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Global existence, decay and blow up solutions for coupled nonlinear wave equations with damping and source terms

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Abstract: We study the initial-boundary value problem for a system of nonlinear wave equations with nonlinear damping and source terms, in a bounded domain. The decay estimates of the energy function are established by using Nakao's inequality. The nonexistence of global solutions is discussed under some conditions on the given parameters.

Key words: Decay rate, blow up, initial boundary value problem, nonlinear wave equations

1. Introduction

In this paper we consider the following initial-boundary value problem:

$$\begin{cases} u_{tt}+|u_t|^{m-1}u_t=\operatorname{div}\left(\rho\left(|\nabla u|^2\right)\nabla u\right)+f_1(u,v), & (x,t)\in\Omega\times(0,T), \\ v_{tt}+|v_t|^{r-1}v_t=\operatorname{div}\left(\rho\left(|\nabla v|^2\right)\nabla v\right)+f_2(u,v), & (x,t)\in\Omega\times(0,T), \\ u=v=0, & (x,t)\in\partial\Omega\times(0,T), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in\Omega, \\ v(x,0)=v_0(x),\ v_t(x,0)=v_1(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n, $n=1,2,3$; $m,r\geq 1$; $f_i(.\,,.):R^2\longrightarrow R$ are given functions to be specified later. Problems of this type arise in material science and physics.

We assume that ρ is a function which satisfies the relation

$$\rho(s)\in C^1,\ \rho(s)>0,\ \rho(s)+2s\rho'(s)>0 \tag{1.2}$$

for $s>0$.

(**A1**). Let $F(u,v)=a|u+v|^{p+1}+2b|uv|^{\frac{p+1}{2}}$ with $a,b>0$, $p\geq 3$ if $n=1,2$ and $p=3$ if $n=3$; $f_1(u,v)=\frac{\partial F}{\partial u}$, $f_2(u,v)=\frac{\partial F}{\partial v}$; $m,r\geq 1$ if $n=1,2$ and $1\leq m,r\leq 5$ if $n=3$.

One can easily verify that

$$u\ f_1(u,v)+vf_2(u,v)=(p+1)F(u,v),\ \forall(u,v)\in R^2. \tag{1.3}$$

Hao, Zhang and Li [6] considered the single wave equation of the form

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$$u_{tt} - \operatorname{div}\left(\rho\left(|\nabla u|^2\right)\nabla u\right) + h(u_t) = f(u), \ x \in \Omega, \ t > 0 \tag{1.4}$$

with initial and Dirichlet boundary condition, where ρ satisfies condition (1.2) and

$$\rho(s) \geq b_1 + b_2 s^q, \ q \geq 0, \tag{1.5}$$

where b_1, b_2 are nonnegative constants and $b_1 + b_2 > 0$.

Lemma 1.1 *[13]. There exist two positive constants c_0 and c_1 such that*

$$c_0\left(|u|^{p+1} + |v|^{p+1}\right) \leq F(u,v) \leq c_1\left(|u|^{p+1} + |v|^{p+1}\right). \tag{1.6}$$

Throughout this paper we define ρ by

$$\rho(s) = b_1 + b_2 s^q, \ q \geq 0, \tag{1.7}$$

where b_1, b_2 are nonnegative constants and $b_1 + b_2 > 0$. Obviously, ρ satisfies condition (1.2) and (1.5).

In the case of $\rho = 1$, equation (1.4) can be written in the form

$$u_{tt} - \triangle u + h(u_t) = f(u), \ x \in \Omega, \ t > 0. \tag{1.8}$$

The local existence, global existence, and blow up in finite time of solution for (1.8) were established (see [7, 8, 5, 11, 12] and references therein). The interaction between the damping $h(u_t) = |u_t|^{m-1} u_t$ and the source term $f(u) = |u|^{r-1} u$ makes the problem more interesting. Levine [7, 8] first considered the interaction between the linear damping ($m = 1$) and source term by using the concavity method. He showed that solutions with negative initial energy blow up in finite time. But this method can not be applied in the case of a nonlinear damping term.

Georgiev and Todorova in [5] extended Levine's result to the nonlinear case $(m > 1)$. For further knowledge, see [11, 12, 20] and references therein.

Agre and Rammaha [3] studied the global existence and the blow up of solutions of problem (1.1) with $\rho = 1$ using the same techniques as in [5], and also Alves et al. [2] investigated the existence, uniform decay rates and blow up of solutions to systems. After that, the blow up result was improved by Said-Houari [17]. Also, Said-Houari [18] showed that the local solution obtained in [3] is global and this solution has decay property.

Recently, Wu et al. [21] obtained the global existence and blow up of the solution of problem (1.1) under some suitable conditions. Also, Fei and Hongjun [4] considered problem (1.1) and improved the blow up result obtained in [21], for a large class of initial data in positive initial energy, using some techniques as in Payne and Sattinger [15] and some estimates used firstly by Vitillaro [20].

In this paper, under some restrictions on the initial data, we establish the uniform decay rates by means of Nakao's inequality. After that, we show blow up of solution with negative and nonnegative initial energy, using the same techniques as in [9].

Throughout this paper, $\|.\|$ and $\|.\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively.

This paper is organized as follows. In section 2, we present some lemmas. In section 3, we state and prove the local existence result. In section 4, the global existence and the decay of the solution are given. In section 5, we show the blow up properties of solution in cases $m = r = 1$ and $m, r > 1$.

2. Preliminaries

Let us begin stating the following lemmas which will be used later.

Lemma 2.1 *(Sobolev-Poincare inequality) [1]. Let* q *be a number with* $2 \leq q < \infty$ $(n = 1, 2)$ *or* $2 \leq q \leq 2n/(n-2)$ $(n \geq 3)$*, then there is a constant* $C_* = C_*(\Omega, q)$ *such that*

$$\|u\|_q \leq C_* \|\nabla u\| \text{ for } u \in H_0^1(\Omega). \tag{2.1}$$

Lemma 2.2 *[14]. Let* $\phi(t)$ *be nonincreasing and nonnegative function defined on* $[0, T]$*,* $T > 1$*, satisfying*

$$\phi^{1+\alpha}(t) \leq w_0 (\phi(t) - \phi(t+1)), \ t \in [0, T] \tag{2.2}$$

for w_0 *is a positive constant,* α *is a nonnegative constant. Then we have for each* $t \in [0, T]$

$$\begin{cases} \phi(t) \leq \phi(0) e^{-w_1 [t-1]^+}, & \alpha = 0, \\ \phi(t) \leq \left(\phi(0)^{-\alpha} + w_0^{-1} \alpha [t-1]^+\right)^{-\frac{1}{\alpha}}, & \alpha > 0, \end{cases} \tag{2.3}$$

where $[t-1]^+ = \max\{t-1, 0\}$*, and* $w_1 = \ln\left(\frac{w_0}{w_0 - 1}\right)$*.*

Lemma 2.3 *[9]. Let* $\delta > 0$ *and* $B(t) \in C^2(0, \infty)$ *be a nonnegative function satisfying*

$$B''(t) - 4(\delta+1) B'(t) + 4(\delta+1) B(t) \geq 0. \tag{2.4}$$

If

$$B'(0) > r_2 B(0) + K_0 \tag{2.5}$$

with $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$*, then* $B'(t) > K_0$ *for* $t > 0$*, where* K_0 *is a constant.*

Lemma 2.4 *[9]. If* $H(t)$ *is a nonincreasing function on* $[t_0, \infty)$ *and satisfies the differential inequality*

$$[H'(t)]^2 \geq a + b [H(t)]^{2+\frac{1}{\delta}}, \text{ for } t \geq t_0, \tag{2.6}$$

where $a > 0$*,* $b \in R$*, then there exists a finite time* T^* *such that*

$$\lim_{t \longrightarrow T^{*-}} H(t) = 0.$$

Upper bounds for T^* *are estimated as follows:*

(i) If $b < 0$ *and* $H(t_0) < \min\left\{1, \sqrt{-\frac{a}{b}}\right\}$*, then*

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{-\frac{a}{b}}}{\sqrt{-\frac{a}{b}} - H(t_0)}.$$

(ii) If $b = 0$*, then*

$$T^* \leq t_0 + \frac{H(t_0)}{H'(t_0)}.$$

(iii) If $b > 0$, *then*

$$T^* \leq \frac{H(t_0)}{\sqrt{a}} \text{ or } T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \left[1 - (1 + cH(t_0))^{-\frac{1}{2\delta}}\right],$$

where $c = \left(\frac{a}{b}\right)^{2+\frac{1}{\delta}}$.

3. Local existence

In this section, we state and prove the local existence and uniqueness of the solution of problem (1.1).

Definition 3.1 *A pair of functions* (u, v) *is said to be a weak solution of (1.1) on* $[0, T]$ *if* $u, v \in C\left([0, T]; W_0^{1,2(q+1)}(\Omega) \cap L^{p+1}(\Omega)\right)$, $u_t \in C\left([0, T]; L^2(\Omega)\right) \cap L^{m+1}(\Omega \times (0, T))$ *and* $v_t \in C\left([0, T]; L^2(\Omega)\right) \cap L^{r+1}(\Omega \times (0, T))$. *In additon,* (u, v) *satisfies*

$$\begin{aligned} & \int_\Omega u'(t)\phi dx - \int_\Omega u_1(t)\phi dx + \int_\Omega \left(\rho\left(|\nabla u|^2\right)\nabla u\right)\nabla\phi dx + \int_0^t \int_\Omega |u'|^{m-1} u'\phi dx d\tau \\ = & \int_0^t \int_\Omega f_1(u(\tau), v(\tau))\phi dx d\tau, \end{aligned} \tag{3.1}$$

$$\begin{aligned} & \int_\Omega v'(t)\varphi dx - \int_\Omega v_1(t)\varphi dx + \int_\Omega \left(\rho\left(|\nabla v|^2\right)\nabla v\right)\nabla\varphi dx + \int_0^t \int_\Omega |v'|^{r-1} v'\varphi dx d\tau \\ = & \int_0^t \int_\Omega f_2(u(\tau), v(\tau))\varphi dx d\tau \end{aligned} \tag{3.2}$$

for all test functions $\phi \in W_0^{1,2(q+1)}(\Omega) \cap L^{m+1}(\Omega)$, $\varphi \in W_0^{1,2(q+1)}(\Omega) \cap L^{r+1}(\Omega)$ *and for almost all* $t \in [0, T]$.

Theorem 3.2 *(Local existence). Assume (A1) holds. Then, for any initial data* $u_0, v_0 \in W_0^{1,2(q+1)}(\Omega) \cap L^{p+1}(\Omega)$ *and* $u_1, v_1 \in L^2(\Omega)$, *there exists a unique local weak solution* (u, v) *of problem (1.1) (in the sense of Definition 3.1) defined in* $[0, T]$ *for some* $T > 0$, *and satisfies the energy identity*

$$E(t) + \int_0^t \left(\|u_\tau(\tau)\|_{m+1}^{m+1} + \|v_\tau(\tau)\|_{r+1}^{r+1}\right) d\tau = E(0), \tag{3.3}$$

where $E(t)$ *is defined in (4.3).*

Proof The proof will be done by applying the Faedo-Galerkin method.

Approximate solution:

Let $\{w_j\}_{j=1}^\infty$ be a basis for $W_0^{1,2(q+1)}(\Omega)$. Let us define V_m =the linear span of $\{w_j\}_{j=1}^\infty$, $m \geq 1$. We look for an approximate solution of our problem in the form

$$u_m(t) = \sum_{j=1}^m u_{m,j}(t) w_j, \ v_m(t) = \sum_{j=1}^m v_{m,j}(t) w_j, \tag{3.4}$$

where $u_{m,j}(t)$ and $v_{m,j}(t)$ are the solutions of the ODE system

$$\int_\Omega \left\{u''_m - \operatorname{div}\left(\rho\left(|\nabla u_m|^2\right)\nabla u_m\right) + |u'_m|^{m-1} u'_m\right\} w_j dx = \int_\Omega f_1(u_m, v_m) w_j dx, \tag{3.5}$$

$$\int_\Omega \left\{v''_m - \operatorname{div}\left(\rho\left(|\nabla v_m|^2\right)\nabla v_m\right) + |v'_m|^{r-1} v'_m\right\} w_j dx = \int_\Omega f_2(u_m, v_m) w_j dx \tag{3.6}$$

with initial conditions

$$u_m(0) = u_{0m};\ u'_m(0) = u_{1m},\ v_m(0) = v_{0m};\ v'_m(0) = v_{1m}, \tag{3.7}$$

where $u_{0m},\ u_{1m},\ v_{0m}$ and v_{1m} are chosen in V_m such that

$$u_{0m} \longrightarrow u_0,\ v_{0m} \longrightarrow v_0 \text{ strongly in } W_0^{1,2(q+1)}(\Omega) \tag{3.8}$$

and

$$u_{1m} \longrightarrow u_1,\ v_{1m} \longrightarrow v_1 \text{ strongly in } L^2(\Omega). \tag{3.9}$$

Well-known results on the solvability of nonlinear systems of ODE provide the existence of a solution to problem (3.5)–(3.7) on some interval $[0, t_m)$. Such a solution can be extended to the closed interval $[0, T]$ by using the first a priori estimate below.

A priori estimate I:

Multiply (3.5) by $u'_{m,j}(t)$, (3.6) by $v'_{m,j}(t)$, and sum for $j = 1, ..., m$. One then has

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u'_m(t)\|^2 + b_1\|\nabla u_m(t)\|^2 + \frac{b_2}{q+1}\|\nabla u_m(t)\|_{2(q+1)}^{2(q+1)}\right) + \int_\Omega |u'_m(t)|^{m+1} dx \\ = &\int_\Omega f_1(u_m, v_m) u'_m dx, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|v'_m(t)\|^2 + b_1\|\nabla v_m(t)\|^2 + \frac{b_2}{q+1}\|\nabla v_m(t)\|_{2(q+1)}^{2(q+1)}\right) + \int_\Omega |v'_m(t)|^{r+1} dx \\ = &\int_\Omega f_2(u_m, v_m) v'_m dx. \end{aligned} \tag{3.11}$$

By summing (3.10) and (3.11) and integrating the resulting identity from 0 to t, we obtain

$$\begin{aligned} &\frac{1}{2}\left(\|u'_m(t)\|^2 + \|v'_m(t)\|^2 + b_1\|\nabla u_m(t)\|^2 + b_1\|\nabla v_m(t)\|^2\right) \\ &+\frac{1}{2}\left(\frac{b_2}{q+1}\|\nabla u_m(t)\|_{2(q+1)}^{2(q+1)} + \frac{b_2}{q+1}\|\nabla v_m(t)\|_{2(q+1)}^{2(q+1)}\right) \\ &+\int_0^t\int_\Omega |u'_m(\tau)|^{m+1} dx d\tau + \int_0^t\int_\Omega |v'_m(\tau)|^{r+1} dx d\tau \\ \leq\ & C_0 + \int_0^t\int_\Omega \left(f_1(u_m, v_m) u'_m + f_2(u_m, v_m) v'_m\right) dx d\tau. \end{aligned} \tag{3.12}$$

We need to estimate the right-hand terms of (3.12). Applying (A1), Hölder inequality, Sobolev emmedding theorem and Young inequality, we obtain

$$\begin{aligned}
&\left|\int_0^t \int_\Omega f_1\left(u_m(\tau), v_m(\tau)\right) u_m'(\tau)\, dx d\tau\right| \\
\leq\ & C\int_0^t \int_\Omega \left(\left|u_m(\tau)\right|^p + \left|v_m(\tau)\right|^p + \left|u_m(\tau)\right|^{\frac{p-1}{2}} \left|v_m(\tau)\right|^{\frac{p+1}{2}}\right) \left|u_m'(\tau)\right| dx d\tau \\
\leq\ & C\int_0^t \left(\left\|u_m(\tau)\right\|_{2p}^p + \left\|v_m(\tau)\right\|_{2p}^p + \left\|u_m(\tau)\right\|_{3(p-1)}^{\frac{p-1}{2}} \left\|v_m(\tau)\right\|_{\frac{3(p+1)}{2}}^{\frac{p+1}{2}}\right) \left\|u_m'(\tau)\right\| d\tau \\
\leq\ & C\int_0^t \left(\left\|\nabla u_m(\tau)\right\|^p + \left\|\nabla v_m(\tau)\right\|^p + \left\|\nabla u_m(\tau)\right\|^{\frac{p-1}{2}} \left\|\nabla v_m(\tau)\right\|^{\frac{p+1}{2}}\right) \left\|u_m'(\tau)\right\| d\tau \\
\leq\ & C\int_0^t \left(\left\|u_m'(\tau)\right\|^2 + \left\|\nabla u_m(\tau)\right\|^p + \left\|\nabla v_m(\tau)\right\|^p + \left\|\nabla u_m(\tau)\right\|^{p-1} \left\|\nabla v_m(\tau)\right\|^{p+1}\right) d\tau. \qquad (3.13)
\end{aligned}$$

Likewise, we obtain

$$\begin{aligned}
&\left|\int_0^t \int_\Omega f_2\left(u_m(\tau), v_m(\tau)\right) v_m'(\tau)\, dx d\tau\right| \\
\leq\ & C\int_0^t \left(\left\|v_m'(\tau)\right\|^2 + \left\|\nabla u_m(\tau)\right\|^p + \left\|\nabla v_m(\tau)\right\|^p + \left\|\nabla u_m(\tau)\right\|^{p+1} \left\|\nabla v_m(\tau)\right\|^{p-1}\right) d\tau. \qquad (3.14)
\end{aligned}$$

Let $y_m(t) = 1 + \left\|u_m'(t)\right\|^2 + \left\|v_m'(t)\right\|^2 + \int_\Omega \left(P\left(\left|\nabla u_m(t)\right|^2\right) + P\left(\left|\nabla v_m(t)\right|^2\right)\right) dx$. Then, it follows from (3.12)-(3.14) that

$$y_m(t) + 2\int_0^t \int_\Omega \left|u_m'(\tau)\right|^{m+1} dx d\tau + 2\int_0^t \int_\Omega \left|v_m'(\tau)\right|^{r+1} dx d\tau \leq C_0 + C\int_0^t y_m(\tau)^p\, d\tau. \qquad (3.15)$$

Particularly, we have

$$y_m(t) \leq C_0 + C\int_0^t y_m(\tau)^p\, d\tau.$$

Using a Gronwall type inequality, we can state that

$$y_m(t) \leq \left[C_0 - (p-1)\, Ct\right]^{-\frac{1}{p-1}}. \qquad (3.16)$$

Thus, we deduce from (3.16) that there exists a time $T > 0$ such that

$$y_m(t) \leq C_1,\ \forall t \in [0, T], \qquad (3.17)$$

where C_1 is a positive constant independent of m.

Combining (3.15) and (3.17), we easily have

$$\int_0^t \int_\Omega \left|u_m'(\tau)\right|^{m+1} dx d\tau + 2\int_0^t \int_\Omega \left|v_m'(\tau)\right|^{r+1} dx d\tau \leq C_2,\ \forall t \in [0, T]. \qquad (3.18)$$

It follows of (3.17) and (3.18) that we have

$$u_m,\ v_m \text{ are bounded in } L^\infty\left([0,T];W_0^{1,2(q+1)}(\Omega)\right), \tag{3.19}$$

$$u_m',\ v_m' \text{ are bounded in } L^\infty\left([0,T];L^2(\Omega)\right), \tag{3.20}$$

$$u_m' \text{ is bounded in } L^2\left([0,T];L^{m+1}(\Omega)\right), \tag{3.21}$$

$$v_m' \text{ is bounded in } L^2\left([0,T];L^{r+1}(\Omega)\right),$$

and once A is a bounded operator from $W_0^{1,2(q+1)}(\Omega) \longrightarrow \left(W_0^{1,2(q+1)}(\Omega)\right)'$, defined by $A\omega = \operatorname{div}\left(\rho\left(|\nabla\omega|^2\right)\nabla\omega\right)$. It follows from (3.19) that

$$A(u_m),\ A(v_m) \text{ are bounded in } L^\infty\left([0,T];\left(W_0^{1,2(q+1)}(\Omega)\right)'\right). \tag{3.22}$$

Using a similar priori estimate II to [19], we obtain $u,v \in L^\infty\left([0,T];W_0^{1,2(q+1)}(\Omega)\cap L^{p+1}(\Omega)\right)$, $u_t \in L^\infty\left([0,T];L^2(\Omega)\right)\cap L^{m+1}(\Omega\times(0,T))$ and $v_t \in L^\infty\left([0,T];L^2(\Omega)\right)\cap L^{r+1}(\Omega\times(0,T))$. By use of a well-known result (Lemma 8.1–8.2, Lions and Magenes [10]) it follows that $u,v \in C_w\left([0,T];W_0^{1,2(q+1)}(\Omega)\cap L^{p+1}(\Omega)\right)$, $u_t \in C_w\left([0,T];L^2(\Omega)\right)\cap L^{m+1}(\Omega\times(0,T))$ and $v_t \in C_w\left([0,T];L^2(\Omega)\right)\cap L^{r+1}(\Omega\times(0,T))$. By appealing to Lemma 2.11 in [16] we obtain the regularity. The proof of Theorem 3.1 is completed. □

4. Global existence and decay of solution

In this section, we discuss the global existence and decay of the solution for problem (1.1). In order to do so, let us first introduce the functionals

$$J(t) = J(u(t),v(t)) = \frac{1}{2}\int_\Omega \left(P\left(|\nabla u|^2\right) + P\left(|\nabla v|^2\right)\right)dx - \int_\Omega F(u,v)\,dx, \tag{4.1}$$

and

$$I(t) = I(u(t),v(t)) = \int_\Omega \left(P\left(|\nabla u|^2\right) + P\left(|\nabla v|^2\right)\right)dx - (p+1)\int_\Omega F(u,v)\,dx. \tag{4.2}$$

The energy functional $E(t) = E(t,u(t),v(t))$ associated to problem (1.1) is

$$E(t) = \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{1}{2}\int_\Omega \left(P\left(|\nabla u|^2\right) + P\left(|\nabla v|^2\right)\right)dx - \int_\Omega F(u,v)\,dx. \tag{4.3}$$

where $P(s) = \int_0^s \rho(\xi)\,d\xi, \quad s \geq 0.$

We also define

$$W = \left\{(u,v) : (u,v) \in W_0^{1,2(q+1)}(\Omega) \times W_0^{1,2(q+1)}(\Omega),\ I(u,v) > 0\ \right\} \cup \{(0,0)\}. \tag{4.4}$$

The next lemma shows that our energy functional (4.3) is a nonincreasing function along the solutions of (1.1).

Lemma 4.1 *$E(t)$ is a nonincreasing function for $t \geq 0$ and*

$$\frac{d}{dt}E(t) = -\|u_t(t)\|_{m+1}^{m+1} - \|v_t(t)\|_{r+1}^{r+1}. \tag{4.5}$$

Proof Multiplying the first equation of (1.1) by u_t, and the second equation by v_t, integrating over Ω using integrating by parts and summing up to the product results, we obtain

$$E(t) - E(0) = -\int_0^t \left(\|u_\tau(\tau)\|_{m+1}^{m+1} + \|v_\tau(\tau)\|_{r+1}^{r+1}\right) d\tau \text{ for } t \geq 0. \tag{4.6}$$

□

Lemma 4.2 *Suppose that*

$$\begin{cases} p \geq 3, & \text{if } n = 1, 2, \\ p = 3, & \text{if } n = 3 \end{cases} \tag{4.7}$$

holds. If $(u_0, v_0) \in W$ and $(u_1, v_1) \in L^2(\Omega)$ such that

$$\beta = c_1 C_*^{p+1}(p+1)\left(\frac{2(p+1)}{b_1(p-1)}E(0)\right)^{\frac{p-1}{2}} < 1, \tag{4.8}$$

then $(u, v) \in W$ for each $t \geq 0$.

Proof Since $I(0) > 0$, then by continuity, there exists $T_m < T$, such that

$$I(t) > 0, \quad \forall t \in [0, T_m],$$

which implies that for all $t \in [0, T_m]$,

$$\begin{aligned} J(t) &= \frac{p-1}{2(p+1)}\int_\Omega \left(P\left(|\nabla u|^2\right) + P\left(|\nabla v|^2\right)\right) dx + \frac{1}{p+1}I(t) \\ &\geq \frac{p-1}{2(p+1)}\int_\Omega \left(P\left(|\nabla u|^2\right) + P\left(|\nabla v|^2\right)\right) dx \\ &= \frac{p-1}{2(p+1)}\left[b_1\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + \frac{b_2}{q+1}\left(\|\nabla u\|_{2(q+1)}^{2(q+1)} + \|\nabla v\|_{2(q+1)}^{2(q+1)}\right)\right]. \end{aligned} \tag{4.9}$$

Hence, we get

$$\begin{aligned} \|\nabla u\|^2 + \|\nabla v\|^2 &\leq \frac{2(p+1)}{b_1(p-1)}J(t) \\ &\leq \frac{2(p+1)}{b_1(p-1)}E(t) \\ &\leq \frac{2(p+1)}{b_1(p-1)}E(0). \end{aligned} \tag{4.10}$$

By recalling (1.6) and (4.8), we have

$$\begin{aligned} c_1(p+1)\|u\|_{p+1}^{p+1} &\leq c_1 C_*^{p+1}(p+1)\|\nabla u\|^{p+1} \\ &\leq c_1 C_*^{p+1}(p+1)\|\nabla u\|^{p-1}\|\nabla u\|^2 \\ &\leq c_1 C_*^{p+1}(p+1)\left(\frac{2(p+1)}{b_1(p-1)}E(0)\right)^{\frac{p-1}{2}}\|\nabla u\|^2 \\ &< \|\nabla u\|^2 \text{ on } t\in[0,T_m]. \end{aligned} \tag{4.11}$$

Similarly, we get

$$c_1(p+1)\|v\|_{p+1}^{p+1} < \|\nabla v\|^2 \text{ on } t\in[0,T_m].$$

Therefore, by using (4.2), we get $I(t)>0$ for all $t\in[0,T_m]$. By repeating the procedure, T_m is extended to T. The proof of Lemma 4.2 is completed.

Also, the following inequality can be written:

$$\int_\Omega \left(P\left(|\nabla u|^2\right)+P\left(|\nabla v|^2\right)\right)dx \leq \frac{1}{1-c_1 C_*^{p+1}(p+1)\left(\frac{2(p+1)}{b_1(p-1)}E(0)\right)^{\frac{p-1}{2}}} I(t). \tag{4.12}$$

□

Theorem 4.3 *Suppose that (4.7) holds. If* $(u_0,v_0)\in W$ *satisfying (4.8). Then the solution of problem (1.1) is global.*

Proof It is sufficient to show that $\int_\Omega \left(P\left(|\nabla u|^2\right)+P\left(|\nabla v|^2\right)\right)dx+\|u_t\|^2+\|v_t\|^2$ is bounded independently of t. To achieve this we use (4.4) and (4.6) to obtain

$$\begin{aligned} E(0) &\geq E(t)=\frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)+\frac{1}{2}\int_\Omega\left(P\left(|\nabla u|^2\right)+P\left(|\nabla v|^2\right)\right)dx-\int_\Omega F(u,v)\,dx \\ &= \frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)+\frac{p-1}{2(p+1)}\int_\Omega\left(P\left(|\nabla u|^2\right)+P\left(|\nabla v|^2\right)\right)dx+\frac{1}{p+1}I(t) \\ &\geq \frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)+\frac{p-1}{2(p+1)}\int_\Omega\left(P\left(|\nabla u|^2\right)+P\left(|\nabla v|^2\right)\right)dx \end{aligned}$$

since $I(t)\geq 0$. Therefore

$$\left(\|u_t\|^2+\|v_t\|^2\right)+\int_\Omega\left(P\left(|\nabla u|^2\right)+P\left(|\nabla v|^2\right)\right)dx \leq CE(0),$$

where $C=\max\left\{2,\frac{2(p+1)}{p-1}\right\}$. Then by Theorem 3.1, we have the global existence result. □

Theorem 4.4 *Suppose that (A1), (1.6) and (4.8) hold, and further* $(u_0,v_0)\in W$. *Thus, we have the following decay estimates:*

$$E(t)\leq\begin{cases} E(0)e^{-w_1[t-1]^+}, & \text{if } m=r=1, \\ \left(E(0)^{-\alpha}+C_9^{-1}\alpha[t-1]^+\right)^{-\frac{1}{\alpha}}, & \text{if } m,r>1, \end{cases}$$

where w_1, α and C_9 are positive constants which will be defined later.

Proof By integrating

$$\frac{d}{dt}E(t) = -\|u_t(t)\|_{m+1}^{m+1} - \|v_t(t)\|_{r+1}^{r+1}, \tag{4.13}$$

over $[t, t+1]$, we have

$$\begin{aligned} E(t) - E(t+1) &= \int_t^{t+1} \left(\|u_\tau(\tau)\|_{m+1}^{m+1} + \|v_\tau(\tau)\|_{r+1}^{r+1} \right) d\tau \\ &= D_1^{m+1}(t) + D_2^{r+1}(t). \end{aligned} \tag{4.14}$$

By virtue of (4.14) and Hölder inequality, we observe that

$$\int_t^{t+1} \int_\Omega |u_t|^2 \, dxdt \le |\Omega|^{\frac{m-1}{m+1}} D_1^2(t) = CD_1^2(t). \tag{4.15}$$

Similarly, we get

$$\int_t^{t+1} \int_\Omega |v_t|^2 \, dxdt \le |\Omega|^{\frac{r-1}{r+1}} D_2^2(t) = CD_2^2(t). \tag{4.16}$$

Hence, from (4.15), there exist $t_1 \in \left[t, t+\frac{1}{4}\right]$ and $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$\|u_t(t_i)\| \le CD_1(t), \; i = 1, 2 \tag{4.17}$$

and

$$\|v_t(t_i)\| \le CD_2(t), \; i = 1, 2. \tag{4.18}$$

Multiplying the first equation of (1.1) by u, the second equation by v, and integrating the result over $\Omega \times [t_1, t_2]$, we get

$$\begin{aligned} \int_{t_1}^{t_2} I(t)\, dt &= -\int_{t_1}^{t_2} \int_\Omega [uu_{tt} + vv_{tt}]\, dxdt \\ &\quad - \int_{t_1}^{t_2} \int_\Omega |u_t|^{m-1} u_t u dxdt - \int_{t_1}^{t_2} \int_\Omega |v_t|^{r-1} v_t v dxdt. \end{aligned} \tag{4.19}$$

Integrating by parts and Cauchy-Schwarz inequality in the first term of the right-hand side of (4.19), we obtain

$$\begin{aligned} \int_{t_1}^{t_2} I(t)\, dt &\le \|u_t(t_1)\| \|u(t_1)\| + \|u_t(t_2)\| \|u(t_2)\| \\ &\quad + \|v_t(t_1)\| \|v(t_1)\| + \|v_t(t_2)\| \|v(t_2)\| \\ &\quad + \int_{t_1}^{t_2} \|u_t(t)\|^2 dt + \int_{t_1}^{t_2} \|v_t(t)\|^2 dt \\ &\quad - \int_{t_1}^{t_2} \int_\Omega |u_t|^{m-1} u_t u dxdt - \int_{t_1}^{t_2} \int_\Omega |v_t|^{r-1} v_t v dxdt. \end{aligned} \tag{4.20}$$

Now our goal is to estimate the last two terms in the right-hand side of inequality (4.20). By using the Hölder inequality, we obtain

$$\int_{t_1}^{t_2}\int_{\Omega}|u_t|^{m-1}u_t u dxdt \le \int_{t_1}^{t_2}\|u_t(t)\|_{m+1}^m\|u(t)\|_{m+1}dt \tag{4.21}$$

and

$$\int_{t_1}^{t_2}\int_{\Omega}|v_t|^{r-1}v_t v dxdt \le \int_{t_1}^{t_2}\|v_t(t)\|_{r+1}^r\|v(t)\|_{r+1}dt. \tag{4.22}$$

By applying the Sobolev-Poincare inequality and (4.10), we find

$$\begin{aligned}\int_{t_1}^{t_2}\|u_t\|_{m+1}^m\|u\|_{m+1}dt &\le C_*\int_{t_1}^{t_2}\|u_t\|_{m+1}^m\|\nabla u\|dt\\ &\le C_*\left(\frac{2(p+1)}{b_1(p-1)}\right)^{\frac12}\int_{t_1}^{t_2}\|u_t\|_{m+1}^m E^{\frac12}(s)dt\\ &\le C_*\left(\frac{2(p+1)}{b_1(p-1)}\right)^{\frac12}\sup_{t_1\le s\le t_2}E^{\frac12}(s)\int_{t_1}^{t_2}\|u_t\|_{m+1}^m dt\\ &= C_*\left(\frac{2(p+1)}{b_1(p-1)}\right)^{\frac12}\sup_{t_1\le s\le t_2}E^{\frac12}(s)D_1^m(t). \end{aligned}\tag{4.23}$$

From (4.10), (4.17) and the Sobolev-Poincare inequality, we have

$$\|u_t(t_i)\|\|u(t_i)\|\le C_1D_1(t)\sup_{t_1\le s\le t_2}E^{\frac12}(s), \tag{4.24}$$

where $C_1=2C_*\sqrt{\frac{2(p+1)}{b_1(p-1)}C}$. Similarly, we get

$$\int_{t_1}^{t_2}\|v_t(t)\|_{r+1}^r\|v(t)\|_{r+1}dt\le C_*\left(\frac{2(p+1)}{b_1(p-1)}\right)^{\frac12}\sup_{t_1\le s\le t_2}E^{\frac12}(s)D_2^r(t), \tag{4.25}$$

$$\|v_t(t_i)\|\|v(t_i)\|\le C_2D_2(t)\sup_{t_1\le s\le t_2}E^{\frac12}(s), \tag{4.26}$$

where $C_2=2C_*'\sqrt{\frac{2(p+1)}{b_1(p-1)}C}$. Then by (4.23)–(4.26) we have

$$\begin{aligned}\int_{t_1}^{t_2}I(t)dt &\le C_3\left\{\sup_{t_1\le s\le t_2}E^{\frac12}(s)(D_1(t)+D_2(t))+D_1^2(t)+D_2^2(t)\right.\\ &\left.+C_*\sqrt{\frac{2(p+1)}{b_1(p-1)}}\sup_{t_1\le s\le t_2}E^{\frac12}(s)(D_1^m(t)+D_2^r(t))\right\}. \end{aligned}\tag{4.27}$$

On the other hand, from (4.12) we obtain

$$E(t)\le\frac12\left(\|u_t\|^2+\|v_t\|^2\right)+C_4I(t), \tag{4.28}$$

where $C_4 = \frac{p-1}{2(p+1)\left[1-c_1 C_*^{p+1}(p+1)\left(\frac{2(p+1)}{b_1(p-1)}E(0)\right)^{\frac{p-1}{2}}\right]} + \frac{1}{p+1}$.

Integrating (4.28) over $[t_1, t_2]$, we have

$$\int_{t_1}^{t_2} E(t)\,dt \leq \frac{1}{2}\int_{t_1}^{t_2}\left[\|u_t\|^2 + \|u_t\|^2\right]dt + C_4\int_{t_1}^{t_2} I(t)\,dt.$$

Then by (4.9), (4.10) and (4.27), we get

$$\begin{aligned}\int_{t_1}^{t_2} E(t)\,dt &\leq \frac{1}{2}CD_1^2(t) + \frac{1}{2}CD_2^2(t) \\ &\quad + C_4 C_3\left\{\sup_{t_1\leq s\leq t_2} E^{\frac{1}{2}}(s)\left(D_1(t)+D_2(t)\right) + D_1^2(t) + D_2^2(t)\right. \\ &\quad \left. + C_*\sqrt{\frac{2(p+1)}{b_1(p-1)}}\sup_{t_1\leq s\leq t_2} E^{\frac{1}{2}}(s)\left(D_1^m(t) + D_2^r(t)\right)\right\}. \end{aligned} \tag{4.29}$$

By integrating (4.5) over $[t, t_2]$, we obtain

$$E(t) = E(t_2) + \int_t^{t_2}\left[\|u_\tau(\tau)\|_{m+1}^{m+1} + \|v_\tau(\tau)\|_{r+1}^{r+1}\right]d\tau. \tag{4.30}$$

Therefore, since $t_2 - t_1 \geq \frac{1}{2}$, we conclude that

$$\int_{t_1}^{t_2} E(t)\,dt \geq (t_2 - t_1)E(t_2) \geq \frac{1}{2}E(t_2).$$

That is,

$$E(t_2) \leq 2\int_{t_1}^{t_2} E(t)\,dt. \tag{4.31}$$

Consequently, exploiting (4.14), (4.29), (4.30), and (4.31), and since $t_1, t_2 \in [t, t+1]$, we get

$$\begin{aligned} E(t) &\leq 2\int_{t_1}^{t_2} E(t)\,dt + \int_t^{t+1}\left(\|u_\tau(\tau)\|_{m+1}^{m+1} + \|v_\tau(\tau)\|_{r+1}^{r+1}\right)d\tau \\ &= 2\int_{t_1}^{t_2} E(t)\,dt + D_1^{m+1}(t) + D_2^{r+1}(t). \end{aligned} \tag{4.32}$$

Then, by (4.29), we have

$$\begin{aligned} E(t) &\leq \left(\frac{1}{2}C + C_4 C\right)\left(D_1^2(t) + D_2^2(t)\right) + D_1^{m+1}(t) + D_2^{r+1}(t) \\ &\quad + C_5\left[D_1(t) + D_2(t) + D_1^m(t) + D_2^r(t)\right]E^{\frac{1}{2}}(t). \end{aligned}$$

Hence, by Young inequality, we obtain

$$E(t) \leq C_6\left[D_1^2(t) + D_2^2(t) + D_1^{m+1}(t) + D_2^{r+1}(t) + D_1^{2m}(t) + D_2^{2r}(t)\right]. \tag{4.33}$$

Case 1: When $m = r = 1,$ from (4.33), we obtain

$$E(t) \leq 3C_6 \left[D_1^2(t) + D_2^2(t)\right] = 3C_6 \left[E(t) - E(t+1)\right].$$

By Lemma 2.2, we get

$$E(t) \leq E(0)\, e^{-w_1[t-1]^+},$$

where $w_1 = \ln \frac{3C_6}{3C_6-1}$.

Case 2: When $m, r > 1,$ from (4.33), we obtain

$$\begin{aligned} E(t) &\leq C_6 D_1^2(t)\left(1 + D_1^{m-1}(t) + D_1^{2(m-1)}(t)\right) + C_6 D_2^2(t)\left(1 + D_2^{r-1}(t) + D_2^{2(r-1)}(t)\right) \\ &\leq C_6\left(1 + D_1^{m-1}(t) + D_1^{2(m-1)}(t) + D_2^{r-1}(t) + D_2^{2(r-1)}(t)\right)\left(D_1^2(t) + D_2^2(t)\right). \end{aligned}$$

Then since $E(t) \leq E(0),\ \forall t \geq 0,$ we see from (4.14)

$$\begin{aligned} E(t) &\leq C_6\left(1 + E^{\frac{m-1}{m+1}}(0) + E^{\frac{2(m-1)}{m+1}}(0) + E^{\frac{r-1}{r+1}}(0) + E^{\frac{2(r-1)}{r+1}}(0)\right)\left(D_1^2(t) + D_2^2(t)\right) \\ &\leq C_7\left(D_1^2(t) + D_2^2(t)\right), \quad t \geq 0. \end{aligned}$$

Then we obtain

$$\begin{aligned} E(t)^{1+\max\left\{\frac{m-1}{2},\frac{r-1}{2}\right\}} &\leq \left[C_7\left(D_1^2(t) + D_2^2(t)\right)\right]^{1+\max\left\{\frac{m-1}{2},\frac{r-1}{2}\right\}} \\ &\leq C_8\left(D_1^{\max\{m+1,r+1\}}(t) + D_2^{\max\{m+1,r+1\}}(t)\right). \end{aligned} \tag{4.34}$$

We set $\alpha = \max\left\{\frac{m-1}{2}, \frac{r-1}{2}\right\}$; then (4.34) is equal to

$$\begin{aligned} E(t)^{1+\alpha} &\leq C_8\left(D_1^{m+1}(t)\, D_1^{2\alpha-m+1}(t) + D_2^{r+1}(t)\, D_2^{2\alpha-r+1}(t)\right) \\ &\leq C_8\left(D_1^{m+1}(t)\, E^{\frac{2\alpha-m+1}{m+1}}(0) + D_2^{r+1}(t)\, E^{\frac{2\alpha-r+1}{r+1}}(0)\right) \\ &\leq C_9\left(D_1^{m+1}(t) + D_2^{r+1}(t)\right) \\ &= C_9\left[E(t) - E(t+1)\right], \end{aligned} \tag{4.35}$$

where $C_9 = C_8 \max\left\{E^{\frac{2\alpha-m+1}{m+1}}(0), E^{\frac{2\alpha-r+1}{r+1}}(0)\right\}$. Thus, from (4.35) and Lemma 2.2, we have

$$E(t) \leq \left(E(0)^{-\alpha} + C_9^{-1}\alpha\,[t-1]^+\right)^{-\frac{1}{\alpha}}.$$

The proof of Theorem 4.2 is completed. □

5. Blow up of solution

In this section, we deal with the blow up of the solution of problem (1.1).

5.1. Case 1: $m = r = 1$

We consider problem (1.1) with $m = r = 1$.

$$\begin{cases} u_{tt} + u_t = \operatorname{div}\left(\rho\left(|\nabla u|^2\right)\nabla u\right) + f_1(u,v), & (x,t) \in \Omega \times (0,T), \\ v_{tt} + v_t = \operatorname{div}\left(\rho\left(|\nabla v|^2\right)\nabla v\right) + f_2(u,v), & (x,t) \in \Omega \times (0,T). \end{cases} \tag{5.1}$$

Definition 5.1 *A solution (u, v) of (1.1) with $m = r = 1$ is called blow up if there exists a finite time T^* such that*

$$\lim_{t \longrightarrow T^{*-}} \left\{ \int_\Omega \left(u^2 + v^2\right) dx + \int_0^t \int_\Omega \left(u^2 + v^2\right) dxds \right\} = \infty. \tag{5.2}$$

Let

$$a(t) = \int_\Omega \left(u^2 + v^2\right) dx + \int_0^t \int_\Omega \left(u^2 + v^2\right) dxds, \ \text{for } t \geq 0. \tag{5.3}$$

Lemma 5.2 *Assume (A1), and that $\frac{q}{2} \leq \delta \leq \frac{p-1}{4}$, then we have*

$$a''(t) \geq 4(\delta+1)\int_\Omega \left(u_t^2 + v_t^2\right) dx + (-4 - 8\delta) E(0) + (4 + 8\delta)\int_0^t \left(\|u_t\|^2 + \|v_t\|^2\right) dt. \tag{5.4}$$

Proof From (5.3), we have

$$a'(t) = 2\int_\Omega (uu_t + vv_t)\, dx + \|u\|^2 + \|v\|^2. \tag{5.5}$$

By (5.1) and the divergence theorem, we get

$$\begin{aligned} a''(t) &= 2\int_\Omega \left(u_t^2 + v_t^2\right) dx + 2\int_\Omega (uu_{tt} + vv_{tt})\, dx + 2\int_\Omega (uu_t + vv_t)\, dx \\ &= 2\int_\Omega \left(u_t^2 + v_t^2\right) dx - 2\int_\Omega \left(\rho\left(|\nabla u|^2\right)|\nabla u|^2 + \rho\left(|\nabla v|^2\right)|\nabla v|^2\right) dx \\ &\quad + 2(p+1)\int_\Omega F(u,v)\, dx. \end{aligned} \tag{5.6}$$

Then from (4.6) and (5.6), we have

$$\begin{aligned} a''(t) &= 4(\delta+1)\int_\Omega \left(u_t^2 + v_t^2\right) dx + (-4 - 8\delta) E(0) + (4 + 8\delta)\int_0^t \left(\|u_t\|^2 + \|v_t\|^2\right) dt \\ &\quad + (4\delta + 2)\int_\Omega \left(P\left(|\nabla u|^2\right) + P\left(|\nabla v|^2\right)\right) dx - 2\int_\Omega \left(\rho\left(|\nabla u|^2\right)|\nabla u|^2 + \rho\left(|\nabla v|^2\right)|\nabla v|^2\right) dx \\ &\quad + (2p - 8\delta - 2)\int_\Omega F(u,v)\, dx \\ &= 4(\delta+1)\int_\Omega \left(u_t^2 + v_t^2\right) dx + (-4 - 8\delta) E(0) + (4 + 8\delta)\int_0^t \left(\|u_t\|^2 + \|v_t\|^2\right) dt \\ &\quad + 4\delta b_1\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + b_2\left(\frac{4\delta + 2}{q+1} - 2\right)\left(\|\nabla u\|_{2(q+1)}^{2(q+1)} + \|\nabla v\|_{2(q+1)}^{2(q+1)}\right) \\ &\quad + (2p - 8\delta - 2)\int_\Omega F(u,v)\, dx. \end{aligned}$$

Since $0<\delta\leq\frac{p-1}{4}$, $2p-8\delta-2\geq 0$, and $\frac{q}{2}\leq\delta$, $\frac{4\delta+2}{q+1}-2\geq 0$, consequently $\frac{q}{2}\leq\delta\leq\frac{p-1}{4}$, and we obtain (5.4). □

Lemma 5.3 *Assume (A1) and one of the following statements are satisfied:*

(i) $E(0)<0,$

(ii) $E(0)=0,$ *and* $\int_\Omega (u_0u_1+v_0v_1)\,dx>0,$

(iii) $E(0)>0,$ *and*

$$a'(0)>r_2\left[a(0)+\frac{K_1}{4(\delta+1)}\right]+\left(\|u_0\|^2+\|v_0\|^2\right) \tag{5.7}$$

holds.

Then $a'(t)>\|u_0\|^2+\|v_0\|^2$ *for* $t>t^*,$ *where* $t_0=t^*$ *is given by (5.8) in case (i) and* $t_0=0$ *in cases (ii) and (iii).*

Where K_1 *and* t^* *are defined in (5.13) and (5.8), respectively.*

Proof (i) If $E(0)<0,$ then from (5.4), we have

$$a'(t)\geq a'(0)-4(1+2\delta)E(0)\,t,\ t\geq 0.$$

Thus we get $a'(t)>\|u_0\|^2+\|v_0\|^2$ for $t>t^*,$ where

$$t^*=\max\left\{\frac{a'(0)-\left(\|u_0\|^2+\|v_0\|^2\right)}{4(1+2\delta)E(0)},0\right\}. \tag{5.8}$$

(ii) If $E(0)=0,$ and $\int_\Omega (u_0u_1+v_0v_1)\,dx>0,$ then $a''(t)\geq 0$ for $t\geq 0.$ We have $a'(t)>\|u_0\|^2+\|v_0\|^2,\ t\geq 0.$

(iii) If $E(0)>0,$ we first note that

$$2\int_0^t\int_\Omega uu_t\,dx\,dt=\|u\|^2-\|u_0\|^2. \tag{5.9}$$

By Hölder inequality and Young inequality, we have from (5.9)

$$\|u\|^2\leq\|u_0\|^2+\int_0^t\|u\|^2dt+\int_\Omega\|u_t\|^2dt. \tag{5.10}$$

Similarly,

$$\|v\|^2\leq\|v_0\|^2+\int_0^t\|v\|^2dt+\int_\Omega\|v_t\|^2dt. \tag{5.11}$$

By Hölder inequality, Young inequality and inequalities (5.10) and (5.11), we have

$$a'(t)\leq a(t)+\|u_0\|^2+\|v_0\|^2+\int_\Omega\left(u_t^2+v_t^2\right)dx+\int_0^t\left(\|u_t\|^2+\|v_t\|^2\right)dt. \tag{5.12}$$

Hence, by (5.4) and (5.12), we obtain

$$a''(t)-4(\delta+1)a'(t)+4(\delta+1)a(t)+K_1\geq 0,$$

where

$$K_1 = (4+8\delta) E(0) + 4(\delta+1)\left(\|u_0\|^2 + \|v_0\|^2\right). \tag{5.13}$$

Let

$$b(t) = a(t) + \frac{K_1}{4(\delta+1)},\ t > 0.$$

Then $b(t)$ satisfies Lemma 2.3. Consequently, we get from (5.7) $a'(t) > \left(\|u_0\|^2 + \|v_0\|^2\right),\ t > 0,$ where r_2 is given in Lemma 2.3. □

Theorem 5.4 *Assume (A1) and one of the following statements are satisfied (for $\frac{q}{2} \le \delta \le \frac{p-1}{4}$):*

(i) $E(0) < 0,$

(ii) $E(0) = 0,$ *and* $\int_\Omega (u_0 u_1 + v_0 v_1)\, dx > 0,$

(iii) $0 < E(0) < \frac{\left(a'(t_0) - \left(\|u_0\|^2 + \|v_0\|^2\right)\right)^2}{8\left[a(t_0) + (T_1 - t_0)\left(\|u_0\|^2 + \|v_0\|^2\right)\right]},$ *and (5.7) holds.*

Then the solution (u,v) blows up in finite time T^* in the sense of (5.2). In case (i),

$$T^* \le t_0 - \frac{H(t_0)}{H'(t_0)}. \tag{5.14}$$

Furthermore, if $H(t_0) < \min\left\{1, \sqrt{-\frac{a}{b}}\right\},$ we have

$$T^* \le t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{-\frac{a}{b}}}{\sqrt{-\frac{a}{b}} - H(t_0)}, \tag{5.15}$$

where

$$a = \delta^2 H^{2+\frac{2}{\delta}}(t_0)\left[\left(a'(t_0) - \|u_0\|^2 - \|v_0\|^2\right)^2 - 8E(0) H^{-\frac{1}{\delta}}(t_0)\right] > 0, \tag{5.16}$$

$$b = 8\delta^2 E(0). \tag{5.17}$$

In case (ii),

$$T^* \le t_0 - \frac{H(t_0)}{H'(t_0)}. \tag{5.18}$$

In case (iii),

$$T^* \le \frac{H(t_0)}{\sqrt{a}} \text{ or } T^* \le t_0 + 2^{\frac{3\delta+1}{2\delta}} \left(\frac{a}{b}\right)^{2+\frac{1}{\delta}} \frac{\delta}{\sqrt{a}} \left\{1 - \left[1 + \left(\frac{a}{b}\right)^{2+\frac{1}{\delta}} H(t_0)\right]^{-\frac{1}{2\delta}}\right\}, \tag{5.19}$$

where a and b are given in (5.16) and (5.17).

Proof Let

$$H(t) = \left[a(t) + (T_1 - t)\left(\|u_0\|^2 + \|v_0\|^2\right)\right]^{-\delta},\ \text{for } t \in [0, T_1], \tag{5.20}$$

where $T_1 > 0$ is a certain constant which will be specified later. Then we get

$$\begin{aligned} H'(t) &= -\delta\left[a(t) + (T_1 - t)\left(\|u_0\|^2 + \|v_0\|^2\right)\right]^{-\delta-1}\left[a'(t) - \left(\|u_0\|^2 + \|v_0\|^2\right)\right] \\ &= -\delta H^{1+\frac{1}{\delta}}(t)\left[a'(t) - \left(\|u_0\|^2 + \|v_0\|^2\right)\right], \end{aligned} \tag{5.21}$$

$$\begin{aligned} H''(t) &= -\delta H^{1+\frac{2}{\delta}}(t)\, a''(t)\left[a(t) + (T_1 - t)\left(\|u_0\|^2 + \|v_0\|^2\right)\right] \\ &\quad + \delta H^{1+\frac{2}{\delta}}(t)(1+\delta)\left[a'(t) - \left(\|u_0\|^2 + \|v_0\|^2\right)\right]^2, \end{aligned} \tag{5.22}$$

and

$$H''(t) = -\delta H^{1+\frac{2}{\delta}}(t)\, V(t), \tag{5.23}$$

where

$$V(t) = a''(t)\left[a(t) + (T_1 - t)\left(\|u_0\|^2 + \|v_0\|^2\right)\right] - (1+\delta)\left[a'(t) - \left(\|u_0\|^2 + \|v_0\|^2\right)\right]^2. \tag{5.24}$$

For simplicity of calculation, we define

$$\begin{aligned} &P_u = \textstyle\int_\Omega u^2 dx, \quad R_u = \int_\Omega u_t^2 dx, \quad Q_u = \int_0^t \|u\|^2 dt, \quad S_u = \int_0^t \|u_t\|^2 dt, \\ &P_v = \textstyle\int_\Omega v^2 dx, \quad R_v = \int_\Omega v_t^2 dx, \quad Q_v = \int_0^t \|v\|^2 dt, \quad S_v = \int_0^t \|v_t\|^2 dt. \end{aligned}$$

From (5.5), (5.9) and Hölder inequality, we get

$$\begin{aligned} a'(t) &= 2\int_\Omega (uu_t + vv_t)\, dx + \|u_0\|^2 + \|v_0\|^2 + 2\int_0^t \int_\Omega (uu_t + vv_t)\, dxdt \\ &\leq 2\left(\sqrt{R_u P_u} + \sqrt{Q_u S_u} + \sqrt{R_v P_v} + \sqrt{Q_v S_v}\right) + \|u_0\|^2 + \|v_0\|^2. \end{aligned} \tag{5.25}$$

If case (i) or (ii) holds, by (5.4) we have

$$a''(t) \geq (-4 - 8\delta)\, E(0) + 4(1+\delta)(R_u + S_u + R_v + S_v). \tag{5.26}$$

Thus, from (5.24)–(5.26) and (5.20), we obtain

$$\begin{aligned} V(t) \geq\ & [(-4 - 8\delta)\, E(0) + 4(1+\delta)(R_u + S_u + R_v + S_v)]\, H^{-\frac{1}{\delta}}(t) \\ & -4(1+\delta)\left(\sqrt{R_u P_u} + \sqrt{Q_u S_u} + \sqrt{R_v P_v} + \sqrt{Q_v S_v}\right)^2. \end{aligned}$$

From (5.3),

$$a(t) = \int_\Omega \left(u^2 + v^2\right) dx + \int_0^t \int_\Omega \left(u^2 + v^2\right) dxds = P_u + P_v + Q_u + Q_v$$

and (5.20), we get

$$V(t) \geq (-4 - 8\delta)\, E(0)\, H^{-\frac{1}{\delta}}(t) + 4(1+\delta)\left[(R_u + S_u + R_v + S_v)(T_1 - t)\left(\|u_0\|^2 + \|v_0\|^2\right) + \Theta(t)\right],$$

where

$$\Theta(t) = (R_u + S_u + R_v + S_v)(P_u + Q_u + P_v + Q_v) - \left(\sqrt{R_u P_u} + \sqrt{Q_u S_u} + \sqrt{R_v P_v} + \sqrt{Q_v S_v}\right)^2 .$$

By the Schwarz inequality, and $\Theta(t)$ being nonnegative, we have

$$V(t) \geq (-4 - 8\delta) E(0) H^{-\frac{1}{\delta}}(t), \ t \geq t_0. \tag{5.27}$$

Therefore, by (5.23) and (5.27), we get

$$H''(t) \leq 4\delta (1 + 2\delta) E(0) H^{1+\frac{1}{\delta}}(t), \ t \geq t_0. \tag{5.28}$$

By Lemma 5.2, we know that $H'(t) < 0$ for $t \geq t_0$. Multiplying (5.28) by $H'(t)$ and integrating it from t_0 to $t,$ we get

$$H'^2(t) \geq a + bH^{2+\frac{1}{\delta}}(t)$$

for $t \geq t_0,$ where a, b are defined in (5.16) and (5.17) respectively.

If case (iii) holds, by the steps of case (i), we get $a > 0$ if and only if

$$E(0) < \frac{\left(a'(t_0) - \left(\|u_0\|^2 + \|v_0\|^2\right)\right)^2}{8\left[a(t_0) + (T_1 - t_0)\left(\|u_0\|^2 + \|v_0\|^2\right)\right]}.$$

Then by Lemma 2.4, there exists a finite time T^* such that $\lim_{t \longrightarrow T^{*-}} H(t) = 0$ and the upper bound of T^* is estimated according to the sign of $E(0)$. This means that (5.2) holds. □

5.2. Case 2: $1 < m, r < p$

We consider problem (1.1) with $1 < m, r < p$ and $q = 0$.

Theorem 5.5 *Suppose that (A1), $1 < m, r < p$ and $q = 0$ holds, and further assume that $E(0) < 0$; then the solution of (1.1) blows up at a finite time T^*,*

$$0 < T^* \leq \frac{z^{1-r}(0)}{c(1-r)},$$

where $z(0) = k_1\left(-E(0)^{1-\alpha_1}\right) + \int_\Omega (u_0 u_1 + v_0 v_1)\,dx,$ here $k_1,$ α_1 and r are positive constants.

Proof can be done by following the arguments in [3, 22].

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