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On the nilpotent graph of a ring

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Abstract: Let R be a ring with unity. The nilpotent graph of R, denoted by $\Gamma_N(R)$, is a graph with vertex set $\mathcal{Z}_N(R)^* = \{0 \neq x \in R \mid xy \in N(R)$ for some $0 \neq y \in R\}$; and two distinct vertices x and y are adjacent if and only if $xy \in N(R)$, where $N(R)$ is the set of all nilpotent elements of R. Recently, it has been proved that if R is a left Artinian ring, then $\mathrm{diam}(\Gamma_N(R)) \leq 3$. In this paper, we present a new proof for the above result, where R is a finite ring. We study the diameter and the girth of matrix algebras. We prove that if F is a field and $n \geq 3$, then $\mathrm{diam}(\Gamma_N(M_n(F))) = 2$. Also, we determine $\mathrm{diam}(\Gamma_N(M_2(F)))$ and classify all finite rings whose nilpotent graphs have diameter at most 3. Finally, we determine the girth of the nilpotent graph of matrix algebras.

Key words: Nilpotent graph, diameter, girth

1. Introduction

The study of algebraic structures using the properties of graphs has become an exciting research topic in the last twenty years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring, see for example [1-6] and [12]. Throughout this paper, all graphs are simple with no loops and multiple edges. The *zero-divisor graph* of a commutative ring R, denoted by $\Gamma(R)$, is a graph with vertex set $Z(R)^*$, the set of non-zero zero-divisors of R, and two distinct vertices x and y are adjacent if and only if $xy = 0$. The zero-divisor graph has been studied extensively in recent years, see for example [1–5]. In [7], Chen defined a graph structure on a ring R whose vertices are all the elements of R, and two distinct vertices x and y are adjacent if and only if $xy \in N(R)$, where $N(R)$ denotes the set of all nilpotent elements of R.

Throughout, R is a ring with unity. The *nilpotent graph* of R was introduced in [10]. For every $X \subseteq R$, we denote $X \setminus \{0\}$ by X^*. The vertex set of $\Gamma_N(R)$ is $\mathcal{Z}_N(R)^*$, where $\mathcal{Z}_N(R) = \{x \in R \mid xy \in N(R)$ for some $y \in R^*\}$, and two distinct vertices x and y in $\mathcal{Z}_N(R)^*$ are adjacent if and only if $xy \in N(R)$, or equivalently, $yx \in N(R)$. It is easy to see that the usual zero-divisor graph $\Gamma(R)$ is a subgraph of $\Gamma_N(R)$. Let G be a graph with vertex set $V(G)$. A *path* from x to y is a series of adjacent vertices $x — x_1 — x_2 — \cdots — x_n — y$. For $x, y \in V(G)$ with $x \neq y$, $d(x, y)$ denotes the length of a shortest path from x to y, if there is no such path, we will make the convention $d(x, y) = \infty$. The *diameter* of G is defined as $\mathrm{diam}(G) = \sup\{d(x, y) | x$ and y are vertices of $G\}$. For any $x \in V(G), d(x)$ denotes the number of edges incident with x, called the *degree* of x. A *cycle* is a path that begins and ends at the same vertex in which no edge is repeated and all vertices other than the starting and ending vertex are distinct. An n-cycle is a cycle with n vertices, where $n \geq 3$. We denote

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the complete graph with n vertices by K_n. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G ($\mathrm{gr}(G) = \infty$ if G contains no cycles). A *bipartite graph* is one whose vertex set can be partitioned into two subsets X and Y so that each edge has one end in X and one end in Y. A *complete bipartite graph* is a bipartite graph with two partitions X and Y in which every vertex in X is joined to every vertex in Y. A *star graph* is a bipartite graph with part sizes 1 and n for some positive integer n.

For a ring R, we denote by $M_n(R)$, R^n and I, the ring of all $n \times n$ matrices, the set of $n \times 1$ matrices over R and the identity matrix, respectively. Also, for any i and j, $1 \leq i, j \leq n$, we use E_{ij} to denote the element of $M_n(R)$ whose (i, j)-entry is 1 and other entries are 0.

We denote the Jacobson radical of R by $J(R)$. A ring R is called *semisimple* if its Jacobson radical is zero. By the Wedderburn-Artin Theorem [8, Theorem 3.5], we know that every semisimple Artinian ring is isomorphic to the direct product of finitely many full matrix rings over division rings. A ring R is called *reduced* if R has no non-zero nilpotent elements.

In [9], the authors proved that if R is a left Artinian ring, then $\mathrm{diam}(\Gamma_N(R)) \leq 3$. In Section 2, we prove that if F is a field and $n \geq 3$, then $\mathrm{diam}(\Gamma_N(M_n(F))) = 2$. We show that if F is a field, then $\mathrm{diam}(\Gamma_N(M_2(F))) = 3$. Furthermore, we present a new proof for Theorem 2.3 of [9] when R is a finite ring. Also, we classify the diameter of the nilpotent graph for all finite rings with $J(R) = 0$. In Section 3, we determine the girth of the nilpotent graph for products of matrix rings.

2. The diameter of nilpotent graph of matrix algebras

In [9], the authors proved that if R is a left Artinian ring, then $\mathrm{diam}(\Gamma_N(R)) \leq 3$. In this section, we prove that if F is a field and $n \geq 3$, then $\mathrm{diam}(\Gamma_N(M_n(F))) = 2$. We show that if F is a field, then $\mathrm{diam}(\Gamma_N(M_2(F))) = 3$. Furthermore, we present a new proof for Theorem 2.3 of [9] when R is a finite ring and classify the diameter of the nilpotent graph for all finite rings with $J(R) = 0$. We start with the following remark.

Remark 1 If $R = M_n(F)$, where F is a field and $n \geq 2$, then every nonzero element of R is a vertex of $\Gamma_N(R)$. In fact, if A is a non-singular matrix, then A is adjacent to $A^{-1}E_{1n}$ and so $A \in V(\Gamma_N(R))$. Also, if A is a singular matrix, then $AY = 0$ for some $0 \neq Y \in R$. Therefore $A \in V(\Gamma_N(R))$.

Theorem 1 *If F is a field and $n \geq 3$, then* $\mathrm{diam}(\Gamma_N(M_n(F))) = 2$.

Proof Suppose that $A, B \in M_n(F)$ and $C = \left[\ 0 \mid X\ \right]$, where $X \in F^n$. Then $AC = \left[\ 0 \mid AX\ \right]$ and $BC = \left[\ 0 \mid BX\ \right]$. Assume that $W_1 = \{X \in F^n \mid A_nX = 0\}$ and $W_2 = \{X \in F^n \mid B_nX = 0\}$, where A_n and B_n are the nth rows of A and B, respectively. Both W_1 and W_2 are subspaces of F^n. We have $\dim W_i \geq n - 1$, for $i = 1, 2$. Since $n \geq 3$, there exists $0 \neq X_0 \in W_1 \cap W_2$. Let $C = \left[\ 0 \mid X_0\ \right]$. Obviously, C is adjacent to both A and B. Hence $\mathrm{diam}(\Gamma_N(M_n(F))) \leq 2$. On the other hand, E_{nn} and I are two non-adjacent vertices of $\Gamma_N(M_n(F))$. Therefore $\mathrm{diam}(\Gamma_N(M_n(F))) = 2$. □

Theorem 2 *If F is a field, then* $\mathrm{diam}(\Gamma_N(M_2(F))) \leq 3$.

Proof Let $A, B \in M_2(F)$ and X be a nilpotent matrix in $M_2(F)$. We have the following cases:

Case 1. A and B are non-singular matrices. Then A — XA^{-1} — $B^{-1}X$ — B is a path.

Case 2. A is a non-singular matrix and B is a singular matrix. Then $BY = 0$ for some $0 \neq Y$. If $YX = 0$, then A — XA^{-1} — Y — B is a path. If $YX \neq 0$, then A — XA^{-1} — YX — B is a path.

Case 3. A and B are singular matrices. If AB is nilpotent, then A — B is a path. Otherwise, there exist $X, Y \neq 0$ such that $AX = 0$ and $YB = 0$. If $XY = 0$, then A — X — Y — B is a path. Also, A — XY — B is a path, for $XY \neq 0$. Thus $\operatorname{diam}(\Gamma_N(M_2(F))) \leq 3$. □

Lemma 1 *If F is a field and there exists $c \in F$ such that c is not a square, then* $\operatorname{diam}(\Gamma_N(M_2(F))) = 3$.

Proof Suppose that $c \in F$ such that c is not a square and $A = \begin{bmatrix} 0 & 1 \\ c & 0 \end{bmatrix}$. We claim that $d(I, A) = 3$. By contradiction, assume that X is adjacent to both I and A. Since X is nilpotent, $X = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$, for some $x, y, z \in F$. On the other hand, $XA = \begin{bmatrix} cy & x \\ -cx & z \end{bmatrix}$ is nilpotent. This yields that that $cy = -z$ and $cyz = -z^2$. Since $detX = 0$, we have $x^2 = -yz$. This implies that $cx^2 = z^2$. Note that if $x = 0$, then $yz = 0$ and by $cy = -z$, we conclude that $X = 0$, a contradiction. Therefore $x \neq 0$ and so $c = (zx^{-1})^2$, which is a contradiction. Thus by Theorem 2, the proof is complete. □

Remark 2 It is well known that for a ring R, the set $\{ a \in R \,|\, a \text{ is a unit } \}$ is a group under multiplication and is called the group of units of R. In particular, if F is a field, then F^* is a group under multiplication. Also, by Cauchy's Theorem [11, Theorem 7.2.2], we know that if G is a finite group and p is a prime number dividing the order of G (the number of elements in G), then G contains an element of order p. That is, there is an x in G so that p is the smallest positive integer with $x^p = e$, where e is the identity element.

Lemma 2 *Let F be a finite field. Then $|F|$ is an even number if and only if every element of F is square.*

Proof First suppose that $|F|$ is an even number. Let $f : F \to F$, $f(x) = x^2$. Since $char(F) = 2$, f is one-to-one. Since F is finite, f is onto. Hence every element of F is square. Conversely, assume that every element of F is square. Let $f : F^* \to F^*$, $f(x) = x^2$. We know that f is a group homomorphism. Since every element of F is a square, f is onto. Since F is finite, we conclude that f is one-to-one. If $|F^*|$ is an even number, then by Remark 2, there exists an $a \in F^*$ such that $o(a) = 2$. Therefore $f(a) = 1$, which is a contradiction. Thus $|F^*|$ is an odd number, and so $|F|$ is an even number. This completes the proof. □

Corollary 1 *If F is a finite field and $|F|$ is an odd number, then* $\operatorname{diam}(\Gamma_N(M_2(F))) = 3$.

Proof This is clear by Lemmas 1 and 2. □

Now, we would like to show that $\operatorname{diam}(\Gamma_N(M_2(F))) = 3$ when $|F|$ is an even number. Before starting the proof, we need the following lemma.

Lemma 3 $\operatorname{diam}(\Gamma_N(M_2(\mathbb{Z}_2))) = 3$.

Proof Suppose that $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Obviously, AB is not nilpotent. We claim that $d(A,B) = 3$. By contradiction, assume that $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is adjacent to both A and B. Therefore the following matrices should be nilpotent:

$$C = \begin{bmatrix} z & w \\ x & y \end{bmatrix}, G = \begin{bmatrix} y & x \\ w & z \end{bmatrix}, D = \begin{bmatrix} x & y \\ x+z & y+w \end{bmatrix}, E = \begin{bmatrix} x+y & y \\ z+w & w \end{bmatrix}.$$

Since $xw = yz$, if $x = 0$, then $y = z = 0$. On the the other hand, D is nilpotent, which implies that $w = 0$, a contradiction. Now, suppose that $x \neq 0$. Since C is nilpotent, we conclude that $y = -z$. But D is nilpotent; this implies that $D^2 = 0$ and $x^2 + xy - y^2 = 0$ for some $0 \neq x, y \in \mathbb{Z}_2$, a contradiction. Now, by Theorem 2, $\mathrm{diam}(\Gamma_N(M_2(\mathbb{Z}_2))) = 3$. □

Lemma 4 *If F is a field and $char(F) = 2$, then* $\mathrm{diam}(\Gamma_N(M_2(F))) = 3$.

Proof If $|F| = 2$, then by Lemma 3, we are done. Otherwise, $|F| \geq 4$. Let $S = \{\alpha + \alpha^{-1} \,|\, \alpha \in F^*\}$. Clearly, $|S| \leq \frac{|F|}{2}$. Assume that $y \in F \setminus (S \cup \{0,1\})$. Let $x = 1 + y$, $A = \begin{bmatrix} 1 & 1 \\ 1 & x \end{bmatrix}$ and $B = \begin{bmatrix} x & 1 \\ 1 & 1 \end{bmatrix}$. We claim that $d(A,B) = 3$. If A is adjacent to B, then $det(A)det(B) = 0$. Hence $(1+x)^2 = 0$, and so $y = 0$, a contradiction. Therefore $d(A,B) \geq 2$. By contradiction, suppose that $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is adjacent to A and B. Since $trac(AX) = 0$, we have $a + c = b + dx$. On the other hand, $trac(BX) = 0$, which implies that $ax + c = b + d$. Therefore $a(1+x) = d(1+x)$. We note that $1 + x = y \in F^*$, and so $a = d$. Therefore $X = \begin{bmatrix} a & b \\ a+b+ax & a \end{bmatrix}$. Since $det(AX) = 0$ and $det(A) \neq 0$, we conclude that $det(X) = 0$. Hence $a^2 + b^2 = ab(1+x)$. If $a = 0$, then $b = 0$ and so $X = 0$, a contradiction. Therefore $a \neq 0$, and similarly, $b \neq 0$. This yields that $y = (a^2 + b^2)a^{-1}b^{-1} = ab^{-1} + ba^{-1} \in S$, a contradiction. Therefore $d(A,B) = 3$. By Theorem 2, $\mathrm{diam}(\Gamma_N(M_2(F))) = 3$. □

Now, we are in a position to prove one of the main results.

Theorem 3 *If F is a field, then* $\mathrm{diam}(\Gamma_N(M_2(F))) = 3$.

Proof If $|F|$ is an odd number, then by Corollary 1, $\mathrm{diam}(\Gamma_N(M_2(F))) = 3$. Otherwise, $char(F) = 2$ and by Lemma 4, $\mathrm{diam}(\Gamma_N(M_2(F))) = 3$. □

Lemma 5 *If $R \cong \prod_{i=1}^{k} M_{n_i}(F_i)$, $F_1, \ldots, F_k$ are finite fields, and $n_i \geq 3$ for some i, $1 \leq i \leq k$, then* $\mathrm{diam}(\Gamma_N(R)) = 2$.

Proof Without loss of generality, suppose that $n_1 \geq 3$. Let $x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \in R$. If $x_1 = y_1 = 0$ then x — $(1, 0, \ldots, 0)$ — y. Otherwise, by Theorem 1, there exists $0 \neq \alpha_1 \in M_{n_1}(F_1)$ such that $x_1\alpha_1, y_1\alpha_1 \in N(M_{n_1}(F_1))$. Hence $x(\alpha_1, 0, \ldots, 0), y(\alpha_1, 0, \ldots, 0) \in N(R)$, which implies that $d(x,y) \leq 2$. Note that if $d(x_1, y_1) = 2$ and $x_i = y_i = 0$, for $i \neq 1$, then $d(x,y) = 2$. Thus $\mathrm{diam}(\Gamma_N(R)) = 2$. □

Lemma 6 *If* $R \cong \prod_{i=1}^{k} M_{n_i}(F_i)$, $F_1, \ldots, F_k$ *are finite fields,* $n_i \leq 2$ *for* $i = 1, \ldots, k$, *and* $n_j = 2$ *for some* j, $1 \leq j \leq k$, *then* $\operatorname{diam}(\Gamma_N(R)) = 3$.

Proof If $k = 1$, then by Theorem 3, $\operatorname{diam}(\Gamma_N(R)) = 3$. Now, assume that $k \geq 2$. First we claim that $\operatorname{diam}(\Gamma_N(R)) \leq 3$. Let $x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \in V(\Gamma_N(R))$. Without loss of generality assume that $n_1 = 2$. If $x_1 = y_1 = 0$, then x and y are adjacent to $(1, 0, \ldots, 0)$. If $x_1, y_1 \neq 0$, then by Remark 1 and Theorem 3, $d(x_1, y_1) \leq 3$, and so $d(x, y) \leq 3$. If $x_1 = 0$ and $y_1 \neq 0$, then $y_1 a_1 \in N(M_2(F_1))$, for some $0 \neq a_1 \in M_2(F_1)$. Hence x and y are adjacent to $(a_1, 0, \ldots, 0)$ and the claim is proved. Now, we show that $\operatorname{diam}(\Gamma_N(R)) = 3$. If $n_i = 1$, then let $0 \neq x_i, y_i \in M_{n_i}(F_i)$. Also, there exist $x_i, y_i \in M_{n_i}(F_i)$ such that $d(x_i, y_i) = 3$, where $n_i = 2$. Now, let $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$. It is easy to see that $d(x, y) = 3$. This completes the proof. □

We note that if R is a commutative reduced ring, then $\Gamma_N(R) = \Gamma(R)$ and by [5, Theorem 2.3], $\operatorname{diam}(\Gamma(R)) \leq 3$. In particular, if R is a finite direct product of finite fields, then $\operatorname{diam}(\Gamma_N(R)) \leq 3$. By Theorem 2.3 of [9], if R is a left Artinian ring, then $\operatorname{diam}(\Gamma_N(R)) \leq 3$. In the rest of this section, we present a new proof for the above result when R is a finite ring.

Theorem 4 *If* R *is a finite ring, then* $\operatorname{diam}(\Gamma_N(R)) \leq 3$.

Proof Since R is a finite ring, it is a left Artinian and by [8, Theorem 4.12], $J(R)$ is nilpotent. We have the following cases:

Case 1. $J(R) \neq 0$. Assume that $0 \neq x \in J(R)$. Clearly, every vertex is adjacent to x and hence $\operatorname{diam}(\Gamma_N(R)) \leq 2$.

Case 2. $J(R) = 0$. By the Wedderburn-Artin theorem [8, Theorem 3.5], $R \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$, where $D_1, \ldots, D_k$ are division rings and $n_1, \ldots, n_k, k$ are positive integers. Since R is a finite ring by Wedderburn's little theorem [8, Theorem 13.1], every D_i is a field. Hence $R \cong M_{n_1}(F_1) \times \cdots \times M_{n_k}(F_k)$, where $F_1, \ldots, F_k$ are fields and $n_1, \ldots, n_k, k$ are positive integers. By Lemmas 5 and 6, $\operatorname{diam}(\Gamma_N(R)) \leq 3$. □

According to the proof of Theorem 4, one can easily deduce the next result.

Corollary 2 *Let* R *be a finite ring. If* $J(R) \neq 0$, *then* $\operatorname{diam}(\Gamma_N(R)) \leq 2$.

Also, the following result is a characterization of diameter of the nilpotent graph.

Corollary 3 *If* $R \cong \prod_{i=1}^{k} M_{n_i}(F_i)$, $F_1, \ldots, F_k$ *are fields and* $n_1, \ldots, n_k, k$ *are positive integers, then the following hold:*

(i) If $n_i \geq 3$, *for some* i, *then* $\operatorname{diam}(\Gamma_N(R)) = 2$.

(ii) If $n_i \leq 2$ *for every* i *and* $n_j = 2$, *for some* j, *then* $\operatorname{diam}(\Gamma_N(R)) = 3$.

(iii) If $k = 2$ *and* $n_1 = n_2 = 1$, *then* $\operatorname{diam}(\Gamma_N(R)) \in \{1, 2\}$.

(iv) If $k \geq 3$ *and* $n_1 = \cdots = n_k = 1$, *then* $\operatorname{diam}(\Gamma_N(R)) = 3$.

3. The girth of nilpotent graph of matrix algebras

In this section, we characterize the girth of the nilpotent graph of matrix algebras.

Lemma 7 *If* $R \cong \prod_{i=1}^{k} M_{n_i}(F_i)$, $F_1, \ldots, F_k$ *are fields,* $n_1, \ldots, n_k$ *are positive integers and* $k \geq 3$, *then* $gr(\Gamma_N(R)) = 3$.

Proof Let e_i be the $1 \times n$ vector whose ith component is I and other components are 0. Since $k \geq 3$, e_1 — e_2 — e_3 — e_1 is a 3-cycle of $\Gamma_N(R)$. This implies that $\text{gr}(\Gamma_N(R)) = 3$. □

We note that if F is a field, then $\Gamma_N(F)$ is empty. So in the following lemma, we determine $\text{gr}(\Gamma_N(M_n(F)))$, where $n \geq 2$.

Lemma 8 *If F is a field and $n \geq 2$, then $gr(\Gamma_N(M_n(F))) = 3$.*

Proof It is easy to see that E_{1n} — E_{nn} — $\sum_{i=1}^{n} E_{1i}$ — E_{1n} is a 3-cycle. This implies that $\text{gr}(\Gamma_N(M_n(F))) = 3$. □

We note that if F is a field, then by [6, Theorem 2.4], $\Gamma_N(\mathbb{Z}_2 \times F)$ is a star graph with center $(1,0)$ and $\text{gr}(\Gamma_N(\mathbb{Z}_2 \times F)) = \infty$. Also, if F_i is a field and $|F_i| \geq 3$ for every i, $1 \leq i \leq 2$, then $\Gamma_N(F_1 \times F_2) = \Gamma(F_1 \times F_2)$ and by [6, Remark 2.6], $\text{gr}(\Gamma_N(F_1 \times F_2)) = 4$. In the following results, we determine $\text{gr}(\Gamma_N(M_{n_1}(F_1) \times M_{n_2}(F_2)))$.

Lemma 9 *Let $R \cong M_{n_1}(F_1) \times M_{n_2}(F_2)$, every F_i be a field and $|F_i| \geq 3$ for some i. If $n_1, n_2 \geq 2$, then $gr(\Gamma_N(R)) = 3$.*

Proof Let e_i be the $1 \times n$ vector whose ith component is I and other components are 0. First suppose that $|F_1| \geq 3$. Then e_1 — $(aE_{1n_1}, 0)$ — e_2 — e_1 is a 3-cycle, for some $a \in {F_1}^*$. Therefore $\text{gr}(\Gamma_N(R)) = 3$. The argument for $|F_2| \geq 3$ is similar. □

Lemma 10 *If F_1, F_2 are fields and $n \geq 2$, then $gr(\Gamma_N(F_1 \times M_n(F_2))) = 3$.*

Proof In view of proof of Lemma 8, we find that E_{1n} — E_{nn} — $\sum_{i=1}^{n} E_{1i}$ — E_{1n} is a 3-cycle of $\Gamma_N(M_n(F_2))$. So, $(0, E_{1n})$ — $(0, E_{nn})$ — $(0, \sum_{i=1}^{n} E_{1i})$ — $(0, E_{1n})$ is a 3-cycle $\Gamma_N(F_1 \times M_n(F_2))$. □

Lemma 11 *$gr(\Gamma_N(M_{n_1}(\mathbb{Z}_2) \times M_{n_2}(\mathbb{Z}_2)) \in \{3, \infty\}$.*

Proof Let e_i be the $1 \times n$ vector whose ith component is I and other components are 0. If $n_1 \geq 2$, then e_1 — e_2 — $(E_{1n_1}, 0)$ — e_1 is a 3-cycle. This yields that $\text{gr}(\Gamma_N(R)) = 3$. The argument for $n_2 \geq 2$ is similar. If $n_1 = n_2 = 1$, then $\Gamma_N(R) = K_2$ and so $\text{gr}(\Gamma_N(R)) = \infty$. □

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References

[1] Akbari, S., Mohammadian, A.: On zero-divisor graphs of finite rings, J. Algebra 13, 168–184 (2007).

[2] Akbari, S., Mohammadian, A.: Zero-divisor graphs of non-commutative rings, J. Algebra 296, 462–479 (2006).

[3] Anderson, D.F., Axtell, M.C., Stickles, J.A., Jr.: Zero-divisor graphs in commutative rings, in Commutative Algebra, Noetherian and Non-Noetherian Perspectives (Eds.: M. Fontana, S.-E. Kabbaj, B. Olberding, I. Swanson) Springer-Verlag, New York 23–45 (2011).

[4] Anderson, D.F., Badawi, A.: The total graph of a commutative ring, J. Algebra 320, 2706–2719 (2008).

[5] Anderson, D.F., Livingston, P.S.: The zero-divisor graph of a commutative ring, J. Algebra 217, 434–447 (1999).

[6] Anderson, D.F., Mulay, S.B.: On the diameter and girth of a zero-divisor graph, J. Pure Appl. Algebra 210, 543–550 (2007).

[7] Chen, P.W.: A kind of graph structure of rings, Algebra Colloq. 10:2, 229–238 (2003).

[8] Lam, T.Y.: A First Course in Noncommutative Rings, Springer-Verlag, New York, Inc, 1976.

[9] Li, A.H., Li, Q.H.: A Kind of Graph Structure on Non-reduced Rings, Algebra Colloq. 17:1, 173–180 (2010).

[10] Li, A.H., Li, Q.S.: A kind of graph structure on Von-Neumann Regular Rings, International J. Algebra 4, 291–302 (2010).

[11] Malik, D.S., Mordeson, J.M., Sen, M.K.: Fundamentals of Abstract Algebra, McGraw-Hill, New York, Inc, 1997.

[12] Nikmehr, M.J., Shaveisi, F.: The regular digraph of ideals of a commutative ring, Acta Math. Hungar., 134 (4), 516-528 (2012).