Oscillation of solutions of a neutral pantograph equation with impulsive perturbations

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Abstract: Some sufficient conditions are established on the oscillation of all solutions of a class of neutral pantograph equations with impulsive perturbations of the form

$$\begin{cases} \frac{d}{dt}[x(t) - C(t)x(\gamma t)] + \frac{P(t)}{t}x(\alpha t) - \frac{Q(t)}{t}x(\beta t) = 0, & t \geq t_0 > 0, \quad t \neq t_k, \\ x(t_k^+) = b_k x(t_k), & k = 1, 2, \ldots. \end{cases}$$

Key words: Oscillation, neutral differential equation, pantograph equation, impulses

1. Introduction

Functional differential equations with proportional delays are usually referred to as pantograph equations. The name pantograph originated from the work of Ockendon and Taylor [9] on the collection of current by the pantograph head of an electric locomotive. These equations arise in a variety of applications, such as number theory, electrodynamics, astrophysics, nonlinear dynamical systems, quantum mechanics and cell growth [2, 9, 10]. Therefore, the problems have attracted a great deal of attention [7, 8, 11]. There are also many papers on qualitative properties of solutions of neutral pantograph equations. (See, for example, [4–6] and the references therein.) However, to the best of our knowledge, there is very little in the way of results for the qualitative behavior of solutions of neutral pantograph equations with impulsive perturbations except for [3].

In this paper, we consider the oscillatory behavior of all solutions of the following impulsive neutral pantograph equation with positive and negative coefficients

$$\frac{d}{dt}[x(t) - C(t)x(\gamma t)] + \frac{P(t)}{t}x(\alpha t) - \frac{Q(t)}{t}x(\beta t) = 0, \quad t \geq t_0 > 0, \quad t \neq t_k, \tag{1.1}$$

$$x(t_k^+) = b_k x(t_k), \quad k = 1, 2, \ldots, \tag{1.2}$$

where

(H_1) $0 < \gamma < 1$, $0 < \alpha < \beta \leq 1$ and $0 < t_0 < t_1 < \cdots < t_k < \cdots$ are fixed real numbers with $\lim_{t \to \infty} t_k = \infty$;

(H_2) $\{b_k\}$ is a constant sequence satisfying $0 < b_k \leq 1$, $k = 1, 2, \ldots$;

(H_3) $C \in PC([t_0, \infty), R^+)$, $P, Q \in C([t_0, \infty), (0, +\infty))$, $H(t) = P(t) - Q\left(\frac{\alpha}{\beta}t\right) \geq 0$ and $H(t) \neq 0$ on $(t_{k-1}, t_k](k \geq 1)$, where $R^+ = [0, \infty)$, $PC([t_0, \infty), R^+) = \{f : [t_0, \infty) \to R^+ | f$ is continuous for $t_0 \leq t \leq t_1, t_k < t \leq t_{k+1}$ and $f(t_k^+)$ and $f(t_k^-) = \lim_{t \to t_k^-} f(t)$ exist with $f(t_k^-) = f(t_k)(k = 1, 2, \ldots)\}$.

When all $b_k = 1$ for $k = 1, 2, \ldots$, (1.1) and (1.2) reduce to the neutral pantograph equation with positive and negative coefficients

$$\frac{d}{dt}[x(t) - C(t)x(\gamma t)] + \frac{P(t)}{t}x(\alpha t) - \frac{Q(t)}{t}x(\beta t) = 0, \quad t \geq t_0 > 0. \tag{1.3}$$

For Eq. (1.3), Guan and Shen [4] established Hille type oscillation criteria by considering the three cases $W(t) = 1$, $W(t) \leq 1$ and $W(t) \geq 1$, where

$$W(t) = C(t) + \int_{\frac{\alpha}{\beta}t}^{t} \frac{Q(u)}{u} du. \tag{1.4}$$

The main purpose of this paper is to establish sufficient conditions for the oscillation of all solutions of (1.1) and (1.2) by introducing the function

$$W_s(t) = C(t) + \int_{st}^{t} \frac{Q(u)}{u} du + \int_{t}^{\frac{\beta}{\alpha}st} \frac{P(u)}{u} du, \tag{1.5}$$

where $s \in [\alpha/\beta, 1]$. Obviously, $W_{\alpha/\beta}(t) = W(t)$. Our results improve the known results in the literature and show that the oscillatory properties of all solutions of impulsive neutral pantograph equations may be caused by the impulsive perturbations though the corresponding neutral pantograph equations without impulses admit a nonoscillatory solution.

With Eqs. (1.1) and (1.2), one associates an initial condition of the form

$$x_{t_0} = \phi(s), \quad s \in [\rho t_0, t_0], \tag{1.6}$$

where $\rho = \min\{\alpha, \beta, \gamma\}$, $x_{t_0} = x(s + t_0)$ for $\rho t_0 \leq s \leq t_0$ and $\phi \in PC([\rho t_0, t_0], R) = \{\phi : [\rho t_0, t_0] \to R | \phi$ is continuous everywhere except at a finite number of points $\bar{s}$, and $\phi(\bar{s}^+)$ and $\phi(\bar{s}^-) = \lim_{s \to \bar{s}^-} \phi(s)$ exist with $\phi(\bar{s}^-) = \phi(\bar{s})\}$.

A function $x(t)$ is said to be a solution of (1.1) and (1.2) satisfying the initial value condition (1.6) if

(i) $x(t) = \phi(t)$ for $\rho t_0 \leq t \leq t_0$, $x(t)$ is continuous for $t \geq t_0$ and $t \neq t_k$, $k = 1, 2, \ldots$;

(ii) $x(t) - C(t)x(\gamma t)$ is continuously differentiable for $t > t_0, t \neq t_k, k = 1, 2, \ldots$, so that $x(t)$ satisfies (1.1);

(iii) $x(t_k^+)$ and $x(t_k^-)$ exist with $x(t_k^-) = x(t_k)$ and satisfy (1.2).

As is customary, a solution of (1.1) and (1.2) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, the solution is said to be oscillatory.

2. Main results

Throughout all of this paper, we always assume that $(H_1) - (H_3)$ hold and let

$$\omega(t) = x(t) - C(t)x(\gamma t) - \int_{st}^{t} \frac{Q(u)}{u}x(\beta u)du - \int_{t}^{\frac{\beta}{\alpha}st} \frac{P(u)}{u}x(\alpha u)du, \tag{2.1}$$

where $s \in [\alpha/\beta, 1]$.

Lemma 2.1 *Assume that $b_0 = 1$, $0 < b_k \leq 1$ for $k = 1, 2, \ldots$, and*

$$C(t_k^+) \geq C(t_k), \quad for \ \ k \in E_{1k} = \{k \geq 1 : \gamma t_k \neq t_i, \ \ i < k\} \tag{2.2}$$

$$\overline{b}_k C(t_k^+) \geq C(t_k), \quad for \ \ k \in E_{2k} = \{k \geq 1 : \gamma t_k = t_i, \ \ i < k\}, \tag{2.3}$$

where $\overline{b}_k = b_i$ when $\gamma t_k = t_i (\ \ i < k)$. Let $x(t)$ be a solution of (1.1) and (1.2) such that $x(\rho t) > 0$ for $t \geq t_0$. Then for any fixed $s \in [\alpha/\beta, 1]$, $\omega(t)$ is nonincreasing in $[t_0, \infty)$ and $\omega(t_k^+) \leq b_k \omega(t_k)$ for $k = 1, 2, \ldots$.

Proof   From (1.1) and (2.1), we have

$$\omega'(t) = -\frac{1}{t}H\left(\frac{\beta}{\alpha}st\right)x(\beta st) \leq 0, \ t_k < t \leq t_{k+1}, k \geq 0. \tag{2.4}$$

By (2.1), we obtain

$$\omega(t_k^+) = x(t_k^+) - C(t_k^+)x(\gamma t_k^+) - \int_{st_k}^{t_k} \frac{Q(u)}{u}x(\beta u)du - \int_{t_k}^{\frac{\beta}{\alpha}st_k} \frac{P(u)}{u}x(\alpha u)du. \tag{2.5}$$

If $k \in E_{1k}$, then

$$
\begin{aligned}
\omega(t_k^+) &= b_k x(t_k) - C(t_k^+)x(\gamma t_k) - \int_{st_k}^{t_k} \frac{Q(u)}{u}x(\beta u)du - \int_{t_k}^{\frac{\beta}{\alpha}st_k} \frac{P(u)}{u}x(\alpha u)du \\
&\leq b_k x(t_k) - C(t_k)x(\gamma t_k) - \int_{st_k}^{t_k} \frac{Q(u)}{u}x(\beta u)du - \int_{t_k}^{\frac{\beta}{\alpha}st_k} \frac{P(u)}{u}x(\alpha u)du \\
&\leq x(t_k) - C(t_k)x(\gamma t_k) - \int_{st_k}^{t_k} \frac{Q(u)}{u}x(\beta u)du - \int_{t_k}^{\frac{\beta}{\alpha}st_k} \frac{P(u)}{u}x(\alpha u)du \\
&= \omega(t_k).
\end{aligned}
$$

If $k \in E_{2k}$, then

$$
\begin{aligned}
\omega(t_k^+) &= b_k x(t_k) - C(t_k^+)b_i x(\gamma t_k) - \int_{st_k}^{t_k} \frac{Q(u)}{u}x(\beta u)du - \int_{t_k}^{\frac{\beta}{\alpha}st_k} \frac{P(u)}{u}x(\alpha u)du \\
&\leq x(t_k) - C(t_k)x(\gamma t_k) - \int_{st_k}^{t_k} \frac{Q(u)}{u}x(\beta u)du - \int_{t_k}^{\frac{\beta}{\alpha}st_k} \frac{P(u)}{u}x(\alpha u)du \\
&= \omega(t_k).
\end{aligned}
$$

Since $E_{1k} \bigcup E_{2k} = \{1, 2, \ldots\}$, we have

$$\omega(t_k^+) \leq \omega(t_k).$$

This together with (2.4) implies that $\omega(t)$ is nonincreasing on $[t_0, \infty)$.

Finally, if $k \in E_{1k}$, then

$$C(t_k^+) \geq C(t_k) \geq b_k C(t_k). \tag{2.6}$$

Hence

$$
\begin{aligned}
\omega(t_k^+) &= b_k x(t_k) - C(t_k^+) x(\gamma t_k) - \int_{st_k}^{t_k} \frac{Q(u)}{u} x(\beta u) du - \int_{t_k}^{\frac{\beta}{\alpha} st_k} \frac{P(u)}{u} x(\alpha u) du \\
&\leq b_k x(t_k) - C(t_k) x(\gamma t_k) - \int_{st_k}^{t_k} \frac{Q(u)}{u} x(\beta u) du - \int_{t_k}^{\frac{\beta}{\alpha} st_k} \frac{P(u)}{u} x(\alpha u) du \\
&\leq b_k x(t_k) - b_k C(t_k) x(\gamma t_k) - b_k \int_{st_k}^{t_k} \frac{Q(u)}{u} x(\beta u) du - b_k \int_{t_k}^{\frac{\beta}{\alpha} st_k} \frac{P(u)}{u} x(\alpha u) du \\
&= b_k \omega(t_k).
\end{aligned}
$$

If $k \in E_{2k}$, then

$$C(t_k^+) b_i = \overline{b}_k C(t_k^+) \geq C(t_k) \geq b_k C(t_k). \tag{2.7}$$

It follows from (2.7) that

$$
\begin{aligned}
\omega(t_k^+) &= b_k x(t_k) - C(t_k^+) b_i x(\gamma t_k) - \int_{st_k}^{t_k} \frac{Q(u)}{u} x(\beta u) du - \int_{t_k}^{\frac{\beta}{\alpha} st_k} \frac{P(u)}{u} x(\alpha u) du \\
&\leq b_k x(t_k) - b_k C(t_k) x(\gamma t_k) - b_k \int_{st_k}^{t_k} \frac{Q(u)}{u} x(\beta u) du - b_k \int_{t_k}^{\frac{\beta}{\alpha} st_k} \frac{P(u)}{u} x(\alpha u) du \\
&= b_k \omega(t_k).
\end{aligned}
$$

Therefore, $\omega(t_k^+) \leq b_k \omega(t_k)$ and so the proof is complete. $\qquad\square$

Lemma 2.2 *Let the hypotheses of Lemma 2.1 hold and $\omega(t)$ be defined by (2.1). In addition, assume that there exists a real number $s \in [\alpha/\beta, 1]$ such that*

$$W_s(t) = C(t) + \int_{st}^{t} \frac{Q(u)}{u} du + \int_{t}^{\frac{\beta}{\alpha} st} \frac{P(u)}{u} du \leq 1, \quad t \geq t_0. \tag{2.8}$$

Let $x(t)$ be a solution of (1.1) and (1.2) such that $x(\rho t) > 0$ for $t \geq t_0$. Then $\omega(t) > 0$ for $t \geq t_0$.

Proof By Lemma 2.1, $\omega(t)$ is nonincreasing on $[t_0, \infty)$. We first claim that $\omega(t_k) \geq 0$ for $k = 1, 2, \ldots$. If this is not the case, then there exists some $m \geq 1$ such that $\omega(t_m) = -\mu < 0$. Therefore, $\omega(t) \leq -\mu < 0$ for $t \geq t_m$. From (2.1), we have

$$x(t) \leq -\mu + C(t) x(\gamma t) + \int_{st}^{t} \frac{Q(u)}{u} x(\beta u) du + \int_{t}^{\frac{\beta}{\alpha} st} \frac{P(u)}{u} x(\alpha u) du. \tag{2.9}$$

We consider the following two possible cases.

Case 1. If $\limsup\limits_{t\to\infty} x(t) = +\infty$, then there exists a sequence of points $\{a_n\}_{n=1}^{\infty}$ such that $a_n \geq \frac{t_m}{\rho}$, $\lim\limits_{n\to\infty} x(a_n) = +\infty$ and $x(a_n) = \max\{x(t) : t_m \leq t \leq a_n\}$. From (2.8) and (2.9), we obtain

$$
\begin{aligned}
x(a_n) &\leq -\mu + C(a_n)x(\gamma a_n) + \int_{sa_n}^{a_n} \frac{Q(u)}{u} x(\beta u)du + \int_{a_n}^{\frac{\beta}{\alpha}sa_n} \frac{P(u)}{u} x(\alpha u)du \\
&\leq -\mu + \left(C(a_n) + \int_{sa_n}^{a_n} \frac{Q(u)}{u}du + \int_{a_n}^{\frac{\beta}{\alpha}sa_n} \frac{P(u)}{u}du \right) x(a_n) \\
&\leq -\mu + x(a_n),
\end{aligned}
$$

which is a contradiction.

Case 2. $\limsup\limits_{t\to\infty} x(t) = h < +\infty$. Choose a sequence of points $\{a_n\}_{n=1}^{\infty}$ such that $\lim\limits_{n\to\infty} x(a_n) = h$ and $x(\xi_n) = \max\{x(t) : \rho a_n \leq t \leq a_n\}$. Then $\xi_n \to \infty$ as $n \to \infty$ and $\limsup\limits_{n\to\infty} x(\xi_n) \leq h$. Thus, we have

$$
\begin{aligned}
x(a_n) &\leq -\mu + C(a_n)x(\gamma a_n) + \int_{sa_n}^{a_n} \frac{Q(u)}{u} x(\beta u)du + \int_{a_n}^{\frac{\beta}{\alpha}sa_n} \frac{P(u)}{u} x(\alpha u)du \\
&\leq -\mu + \left(C(a_n) + \int_{sa_n}^{a_n} \frac{Q(u)}{u}du + \int_{a_n}^{\frac{\beta}{\alpha}sa_n} \frac{P(u)}{u}du \right) x(\xi_n) \\
&\leq -\mu + x(\xi_n).
\end{aligned}
$$

Taking the superior limit as $n \to \infty$, we get $h \leq -\mu + h$, which is also a contradiction.

Combining cases 1 and 2, we see that $\omega(t_k) \geq 0$ for $k \geq 1$. From (2.4), $\omega(t_0) \geq 0$.

Now we prove $\omega(t) > 0$ for $t \geq t_0$. For this purpose, we first prove that $\omega(t_k) > 0$ $(k \geq 0)$; if it is not true, then there exists some $m \geq 0$ such that $\omega(t_m) = 0$. Thus, from (2.4), we obtain

$$
\begin{aligned}
\omega(t_{m+1}) &= \omega(t_m^+) - \int_{t_m}^{t_{m+1}} \frac{x(su)}{u} H\left(\frac{\beta}{\alpha}su \right) du \\
&\leq \omega(t_m) - \int_{t_m}^{t_{m+1}} \frac{x(su)}{u} H\left(\frac{\beta}{\alpha}su \right) du < 0.
\end{aligned}
$$

This contradiction shows that $\omega(t_k) > 0$ $(k \geq 0)$. Therefore, from (2.4) it follows that

$$
\omega(t) \geq \omega(t_{k+1}) > 0, \ t \in (t_k, t_{k+1}](k \geq 0).
$$

And thus, $\omega(t) > 0$ for $t \geq t_0$. The proof is complete. $\qquad\square$

Lemma 2.3 *Let all the assumptions of Lemma 2.1 hold. Suppose that there exists a real number $s \in [\alpha/\beta, 1]$ such that*

$$W_s(t) = C(t) + \int_{st}^t \frac{Q(u)}{u} du + \int_t^{\frac{\beta}{\alpha} st} \frac{P(u)}{u} du \geq 1, \quad t \geq t_0. \tag{2.10}$$

Further assume that the impulsive differential inequality

$$\begin{cases} y''(t) + \left(\ln\frac{1}{\rho}\right)^{-1} \frac{1}{t^2} H\left(\frac{\beta}{\alpha} st\right) y(t) \leq 0, & t \geq t_0 > 0, \quad t \neq t_k, \\ y(t_k^+) = y(t_k), & k = 1, 2, \ldots, \\ y'(t_k^+) \leq b_k y'(t_k), & k = 1, 2, \ldots, \end{cases} \tag{2.11}$$

has no eventually positive solution. If $x(t)$ is a solution of (1.1) and (1.2) such that $x(\rho t) > 0$ for $t \geq t_0$, then $\omega(t)$ is eventually negative.

Proof By Lemma 2.1, $\omega(t)$ is nonincreasing for $t \geq t_0$. If $\omega(t) < 0$ does not hold eventually, then $\omega(t)$ is eventually positive. Let $l \geq \min\{k \geq 1 : t_k \geq \frac{t_0}{\rho}\}$ such that $\omega(t) > 0$ for $t \geq t_l$. Set $M = 2^{-1} \min\{x(t) : \rho t_l \leq t \leq t_l\}$, then $M > 0$ and $x(t) > M$ for $\rho t_l \leq t \leq t_l$. We claim that

$$x(t) > M, \quad t \in (t_l, t_{l+1}]. \tag{2.12}$$

If (2.12) does not hold, then there exists a $t^* \in (t_l, t_{l+1}]$ such that $x(t^*) = M$ and $x(t) > M$ for $\rho t_l \leq t < t^*$. From (2.1) and (2.10), we have

$$\begin{aligned} M = x(t^*) &= w(t^*) + C(t^*)x(\gamma t^*) + \int_{st^*}^{t^*} \frac{Q(u)}{u} x(\beta u) du + \int_{t^*}^{\frac{\beta}{\alpha} st^*} \frac{P(u)}{u} x(\alpha u) du \\ &> \left(C(t^*) + \int_{st^*}^{t^*} \frac{Q(u)}{u} du + \int_{t^*}^{\frac{\beta}{\alpha} st^*} \frac{P(u)}{u} du\right) M \geq M, \end{aligned}$$

which is a contradiction and so (2.12) holds. Noting that $w(t_{l+1}^+) > 0$ and using (2.6), (2.7) and (2.10), we have

$$\begin{aligned} x(t_{l+1}^+) &= w(t_{l+1}^+) + C(t_{l+1}^+)x(\gamma t_{l+1}^+) + \int_{st_{l+1}}^{t_{l+1}} \frac{Q(u)}{u} x(\beta u) du + \int_{t_{l+1}}^{\frac{\beta}{\alpha} st_{l+1}} \frac{P(u)}{u} x(\alpha u) du \\ &> C(t_{l+1})x(\gamma t_{l+1}) + \int_{st_{l+1}}^{t_{l+1}} \frac{Q(u)}{u} x(\beta u) du + \int_{t_{l+1}}^{\frac{\beta}{\alpha} st_{l+1}} \frac{P(u)}{u} x(\alpha u) du \\ &> \left(C(t_{l+1}) + \int_{st_{l+1}}^{t_{l+1}} \frac{Q(u)}{u} du + \int_{t_{l+1}}^{\frac{\beta}{\alpha} st_{l+1}} \frac{P(u)}{u} du\right) M \geq M. \end{aligned}$$

Repeating the above argument, by induction, we obtain

$$x(t) > M, \quad t \geq \rho t_l. \tag{2.13}$$

Because $w(t) > 0$ and $w(t)$ is nonincreasing, $\lim_{t \to \infty} w(t)$ exists. Let $\lim_{t \to \infty} w(t) = a$. There are two possible cases.

Case 1. $a = 0$. Let $T_1 > t_l$ be such that $w(t) \leq M/2$ for $t \geq T_1$. Then for any $\bar{t} > T_1$, we have

$$\left(\ln\frac{1}{\rho}\right)^{-1} \int_{\bar{t}}^{\frac{t}{\rho}} \frac{w(s)}{s} ds \leq M < x(t), \quad t \in [\bar{t}, \bar{t}/\rho].$$

Case 2. $a > 0$. Then $w(t) \geq a$ for $t \geq t_l$. From (2.1) and (2.13), we get

$$
\begin{aligned}
x(t) &\geq a + C(t)x(\gamma t) + \int_{st}^{t} \frac{Q(u)}{u} x(\beta u) du + \int_{t}^{\frac{\beta}{\alpha} st} \frac{P(u)}{u} x(\alpha u) du \\
&\geq a + \left(C(t) + \int_{st}^{t} \frac{Q(u)}{u} du + \int_{t}^{\frac{\beta}{\alpha} st} \frac{P(u)}{u} du \right) M \\
&\geq a + M, \quad t \geq t_l.
\end{aligned}
$$

By induction, it is easy to see that $x(t) \geq na + M$ for $t \geq \frac{t_l}{\rho^{n-1}} (n = 1, 2, \ldots)$, and so $\lim_{t \to \infty} x(t) = \infty$, which implies that there exists a $T > T_1$ such that

$$
\left(\ln \frac{1}{\rho} \right)^{-1} \int_{T}^{\frac{t}{\rho}} \frac{w(v)}{v} dv \leq w(T) < x(t), \quad t \in [T, T/\rho].
$$

Cases 1 and 2 show that

$$
x(t) > \left(\ln \frac{1}{\rho} \right)^{-1} \int_{T}^{\frac{t}{\rho}} \frac{w(v)}{v} dv, \quad t \in [T, T/\rho].
$$

Let $l^* = \min\{k \geq l : t_k > T/\rho\}$; we claim that

$$
x(t) > \left(\ln \frac{1}{\rho} \right)^{-1} \int_{T}^{\frac{t}{\rho}} \frac{w(v)}{v} dv, \quad t \in [T/\rho, t_{l^*}]. \tag{2.14}
$$

Otherwise, there exists a $t^* \in (T/\rho, t_{l^*}]$ such that

$$
x(t^*) = \left(\ln \frac{1}{\rho} \right)^{-1} \int_{T}^{\frac{t^*}{\rho}} \frac{w(v)}{v} dv \text{ and } x(t) > \left(\ln \frac{1}{\rho} \right)^{-1} \int_{T}^{\frac{t}{\rho}} \frac{w(v)}{v} dv, \quad t \in (T/\rho, t^*).
$$

Then, from (2.1) and (2.10), we have

$$
\begin{aligned}
\left(\ln \frac{1}{\rho} \right)^{-1} &\int_{T}^{\frac{t^*}{\rho}} \frac{w(v)}{v} dv = x(t^*) \\
&= w(t^*) + C(t^*)x(\gamma t^*) + \int_{st^*}^{t^*} \frac{Q(u)}{u} x(\beta u) du + \int_{t^*}^{\frac{\beta}{\alpha} st^*} \frac{P(u)}{u} x(\alpha u) du \\
&> \left(\ln \frac{1}{\rho} \right)^{-1} \left(\int_{t^*}^{\frac{t^*}{\rho}} \frac{w(v)}{v} dv + C(t^*) \int_{T}^{\frac{\gamma t^*}{\rho}} \frac{w(v)}{v} dv \right. \\
&\quad \left. + \int_{st^*}^{t^*} \frac{Q(u)}{u} \int_{T}^{\frac{\beta}{\rho} u} \frac{\omega(v)}{v} dv du + \int_{t^*}^{\frac{\beta}{\alpha} st^*} \frac{P(u)}{u} \int_{T}^{\frac{\alpha}{\rho} u} \frac{\omega(v)}{v} dv du \right)
\end{aligned}
$$

$$> \left(\ln \frac{1}{\rho} \right)^{-1} \left(\int_{t^*}^{\frac{t^*}{\rho}} \frac{w(v)}{v} dv + C(t^*) \int_T^{t^*} \frac{w(v)}{v} dv \right.$$

$$\left. + \int_{st^*}^{t^*} \frac{Q(u)}{u} \int_T^{t^*} \frac{\omega(v)}{v} dv du + \int_{t^*}^{\frac{\beta}{\alpha} st^*} \frac{P(u)}{u} \int_T^{t^*} \frac{\omega(v)}{v} dv du \right)$$

$$= \left(\ln \frac{1}{\rho} \right)^{-1} \int_{t^*}^{\frac{t^*}{\rho}} \frac{w(v)}{v} dv$$

$$+ \left(\ln \frac{1}{\rho} \right)^{-1} \left(C(t^*) + \int_{st^*}^{t^*} \frac{Q(u)}{u} du + \int_{t^*}^{\frac{\beta}{\alpha} st^*} \frac{P(u)}{u} du \right) \int_T^{t^*} \frac{\omega(v)}{v} dv$$

$$\geq \left(\ln \frac{1}{\rho} \right)^{-1} \int_T^{\frac{t^*}{\rho}} \frac{\omega(v)}{v} dv.$$

This is a contradiction and so (2.14) holds. Similarly, it follows from (2.1)–(2.3) and (2.14) that

$$x(t_{l^*}^+) = w(t_{l^*}^+) + C(t_{l^*}^+) x(\gamma t_{l^*}^+) + \int_{st_{l^*}}^{t_{l^*}} \frac{Q(u)}{u} x(\alpha u) du + \int_{t_{l^*}}^{\frac{\beta}{\alpha} st_{l^*}} \frac{P(u)}{u} x(\beta u) du$$

$$\geq w(t_{l^*}^+) + C(t_{t_{l^*}}) x(\gamma t_{l^*}) + \int_{st_{l^*}}^{t_{l^*}} \frac{Q(u)}{u} x(\alpha u) du + \int_{t_{l^*}}^{\frac{\beta}{\alpha} st_{l^*}} \frac{P(u)}{u} x(\beta u) du$$

$$> \left(\ln \frac{1}{\rho} \right)^{-1} \int_{t_{l^*}}^{\frac{t_{l^*}}{\rho}} \frac{w(v)}{v} dv + \left(\ln \frac{1}{\rho} \right)^{-1} \int_T^{t_{l^*}} \frac{w(v)}{v} dv$$

$$= \left(\ln \frac{1}{\rho} \right)^{-1} \int_T^{\frac{t_{l^*}}{\rho}} \frac{w(v)}{v} dv.$$

Repeating the above procedure, by induction, we can see that

$$x(t) > \left(\ln \frac{1}{\rho} \right)^{-1} \int_T^{\frac{t}{\rho}} \frac{w(v)}{v} dv, \quad t \geq T. \tag{2.15}$$

Thus, by (2.4) and (2.15), we have

$$w'(t) = -\frac{1}{t} H\left(\frac{\beta}{\alpha} st \right) x(\beta st)$$

$$\leq -\left(\ln \frac{1}{\rho} \right)^{-1} \frac{1}{t} H\left(\frac{\beta}{\alpha} st \right) \int_T^{\frac{\beta s}{\rho} t} \frac{w(v)}{v} dv$$

$$\leq -\left(\ln \frac{1}{\rho} \right)^{-1} \frac{1}{t} H\left(\frac{\beta}{\alpha} st \right) \int_T^t \frac{w(v)}{v} dv,$$

$$\leq -\left(\ln \frac{1}{\rho} \right)^{-1} \frac{1}{t^2} H\left(\frac{\beta}{\alpha} st \right) \int_T^t w(v) dv,$$

where $t \geq T/\rho$ and $t \neq t_k$. Let $y(t) = \left(\ln \frac{1}{\rho} \right)^{-1} \int_T^t w(v) dv$, then $y(t_k^+) = y(t_k)$, $y'(t_k^+) = \left(\ln \frac{1}{\rho} \right)^{-1} w(t_k^+) \leq \left(\ln \frac{1}{\rho} \right)^{-1} b_k w(t_k) = b_k y'(t_k)$ for $k = l, l+1, \cdots$. Thus $y(t) > 0$ for $t > T/\rho$ and $y(t)$ satisfies (2.11), which

contradicts the assumption that (2.11) has no eventually positive solution. So $w(t)$ is eventually negative. The proof is complete. □

The following lemma follows from the similar arguments to that of Theorem 1 in [1] by letting $\varphi(x) = x$. We omit the details.

Lemma 2.4 *Consider the impulsive differential inequality*

$$
\begin{cases}
y''(t) + G(t)y(t) \leq 0, & t \geq t_0, \ t \neq t_k, \\
y(t_k^+) \geq y(t_k), & k = 1, 2, \ldots, \\
y'(t_k^+) \leq C_k y'(t_k), & k = 1, 2, \ldots,
\end{cases}
\tag{2.16}
$$

where $0 \leq t_0 < t_1 < \ldots < t_k < \ldots$ are fixed points with $\lim_{t \to \infty} t_k = \infty$, $G(t) \in PC([t_0, \infty), R^+)$ and $C_k > 0$. If

$$
\sum_{i=0}^{\infty} \int_{t_i}^{t_{i+1}} \frac{1}{C_0 C_1 \ldots C_i} G(t) dt = \infty,
$$

where $C_0 = 1$, then inequality (2.16) has no solution $y(t)$ such that $y(t) > 0$ for $t \geq t_0$.

Theorem 2.5 *Assume that all the conditions of Lemma 2.1 hold and there exists a number $s \in [\alpha/\beta, 1]$ such that*

$$
W_s(t) = C(t) + \int_{st}^{t} \frac{Q(u)}{u} du + \int_{t}^{\frac{\beta}{\alpha}st} \frac{P(u)}{u} du \equiv 1, \quad t \geq t_0.
\tag{2.17}
$$

Further assume that (2.11) has no eventually positive solution, then every solution of (1.1) and (1.2) oscillates.

Proof Suppose that (1.1) and (1.2) has a nonoscillatory solution $x(t)$. Without loss of generality, we assume that $x(\rho t) > 0, t \geq t_0$. Then the conditions of Lemma 2.2 imply eventually $\omega(t) > 0$, while Lemma 2.3 implies eventually $\omega(t) < 0$. This is a contradiction and so the proof is complete. □

From Lemma 2.4 and Theorem 2.1, one can easily establish the following theorem.

Theorem 2.6 *Let all the conditions of Lemma 2.1 and (2.17) hold. If*

$$
\sum_{i=0}^{\infty} \int_{t_i}^{t_{i+1}} \frac{1}{b_0 b_1 \cdots b_i} \frac{1}{t^2} H\left(\frac{\beta}{\alpha} st\right) dt = \infty,
$$

where $b_0 = 1$, then every solution of (1.1) and (1.2) oscillates.

3. An example

In this section, we give an example to illustrate the usefulness of our main results.

Example 3.1 *Consider the impulsive neutral pantograph equation*

$$
\left[x(t) - \frac{3}{4}x(t/2)\right]' + \frac{P(t)}{t}x(t/2) - \frac{Q(t)}{t}x(t) = 0, \quad t \geq t_0 = 4, \ t \neq t_k,
\tag{3.1}
$$

$$x(t_k^+) = \left(\frac{k}{k+1}\right)^2 x(t_k), \quad k = 1, 2, \ldots, \tag{3.2}$$

where $t_k = 4k + 1$, $P(t) = \frac{1}{4\ln 2}$ and $Q(t) = \frac{2t+1}{(8\ln 2)(t+1)} + \frac{1}{2t+1} - \frac{3(2t+1)}{8(t+1)^2}$.

Clearly, the conditions $(H_1) - (H_3)$ and (2.2), (2.3) hold. It is easy to see that

$$H(t) = P(t) - Q\left(\frac{\alpha}{\beta}t\right) = \frac{1}{(4\ln 2)(t+2)} + \frac{t^2 - 2t - 5}{2(t+1)(t+2)^2} > \frac{1}{(4\ln 2)(t+2)}.$$

We also have

$$W_1(t) = \frac{3}{4} + \int_t^{2t} \frac{1}{(4\ln 2)u} du \equiv 1, t \geq 4.$$

Computing yields

$$\sum_{i=0}^{\infty} \int_{t_i}^{t_{i+1}} \frac{1}{b_0 b_1 \cdots b_i} \frac{1}{t^2} H\left(\frac{\beta}{\alpha} st\right) dt \geq \frac{1}{8\ln 2} \sum_{i=1}^{\infty} \int_{4i+1}^{4i+5} \frac{1}{b_0 b_1 \cdots b_i} \frac{1}{t^2(st+1)} dt$$

$$\geq \frac{1}{(16\ln 2)s} \sum_{i=1}^{\infty} \int_{4i+1}^{4i+5} \frac{1}{b_0 b_1 \cdots b_i} \frac{1}{t^3} dt$$

$$= \frac{1}{(4\ln 2)s} \sum_{i=1}^{\infty} \frac{(4i+3)(i+1)^2}{(4i+1)^2(4i+5)^2} = \infty.$$

Hence, all the conditions of Theorem 2.2 are satisfied and so every solution of (3.1) and (3.2) is oscillatory by Theorem 2.2.

Remark 3.2 We can verify that Eq. (3.1) has a nonoscillatory solution, $x(t) = \frac{2t}{2t+1}$. Therefore, the oscillatory properties of all solutions of (3.1) and (3.2) are caused by the presence of the impulses. That is, the impulses given by (3.2) play an essential role in the oscillatory behavior of solutions of (3.1) and (3.2).

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