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Growth and distortion theorems for multivalent Janowski close-to-convex harmonic functions with shear construction method

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Abstract: In this paper we introduce the class of m-valent Janowski close to convex harmonic functions. Growth and distortion theorems are obtained for this class.

Our study is based on the harmonic shear methods for harmonic functions.

Key words: Multivalent harmonic functions, distortion theorem, growth theorem

1. Introduction

Let U be a simply connected domain in the complex plane. A harmonic function f has the representation $f = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in U and are called the analytic and co-analytic part of f, respectively. Let $h(z) = z^m + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \cdots$, and $g(z) = b_m z^m + b_{m+1}z^{m+1} + b_{m+2}z^{m+2} + \cdots$ be analytic functions in the open unit disc $\mathbb{D}$. The jacobian J_f of $f = h(z) + \overline{g(z)}$ is defined by $J_f = |f_z|^2 - |f_{\overline{z}}|^2 = |h'(z)|^2 - |g'(z)|^2$. If $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$, then $f = h(z) + \overline{g(z)}$ is called a sense-preserving multivalent harmonic function in $\mathbb{D}$. The class of all sense-preserving multivalent harmonic functions with $|b_m| < 1$ is denoted by $\mathcal{S}_H(m)$ and the class of all sense-preserving multivalent harmonic functions with $b_m = 0$ is denoted by $\mathcal{S}_H^0(m)$. For convenience, we will investigate sense-preserving harmonic functions, that is functions for which $J_f(z) > 0$. If $J_f(z) < 0$, then $\overline{f}$ is sense-preserving. The second analytic dilatation of a harmonic function is given by $w(z) = g'(z)/h'(z)$. We also note that if f is locally univalent and sense-preserving, then $|w(z)| < 1$ for every $z \in \mathbb{D}$, and f is the solution of the differential equation $f_z w(z) = \overline{f_{\overline{z}}}$ (see [3], [1] and [4]).

Let Ω be the family of functions $\varphi(z)$ which are regular and analytic in the open unit disc $\mathbb{D}$ and satisfying the conditions $\varphi(0) = 0$, $|\varphi(z)| < 1$ for every $z \in \mathbb{D}$. For arbitrary fixed numbers $A, B, -1 < A \leq 1, -1 \leq B < 1$, denote by $\mathcal{P}(A, B, m)$ the class of functions $p(z) = m + \sum_{n=1}^{\infty} b_n z^n$ analytic in $\mathbb{D}$ such that $p(z) \in \mathcal{P}(A, B, m)$ if and only if

$$p(z) = m\frac{1 + A\varphi(z)}{1 + B\varphi(z)}, \varphi \in \Omega, z \in \mathbb{D}. \tag{1.1}$$

Moreover, let $\mathcal{S}(A, B, m)$ denote the class of functions $f(z) = z^m + \sum_{n=m+1}^{\infty} a_n z^n$ analytic in $\mathbb{D}$ and satisfying the condition that $f(z) \in \mathcal{S}(A, B, m)$ if and only if $z\frac{f'(z)}{f(z)} = p(z)$ for some $p(z) \in \mathcal{P}(A, B, m)$ and all $z \in \mathbb{D}$.

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Next, denote by $\mathcal{P}(m)$ (with m being a positive integer) the family of functions $p(z)=m+p_1z+p_2z^2+\cdots$ which are regular in $\mathbb{D}$ and satisfying the conditions $p(0)=m$, $\mathrm{Re}p(z)>0$ for all $z\in\mathbb{D}$, and such that $p(z)\in\mathcal{P}(m)$ if and only if for some function $\phi(z)\in\Omega$ and every $z\in\mathbb{D}$([2], [6]).

Let $\mathcal{C}(A,B,m)$ denote the class of functions $f(z)=z^m+\sum_{n=m+1}^{\infty}c_nz^n$ regular in $\mathbb{D}$ and satisfies the condition

$$1+z\frac{f''(z)}{f'(z)}=p(z), \tag{1.2}$$

for some $p(z)\in\mathcal{P}(A,B,m)$ and every $z\in\mathbb{D}$. Finally, a function $f(z)=z^m+\sum_{n=m+1}^{\infty}d_nz^n$ is in the class of $\mathcal{K}(A,B,m)$ if there is a function $\phi(z)$ in $\mathcal{C}(A,B,m)$ such that

$$z\frac{f'(z)}{f(z)}=p(z), \tag{1.3}$$

where $p(z)\in\mathcal{P}(A,B,m)$ and every $z\in\mathbb{D}$.

Let $F(z)=z+a_2z^2+\cdots$ and $G(z)=z+b_2z^2+\cdots$ be analytic functions in $\mathbb{D}$. If there exists a function $\phi(z)\in\Omega$ such that $F(z)=G(\varphi(z))$ for all $z\in\mathbb{D}$, then we say that $F(z)$ subordinate to $G(z)$ and we write $F(z)\prec G(z)$. We also note that if $F(z)\prec G(z)$ then $F(\mathbb{D})\subset G(\mathbb{D})$([5]).

Denote by $\mathcal{S}_H\mathcal{K}(A,B,m)$ the class of all m-valent close to convex harmonic functions in the open unit disc $\mathbb{D}$.

2. Main results

Lemma 2.1 *Let $\phi(z)=z^m+c_{m+1}z^{m+1}+c_{m+2}z^{m+2}+\cdots$ be analytic m-valent Janowski convex function in $\mathbb{D}$. Then the inequalities*

$$\begin{aligned}&\frac{r^{m-1}}{(1+Br)^{\frac{m(B-A)}{B}}}\le|\phi'(z)|\le\frac{r^{m-1}}{(1-Br)^{\frac{m(B-A)}{B}}},\quad B\neq0\\ &r^{m-1}e^{-mAr}\le|\phi'(z)|\le r^{m-1}e^{mAr},\qquad B=0\end{aligned} \tag{2.4}$$

are realized.

Proof Since $\phi(z)\in\mathcal{C}(A,B,m)$ and by using the subordination principle, we have

$$\begin{aligned}&\left|\left(1+z\frac{\phi''(z)}{\phi'(z)}\right)-m\frac{1-ABr^2}{1-B^2r^2}\right|\le\frac{m(A-B)r}{1-B^2r^2},\quad B\neq0\\ &\left|\left(1+z\frac{\phi''(z)}{\phi'(z)}\right)-m\right|\le mAr,\quad B=0\end{aligned} \tag{2.5}$$

for every $|z|=r<1$. Therefore we have

$$\begin{cases} \dfrac{(m-1)-m(A-B)r-(mAB-B^2)r^2}{1-B^2r^2} \leq Re\left(z\dfrac{\phi''(z)}{\phi'(z)}\right) \\ \leq \dfrac{(m-1)+p(A-B)r-(mAB-B^2)r^2}{1-B^2r^2}, B \neq 0 \\ m-1-mAr \leq Re\left(z\frac{\phi''(z)}{\phi'(z)}\right) \leq m-1+mAr, \quad B=0 \end{cases} \tag{2.6}$$

for all $|z| = r < 1$. On the other hand, we know that

$$Re\left(z\frac{\phi''(z)}{\phi'(z)}\right) = r\frac{\partial}{\partial r}\log|\phi'(z)|. \tag{2.7}$$

Thus, by using equality (2.7) in the inequalities (2.6) we obtain that

$$\begin{cases} \dfrac{(m-1)-m(A-B)r-(mAB-B^2)r^2}{r(1-B^2r^2)} \leq \dfrac{\partial}{\partial r}\log|\phi'(z)| \\ \leq \dfrac{(m-1)+m(A-B)r-(mAB-B^2)r^2}{r(1-B^2r^2)}, B \neq 0 \\ \frac{m-1-mAr}{r} \leq \frac{\partial}{\partial r}\log|\phi'(z)| \leq \frac{m-1+mAr}{r}, \quad B=0 \end{cases} \tag{2.8}$$

where $|z| = r < 1$. Integrating from 0 to r of the above inequalities we can get (2.4). □

Lemma 2.2 *Let $w(z)$ be the second analytic dilatation of the class $\mathcal{S}_H\mathcal{K}(A,B,m)$, i.e., $w(z) = \frac{g'(z)}{h'(z)}$. Then*

$$\frac{|b_m|-r}{1-|b_m|r} \leq |w(z)| \leq \frac{|b_m|+r}{1+|b_m|r}, \tag{2.9}$$

$$\frac{(1+|b_m|)(1-r)}{1-|b_m|r} \leq 1+|w(z)| \leq \frac{(1+|b_m|)(1+r)}{1+|b_m|r}, \tag{2.10}$$

and

$$\frac{(1-|b_m|)(1-r)}{1+|b_m|r} \leq 1-|w(z)| \leq \frac{(1-|b_m|)(1+r)}{1-|b_m|r}. \tag{2.11}$$

Proof Since $w(z) = \frac{g'(z)}{h'(z)} = \frac{mb_mz^{m-1}+(m+1)b_{m+1}z^m+\cdots}{mz^{m-1}+(m+1)a_{m+1}z^m+\cdots}$ we have $w(0) = b_m$. Define the function

$$\phi(z) = \frac{w(z)-w(0)}{1-\overline{w(0)}w(z)} = \frac{w(z)-b_m}{1-\overline{b_m}w(z)}.$$

This function satisfies the conditions of Schwarz lemma. Therefore we have

$$w(z) = \frac{b_m+\phi(z)}{1+\overline{b_m}\phi(z)},$$

which shows that the second dilatation $w(z)$ is subordinate to $\left(\frac{z+b_m}{1+\overline{b_m}z}\right)$. On the other hand, the transformation $\left(\frac{z+b_m}{1+\overline{b_m}z}\right)$ maps $|z|=r$ onto the disc with the center $C(r)=\left(\frac{\alpha_1(1-r^2)}{1-|b_m|^2r^2},\frac{\alpha_2(1-r^2)}{1-|b_m|^2r^2}\right)$, and radius $\rho(r)=\frac{(1-|b_m|^2)r}{1-|b_m|^2r^2}$. Using the subordination principle, we can write

$$\left|w(z)-\frac{b_m(1-r^2)}{1-|b_m|^2r^2}\right|\leq\frac{(1-|b_m|^2)r}{1-|b_m|^2r^2}. \tag{2.12}$$

After straightforward calculations from the last inequality, we get (2.9), (2.10) and (2.11). □

Theorem 2.3 *Let $f(z)$ be a m-valent Janowski close to convex function and $\phi(z)$ be a m-valent convex function in $\mathbb{D}$. Thus we obtain those inequalities*

$$\frac{m(1-Ar)}{1-Br}\leq\left|\frac{f'(z)}{\phi'(z)}\right|\leq\frac{m(1+Ar)}{1+Br},\quad B\neq 0$$

$$m(1-Ar)\leq\left|\frac{f'(z)}{\phi'(z)}\right|\leq m(1+Ar),\quad B=0, \tag{2.13}$$

where $|z|=r<1$.

Proof Since $f(z)\in\mathcal{K}(A,B,m)$ and $\phi(z)\in\mathcal{C}(A,B,m)$ then we know that

$$\frac{f'(z)}{\phi'(z)}\prec m\frac{1+Az}{1+Bz},$$

from the last subordination we can write the inequalities

$$\left|\frac{f'(z)}{\phi'(z)}-\frac{m(1-ABr^2)}{1-B^2r^2}\right|\leq\frac{m(A-B)r}{1-B^2r^2},\quad B\neq 0$$

$$\left|\frac{f'(z)}{\phi'(z)}-m\right|\leq mAr,\quad B=0. \tag{2.14}$$

By using the triangle inequality in the inequalities (2.14) we get (2.13). □

Theorem 2.4 *If $f(z)$ is a m-valent Janowski close to convex function and $\phi(z)$ is a m-valent Janowski convex function in $\mathbb{D}$, then the following inequalities*

$$\frac{m(1-Ar)r^{m-1}}{(1-Br)(1+Br)^{\frac{m(B-A)}{B}}}\leq|f'(z)|\leq\frac{m(1+Ar)r^{m-1}}{(1+Br)(1-Br)^{\frac{m(B-A)}{B}}},\quad B\neq 0$$

$$mr^{m-1}e^{-mAr}(1-Ar)\leq|f'(z)|\leq mr^{m-1}e^{mAr}(1+Ar),\qquad B=0 \tag{2.15}$$

are realized.

Proof Using lemma 2.1 in theorem 2.4, we obtain the result. □

Theorem 2.5 *Let* $f = h(z) + \overline{g(z)}$ *be an element of* $\mathcal{S}_H\mathcal{K}(A,B,m)$. *Then*

$$\begin{cases} \dfrac{m(1-Ar)r^{m-1}}{(1-Br)(1+Br)^{\frac{m(B-A)}{B}}}\cdot\dfrac{(1+|b_m|r)}{(1+|b_m|)(1+r)} \le |f_z| \\ \qquad \le \dfrac{m(1+Ar)r^{m-1}}{(1+Br)(1-Br)^{\frac{m(B-A)}{B}}}\cdot\dfrac{(1+|b_m|r)}{(1-|b_m|)(1-r)}, B \neq 0, \\ mr^{m-1}e^{-mAr}(1-Ar).\frac{(1+|b_m|r)}{(1+|b_m|)(1+r)} \le |f_z| \le mr^{m-1}e^{mAr}(1+Ar).\frac{(1+|b_m|r)}{(1-|b_m|)(1-r)}, B = 0, \end{cases} \tag{2.16}$$

and

$$\begin{cases} \dfrac{m(1-Ar)r^{m-1}}{(1-Br)(1+Br)^{\frac{m(B-A)}{B}}}\cdot\dfrac{(|b_m|-r)(1+|b_m|r)}{(1-|b_m|r)(1+|b_m|)(1+r)} \le |f_{\overline{z}}| \\ \qquad \le \dfrac{m(1+Ar)r^{m-1}}{(1+Br)(1-Br)^{\frac{m(B-A)}{B}}}\cdot\dfrac{(|b_m|+r)}{(1-|b_m|)(1-r)}, \qquad B \neq 0, \\ mr^{m-1}e^{-mAr}(1-Ar).\frac{(|b_m|-r)(1+|b_m|r)}{(1-|b_m|r)(1+|b_m|)(1+r)} \le |f_{\overline{z}}| \le mr^{m-1}e^{mAr}(1+Ar).\frac{(|b_m|+r)}{(1-|b_m|)(1-r)}, B = 0. \end{cases} \tag{2.17}$$

Proof If we take $\psi(z) = h(z) - g(z)$, then we have

$$h'(z) = \frac{\psi'(z)}{1-w(z)}, \quad g'(z) = \frac{w(z)\psi'(z)}{1-w(z)}, \quad |w(z)| < 1.$$

Therefore we have

$$\frac{|\psi'(z)|}{1+|w(z)|} \le |f_z| \le \frac{|\psi'(z)|}{1-|w(z)|}, \tag{2.18}$$

$$\frac{|w(z)||\psi'(z)|}{1+|w(z)|} \le |\overline{f_{\overline{z}}}| \le \frac{|w(z)||\psi'(z)|}{1-|w(z)|}. \tag{2.19}$$

Using lemma 2.1 and lemma 2.2 in the inequalities (2.18) and (2.19) we get (2.16) and (2.17), respectively. Since

$$\phi(z) = \frac{w(z) - b_m}{1 - \overline{b_m}w(z)},$$

we have

$$h'(z) = f_z = \frac{\psi'(z)}{1-w(z)},$$

and so

$$h(z) = \int_0^z \frac{\psi'(\xi)}{1-w(\xi)}d\xi.$$

Also, since

$$g'(z) = \overline{f_{\overline{z}}} = \int_0^z \frac{\psi'(\xi)w(\xi)}{1-w(\xi)}d\xi,$$

it follows that

$$g(z) = \int_0^z \frac{\psi'(\xi)w(\xi)}{1-w(\xi)}d\xi.$$

(The solution $h(z)$ and $g(z)$ must be found under the conditions $h(0)=g(0)=0$.) Thus

$$f(z) = h(z) + \overline{g(z)} = \int_0^z \frac{\psi'(\xi)}{1-w(\xi)}d\xi + \overline{\int_0^z \frac{\psi'(\xi)w(\xi)}{1-w(\xi)}d\xi} =$$

$$= \int_0^z \frac{\psi'(\xi)}{1-w(\xi)}d\xi + \overline{\int_0^z \frac{\psi'(\xi)}{1-w(\xi)}d\xi - \int_0^z \psi'(\xi)d\xi} = Re\left(\int_0^z \frac{2\psi'(\xi)}{1-w(\xi)}\right) - \overline{\psi(z)}.$$

□

Corollary 2.6 *If we choose the following values for theorem 2.5, we get the accompanying inequalities:*

- $A = 1, B = -1$:

$$\frac{m(1-r)r^{m-1}}{(1+r)^2(1-r)^{2m}}.\frac{(1+|b_m|r)}{(1+|b_m|)} \leq |f_z| \leq \frac{m(1+r)r^{m-1}}{(1-r)^2(1+r)^{2m}}.\frac{(1+|b_m|r)}{(1-|b_m|)}$$

$$\frac{m(1-r)r^{m-1}}{(1+r)^2(1-r)^{2m}}.\frac{(1+|b_m|r)(|b_m|-r)}{(1+|b_m|)(1-|b_m|r)} \leq |f_{\overline{z}}| \leq \frac{m(1+r)r^{m-1}}{(1-r)^2(1+r)^{2m}}.\frac{(|b_m|+r)}{(1-|b_m|)}$$

- $A = 1-2\alpha, B = -1, 0 \leq \alpha < 1$:

$$\frac{m(1-r+2\alpha r)r^{m-1}}{(1+r)^2(1-r)^{2m(1-\alpha)}}.\frac{(1+|b_m|r)}{(1+|b_m|)} \leq |f_z| \leq \frac{m(1+r-2\alpha r)r^{m-1}}{(1-r)^2(1+r)^{2m(1-\alpha)}}.\frac{(1+|b_m|r)}{(1-|b_m|)}$$

$$\frac{m(1-r+2\alpha r)r^{m-1}}{(1+r)^2(1-r)^{2m(1-\alpha)}}.\frac{(1+|b_m|r)(|b_m|-r)}{(1+|b_m|)(1-|b_m|r)} \leq |f_{\overline{z}}| \leq \frac{m(1+r-2\alpha r)r^{m-1}}{(1-r)^2(1+r)^{2m(1-\alpha)}}.\frac{(|b_m|+r)}{(1-|b_m|)}$$

- $A = 1, B = \frac{1}{M}-1, M > \frac{1}{2}$:

$$\begin{cases} \dfrac{m(1-r)r^{m-1}}{(1+r-\frac{r}{M})(1-r+\frac{r}{M})^{m\frac{1-2M}{1-M}}}.\dfrac{(1+|b_m|r)}{(1+|b_m|)(1+r)} \leq |f_z| \\ \qquad \leq \dfrac{m(1+r)r^{m-1}}{(1-r+\frac{r}{M})(1+r-\frac{r}{M})^{m\frac{1-2M}{1-M}}}.\dfrac{(1+|b_m|r)}{(1-|b_m|)(1-r)} \\ \frac{m(1-r)r^{m-1}}{(1+r-\frac{r}{M})(1-r+\frac{r}{M})^{m\frac{1-2M}{1-M}}}.\frac{(1+|b_m|r)(|b_m|-r)}{(1+|b_m|)(1-|b_m|r)(1+r)} \leq |f_{\overline{z}}| \leq \frac{m(1+r)r^{m-1}}{(1-r+\frac{r}{M})(1+r-\frac{r}{M})^{m\frac{1-2M}{1-M}}}.\frac{(|b_m|+r)}{(1-|b_m|)(1-r)} \end{cases}$$

- $A = \beta, B = -\beta, 0 < \beta \leq 1$:

$$\begin{cases} \frac{mr^{m-1}(1+|b_m|r)}{(1+\beta r)(1-\beta r)^{2m-1}(1+|b_m|)(1+r)} \leq |f_z| \leq \frac{mr^{m-1}(1+|b_m|r)}{(1-\beta r)(1+\beta r)^{2m-1}(1-|b_m|)(1-r)} \\ \dfrac{mr^{m-1}(|b_m|-r)(1+|b_m|r)}{(1+\beta r)(1-\beta r)^{2m-1}(1-|b_m|r)(1+|b_m|)(1+r)} \leq |f_{\overline{z}}| \\ \leq \dfrac{mr^{m-1}(|b_m|+r)}{(1-\beta r)(1+\beta r)^{2m-1}(1-|b_m|)(1-r)} \end{cases}$$

Corollary 2.7 *Let* $f = h(z) + \overline{g(z)}$ *be an element of* $\mathcal{S}_H\mathcal{K}(A,B,m)$. *Then*

$$\begin{cases} \dfrac{m^2(1-Ar)^2 r^{2(m-1)}}{(1-Br)^2(1+Br)^{\frac{2m(B-A)}{B}}} \dfrac{(1-|b_m|^2)(1-r)^2(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1+|b_m|)^2(1+r)^2} \leq J_f(z), \\ \leq \dfrac{m^2(1+Ar)^2 r^{2(m-1)}}{(1+Br)^2(1-Br)^{\frac{2m(B-A)}{B}}} \cdot \dfrac{(1-|b_m|^2)(1+r)^2(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1-|b_m|)^2(1-r)^2}, B \neq 0, \\ \\ m^2 r^{2(m-1)} e^{-2mAr}(1-Ar)^2 \dfrac{(1-|b_m|^2)(1-r)^2(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1+|b_m|)^2(1+r)^2} \leq J_f(z), \\ \leq m^2 r^{2(m-1)} e^{2mAr}(1+Ar)^2 \dfrac{(1-|b_m|^2)(1+r)^2(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1-|b_m|)^2(1-r)^2}, B = 0. \end{cases} \tag{2.20}$$

Proof Since $J_f(z) = |h'(z)|^2 - |g'(z)|^2 = |h'(z)|^2(1-|w(z)|^2)$, then using theorem 2.5 and lemma 2.2, we get (2.20). □

Corollary 2.8 *For the last results, if we take the following values, we get the accompanying inequalities:*

- $A = 1, B = -1$:

$$\frac{m^2(1-r)^4 r^{2(m-1)}}{(1+r)^4(1-r)^{4m}} \cdot \frac{(1-|b_m|^2)(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1+|b_m|)^2} \leq J_f(z) \leq \frac{m^2(1+r)^2 r^{2(m-1)}}{(1-r)^4(1+r)^{4m}} \cdot \frac{(1-|b_m|^2)(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1-|b_m|)^2}$$

- $A = 1-2\alpha, B = -1, 0 \leq \alpha < 1$:

$$\frac{m^2(1-r+2\alpha r)^2 r^{2(m-1)}}{(1+r)^4(1-r)^{4m(1-\alpha)}} \cdot \frac{(1-|b_m|^2)(1-r)^2(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1+|b_m|)^2} \leq J_f(z) \leq \frac{m^2(1+r-2\alpha r)^2 r^{2(m-1)}}{(1-r)^4(1+r)^{4m(1-\alpha)}} \cdot \frac{(1-|b_m|^2)(1+r)^2(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1-|b_m|)^2}.$$

- $A = 1, B = \frac{1}{M} - 1, M > \frac{1}{2}$:

$$\frac{m^2(1-r)^4 r^{2(m-1)}}{(1+r-\frac{r}{M})^2(1-r+\frac{r}{M})^{2m\frac{1-2M}{1-M}}}.\frac{(1-|b_m|^2)(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1+|b_m|)^2(1+r)^2} \leq J_f(z)$$

$$\leq \frac{m^2(1+r)^4 r^{2(m-1)}}{(1-r+\frac{r}{M})^2(1+r-\frac{r}{M})^{2m\frac{1-2M}{1-M}}}.\frac{(1-|b_m|^2)(1+|b_m|r)^2}{(1-|b_m|^2r^2)(1-|b_m|)^2(1-r)^2}.$$

- $A = \beta, B = -\beta$:

$$\frac{m^2 r^{2(m-1)}(1-|b_m|^2)(1-r)^2(1+|b_m|r)^2}{(1+\beta r)^2(1-\beta r)^{4m-2}(1-|b_m|^2r^2)(1+|b_m|)^2(1+r)^2} \leq J_f(z)$$

$$\leq \frac{m^2 r^{2(m-1)}(1-|b_m|^2)(1+r)^2(1+|b_m|r)^2}{(1-\beta r)^2(1+\beta r)^{4m-2}(1-|b_m|^2r^2)(1-|b_m|)^2(1-r)^2}$$

Corollary 2.9 *If* $f = h(z) + \overline{g(z)} \in \mathcal{S}_H(A, B, m)$, *then*

$$\begin{cases} m\displaystyle\int_0^r \rho^{m-1}\left[\frac{(1-A\rho)(1+|b_m|\rho)}{(1-B\rho)(1+B\rho)^{\frac{m(B-A)}{B}}} - \frac{(1+A\rho)(|b_m|+\rho)}{(1+B\rho)(1-B\rho)^{\frac{m(B-A)}{B}}(1-|b_m|)(1-\rho)}\right]d\rho \leq \\ \qquad |f| \leq m\left(\dfrac{1+|b_m|}{1-|b_m|}\right)\displaystyle\int_0^r \rho^{m-1}\frac{(1+A\rho)(1+\rho)}{(1+B\rho)(1-B\rho)^{\frac{m(B-A)}{B}}(1-\rho)}d\rho, B \neq 0, \\ m\displaystyle\int_0^r \rho^{m-1}\left[\frac{(1-A\rho)(1+|b_m|\rho)}{(1+|b_m|)(1+\rho)} - \frac{(1+A\rho)(|b_m|+\rho)}{(1-|b_m|)(1-\rho)}\right]e^{-A\rho}d\rho \leq \\ \qquad |f| \leq m\left(\dfrac{1+|b_m|}{1-|b_m|}\right)\displaystyle\int_0^r \rho^{m-1}\frac{(1+A\rho)(1+\rho)}{(1-\rho)}e^{A\rho}d\rho, B = 0. \end{cases} \tag{2.21}$$

Proof Since $(|f_z| - |f_{\overline{z}}|)|dz| \leq |df| \leq (|f_z| + |f_{\overline{z}}|)|dz|$, it follows that $(|h'(z)| - |g'(z)|)|dz| \leq |df| \leq (|h'(z)| + |g'(z)|)|dz|$, and using theorem 2.5 we can write the result. □

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