T(\Gamma(R))}(x) = \alpha$ for each $x \in Reg(R)$.*

**Proof** (1) Assume that $a \in \mathrm{Reg}(R)$, $z_\alpha = 2a$ and $z_1, z_2, \cdots, z_{\alpha-1} \in Z(R)$, then $a + (z_i - a) \in Z(R); 1 \leq i \leq \alpha - 1$ and proof is completed in this case.

(2) Assume that $z_1, z_2, \cdots z_\alpha \in Z(R)$ and $a \in \mathrm{Reg}(R)$. Then

$$a + (z_i - a) \in Z(R) \quad \text{for each integer } 1 \leq i \leq \alpha,$$

and proof is completed. □

**Corollary 3.3** *Let $R$ be a commutative ring with $|Z(R)| = \alpha$, then if $2 \in Z(R)$ then $T(\Gamma(R))$ is $\alpha - 1$-regular.*

**Proof** It is clear from Theorems 3.1 and 3.2. □

When $Z(R)$ is an ideal of $R$, properties of $T(\Gamma(R))$ and $\mathrm{Reg}(\Gamma(R))$ and $Z(\Gamma(R))$ are mentioned in [2] Theorems 2.1 and 2.2. In the following we intend to investigate properties of finite commutative ring $R$ when $Z(R)$ is not an ideal. Note that if $R$ is finite then for each $x \in R$ either $x \in U(R)$ or $x \in Z(R)$. Also since $Z(R)$ is always closed under multiplication by elements of $R$, this just means that there are distinct $x, y \in Z(R)^*$, such that $x + y \in \mathrm{Reg}(R)$. In this case, $Z(\Gamma(R))$ is always connected (but never complete).

$Z(\Gamma(R))$ and $\mathrm{Reg}(\Gamma(R))$ are never disjoint subgraphs of $T(\Gamma(R))$ by Theorem 3.1 in [2] and since $R$ is a finite commutative ring, $3 \leq |Z(R)| < \infty$ and for each $x \in T(\Gamma(R)); \deg_{T(\Gamma(R))}(x) \geq 2$.

**Theorem 3.4** *Let $R$ be a finite commutative ring such that $Z(R)$ is not an ideal of $R$. Then each $x \in Reg(R)$ is adjacent to* m *elements of $Reg(R)$ and* n *elements of $Z(R)$, where*

$$deg_{T(\Gamma(R))}(x) = \mathrm{m} + \mathrm{n} = \mathrm{k}.$$

**Proof** Let $x \in \mathrm{Reg}(R)$ be a vertex of degree $k$ in $T(\Gamma(R))$, so there is $n \in \mathbb{N}$ such that $x^n = 1$. Then there is $r_i \in R$ ; $1 \leq i \leq k$ such that $x + r_i \in Z(R)$. Let $y \in \mathrm{Reg}(R)$ then:

$$yx^{n-1}(x + r_i) = y + yx^{n-1}r_i \in Z(R),$$

since $y$ was arbitrary; we conclude that $\deg_{T(\Gamma(R))}(y) = k$. Now if $r_i \in \mathrm{Reg}(R)$ $(Z(R))$ then $yx^{n-1}r_i \in \mathrm{Reg}(R)(Z(R))$ and there is a bijection between those vertices of $\mathrm{Reg}(R)(Z(R))$ which are adjacent to $x$ and those vertices of $\mathrm{Reg}(R)(Z(R))$ which are adjacent to $y$ and proof is completed. □

**Remark 3.5** *Let $R$ be a finite commutative ring such that $Z(R)$ is not an ideal of $R$ and let $x \in Z(R) - Nil(R)$; then powers of $x$ will constitute a semigroup, and for each $x \in Z(R) - Nil(R)$, there are similar results as above(i.e $x \in Z(R) - Nil(R)$ is adjacent to* t *elements of $Reg(R)$ and* s *elements of $Z(R)$). Also each $x \in Nill(R)$ is adjacent to all elements of $Z(R)$; therefore elements of $Reg(R), Nill(R)$ and semigroup of powers of $x$ in which $x \in Z(R) - Nil(R)$ have similar properties.*

In the following theorem we show that $diam(\mathrm{Reg}(\Gamma(R)) \leq 2$ and $\mathrm{Reg}(\Gamma(R))$ is m-regular.

**Theorem 3.6** *Let $R$ be a finite commutative ring such that $Z(R)$ is not an ideal of $R$ then:*

*1) $Reg(\Gamma(R))$ is connected and $diam(Reg(\Gamma(R)) \leq 2$.*

*2) $Reg(\Gamma(R))$ is m-regular.*

**Proof** 1) Let $x \in Z(R) - \mathrm{Nil}(R)$; then powers of $x$ will constitute a semigroup and we conclude that $R$ has a nontrivial idempotent element. Since $R$ is finite then according to Lemma 2.5 $R \simeq R_1 \times R_2$. Note that $\mathrm{Reg}(R) = \mathrm{Reg}(R_1 \times R_2) = \mathrm{Reg}(R_1) \times \mathrm{Reg}(R_2)$. So for distinct $(a, b), (c, d) \in \mathrm{Reg}(R_1 \times R_2)$ , $(a, b) - (-a, d) - (c, d)$ is a path of length at most two in $\mathrm{Reg}(R)$. Thus $(\mathrm{Reg}(\Gamma(R))$ is connected with $diam(\mathrm{Reg}(\Gamma(R))) \leq 2$.

2) It is clear from the proof of Theorem 3.4. □

In the following theorem we show that if $R$ is a finite commutative ring and $Z(R)$ is not an ideal in $R$, then for $x \in Z(R) - \mathrm{Nil}(R)$ we can find $y \in Z(R)$ such that $x \pm y \in \mathrm{Reg}(R)$.

**Theorem 3.7** *Let $R$ be a finite commutative ring such that $Z(R)$ is not an ideal of $R$; then there is $p \in \mathbb{N}$ such that*

$$x \pm x^p - 1 \in Reg(R) \quad ; x^p - 1 \in Z(R) \quad \forall x \in Z(R) - Nil(R).$$

**Proof** According to hypothesis we can find $m, n \in \mathbb{N}, m > n$ such that $x^m = x^n$. Without loss of generality we assume that $m$ and $n$ are the smallest numbers with this property. Since $x^m = x^n$, we conclude $x^n(x^{m-n} - 1) = 0$ and consequently $x(x^{m-n} - 1) \in \mathrm{Nil}(R)$. We claim that $m - n = p$. We will consider two cases:

1) If $m - n = 1$ then $(2x - 1)(2x^n - 1) = 2x(x^n - 1) + 1 \in \mathrm{Reg}(R)$ and since $2x(x^n - 1) \in \mathrm{Nil}(R)$ and $\mathrm{Nil}(R) + \mathrm{Reg}(R) \subset \mathrm{Reg}(R)$ then

$$(2x - 1) \in \mathrm{Reg}(R).$$

2) If $m - n > 1$, then

$$(x \pm (x^{m-n} - 1))(x^{(2^t+1)(m-n)-1} \pm x^{2^t(m-n)} \mp 1) =$$

$$x(x^{2^t(m-n)} - 1)(x^{m-n-1} \pm 1) + 1 \in \mathrm{Reg}(R),$$

where $t \in \mathbb{N}$ is chosen such that $2^t(m - n) \geq n$.

Since

$$x(x^{2^t(m-n)} - 1)(x^{m-n-1} \pm 1) \in \mathrm{Nil}(R),$$

and $\mathrm{Nil}(R) + \mathrm{Reg}(R) \subset \mathrm{Reg}(R)$, therefore $x \pm (x^{m-n} - 1) \in \mathrm{Reg}(R)$ and proof is completed.

□

**Corollary 3.8** *Let $R$ be a finite commutative ring such that $Z(R)$ is not an ideal of $R$. Then each $s \in \mathbb{N}$*

$$x^s \pm (x^p - 1) \in Reg(R) \quad \forall x \in Z(R) - Nil(R),$$

*where $p$ is as mentioned in the above theorem.*

**Proof** This is clear from the proof of Theorem 3.7. □

**Remark 3.9** *In Theorem 3.6, if we let $n = 1$ and $m = 2$, then $(2x - 1)^2 = 1$. Let $a = x + (x - 1) = 2x - 1$, then $a^{-1} = a$; moreover, if $(m - n) > 1$ and $a = x + x^{m-1} - 1$, then $a^{-1} = x^{m-1} + x^{m-2} - 1$.*

In the next part we will discuss $\kappa(T(\Gamma(R)))$ and $\lambda(T(\Gamma(R)))$. According to Theorem 4 in [1], if $R \simeq R_1 \times R_2$ is a finite commutative ring then $\kappa(T(\Gamma(R))) \geq |R_1| + |R_2| - 4$, which gives us a lower bound on $\kappa(T(\Gamma(R)))$, but is not very desirable. In the following we will propose an upper bound on $\kappa(T(\Gamma(R)))$ which we guess is exactly equal to $\kappa(T(\Gamma(R)))$. Note that in the next theorem there is no restriction on $Z(R)$.

**Theorem 3.10** *Let $R$ be a finite commutative ring. Then*

$$\kappa(T(\Gamma(R))) \leq |Z(R)| - |Nil(R)|.$$

**Proof** If $R$ is an finite commutative ring such that $Z(R)$ is an ideal in $R$ then by Theorem 2.1 in [2], $T(\Gamma(R))$ is not connected and therefore $\kappa(T(\Gamma(R))) = |Z(R)| - |\mathrm{Nil}(R)| = 0$. On the other hand, if $Z(R)$ is not an ideal in $R$, let $Z(R) - \mathrm{Nil}(R) = S$ then $T(\Gamma(R)) - S$ is disconnected and $\kappa(T(\Gamma(R))) \leq |S|$. □

**Theorem 3.11** *Let $R$ be a finite commutative ring such that $Z(R)$ is not an ideal of $R$ with $|Z(R)| = \alpha$. Then*

*1)* $\lambda(T(\Gamma(R))) = \alpha - 1$.

*2)* $\lambda(Reg(\Gamma(R))) = m$ *($m$ is as mentioned in part 2 Theorem 3.6)*

**Proof** 1) This is clear from Theorem 3.4 in [2], Lemma 2.2 and Theorems 3.1 and 3.2, respectively.

2) It follows from Theorem 3.6 and Lemma 2.2. □

**Example 3.12** *Let $n \geq 2$ be an integer, then $Z(\mathbb{Z}_n)$ is an ideal in $\mathbb{Z}_n$ if and only if $n = p^\alpha$ for some prime $p$ and integer $\alpha \geq 1$. Since $\mathbb{Z}_n$ is a finite ring, then $\mathbb{Z}_n$ is Artinian, the structure Theorem [3, p. 752, Theorem 3] implies that if $n = \prod_{i=1}^m p_i^{\alpha_i}$ for some prime $p_i$ and integer $\alpha_i \geq 1$, then $Z_n \simeq \prod_{i=1}^m \mathbb{Z}_{p_i}^{\alpha_i}$ where $\mathbb{Z}_{p_i}^{\alpha_i}$ is a local ring with $M_i$, its unique maximal ideal for each $i$, and*

$$\varphi(n) = n \prod_{i=1}^m (1 - \frac{1}{p_i}),$$

*then according to Theorems 3.1 and 3.2 for each $x \in Z(\mathbb{Z}_n)$*

$$deg_{T(\Gamma(\mathbb{Z}_n))}(x) = n - \varphi(n) - 1,$$

*and for each $x \in Reg(\mathbb{Z}_n)$*

$$deg_{T(\Gamma(\mathbb{Z}_n))}(x) = \begin{cases} n - \varphi(n) - 1 & 2 \in Z(\mathbb{Z}_n) \\ n - \varphi(n) & 2 \notin Z(\mathbb{Z}_n), \end{cases}$$

*and if $2 \in Z(\mathbb{Z}_n)$ then $Reg(\Gamma(\mathbb{Z}_n))$ is complete and $deg_{Reg(\Gamma(\mathbb{Z}_n))}(x) = \varphi(n) - 1$. In addition $Nil(R) = (\prod_{i=1}^m p_i)R$ and $\lambda(T(\Gamma(\mathbb{Z}_n))) = n - \varphi(n) - 1$,*

$$\kappa(T(\Gamma(\mathbb{Z}_n))) \leqslant n((1 - \frac{1}{\prod_{i=1}^m p_i}) - \prod_{i=1}^m (1 - \frac{1}{p_i})).$$

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