BBM equation with non-constant coefficients

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Abstract: In this article, a model for the propagation of long waves over an uneven bottom is considered. We provide both theoretical and numerical results for this model. We also discuss the changes which occur in a solitary wave solution of the BBM equation as it travels through a channel of decreasing depth.

Key words: Solitary waves, BBM equation, uneven bottom

1. Introduction

Attention is given to the propagation of water waves for uneven bottoms with an incompressible, inviscid, irrotational fluid in a constant gravitational field. Our study investigates the motion of the free surface of fluid under the assumption of small amplitude shallow water. The first study was conducted by Russell in 1834, and he discovered solitary waves, which are stable and do not disperse with time [7, 13, 17]. Russell's observations and experiments gave a motivation to find a mathematical formulation of such waves. Scientists such as Airy, Stokes, Boussinesq and Rayleigh investigated such waves in an attempt to understand this phenomenon. Boussinesq and Rayleigh separately got approximate descriptions of the solitary wave. Boussinesq derived non-linear one-dimensional evolution equation (Boussinesq approximation). Finally Korteweg and de Vries derived a non-linear evolution equation in dimensional form

$$u_t + c_0 u_x + \frac{3}{2}\frac{c_0}{h_0}uu_x + \frac{1}{6}c_0 h_0^2 u_{xxx} = 0, \tag{1.1}$$

which approximates the propagation of unidirectional, small amplitude long waves in non-linear dispersive media. Here h_0 denotes the undisturbed water depth for flat bottom, $c_0 = \sqrt{gh_0}$ is the long wave speed, g is the gravity term, and $u(x,t)$ denotes the elevation of fluid surface from the rest position at a point x and at time t. Korteweg and de Vries found that Russell's solitary wave is a solution of the KdV equation when the waves are long compared to the undisturbed depth of the water, and the average amplitude of the waves is small when compared to h_0. As explained in [3, 4, 8, 12, 20], the KdV equation has certain theoretical difficulties concerning the dispersion relation and the group velocity. There are several noticeable improvements in order to avoid this situation. Benjamin, Bona and Mahony [3] proposed another model,

$$u_t + c_0 u_x + \frac{3}{2}\frac{c_0}{h_0}uu_x - \frac{1}{6}h_0^2 u_{xxt} = 0, \tag{1.2}$$

known as a BBM equation and equally well justified as a model of the same phenomena which replaces the third-order derivative (u_{xxx}) in KdV equation (1.1) by a mixed derivative, $-u_{xxt}$.

Our main work is focused on the solitary waves with long wave length and small amplitude in a channel of slowly varying depth. Many workers have been interested in models which represent the changes that occur in a solitary wave as it travels over an uneven bottom. We shall consider a mathematical model with the following assumptions regarding the physical situation. We shall assume that the solitary wave is long with respect to the depth, that the amplitude of the wave is small, and that the depth of the fluid is slowly varying. These assumptions are indicated in Figure 1.
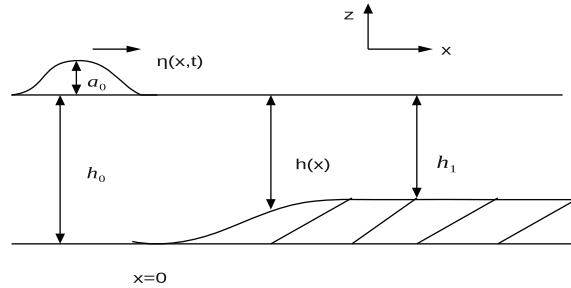

Figure 1. Geometry of our physical system, where $\eta(x,t)$ is the dimensional elevation.

The equation with uneven bottom is not derived formally, but is a reasonable model if the gradient in the bottom profile is small. The equation is

$$u_t + c(x)u_x + \frac{3}{2}\frac{c_0}{h_0}uu_x - \frac{1}{6}h_0^2 u_{xxt} = 0, \tag{1.3}$$

where $c(x) = \sqrt{gh(x)}$, and c_0 is a reference velocity based on a reference value of $h(x)$. The KdV equation with a variable coefficient in the transport term has been indicated to be a good model for surface waves over a gradually varying width or bottom topography in [6]. Since the BBM equation is asymptotically equivalent to the KdV equation, we expect the equation (1.3) to be a valid model in this situation as well. If we use the standard non-dimensional variables $\hat{x} = \frac{x}{h_0}, \hat{u} = \frac{u}{c_0}, \hat{t} = \frac{t}{h_0/c_0}$ and without loss of generality we assume that all the coefficients in this equation (1.3) are one, then our model equation becomes

$$u_t + c(x)u_x + uu_x - u_{xxt} = 0. \tag{1.4}$$

Existence and uniqueness results for the solution of BBM the equation (1.4) in the Sobolev space $H^1(\mathbb{R})$ are established in Section 2. Global well-posedness of the initial value problem for the BBM equation is also established in Section 2. Finally in Section 3 we study the evolution of solitary waves over an uneven bottom numerically. As the solitary wave runs over an uneven bottom, the wave form changes, and the amplitude and total energy are not always constant with time. Amplitude variations of the solitary waves are tabulated with various initial amplitudes. For positive solitary waves, it appears that the system experiences a loss of energy for decreasing height bottom profile and a gain of energy for increasing height bottom profile. For negative solitary waves, the situation is reversed.

2. Local existence of solutions

In this section we consider the long wave equation (1.4) on the unbounded domain $\mathbb{R}$ with the initial condition

$$u(x,0) \quad = \quad u_0(x).$$

Before we present well-posedness theory, we introduce some notations. The space $L^p = L^p(\mathbb{R})$, $1 \leq p < \infty$ is the set of all measurable real-valued functions of a real variable f whose p^{th} powers are integrable over $\mathbb{R}$. For $p = 2$, the norm is denoted by $\|f\|_{L^2}$ and defined by

$$\|f\|_{L^2(\mathbb{R})}^2 = \int\limits_{\mathbb{R}} |f(x)|^2 \, dx.$$

The Sobolev space $H^s(\mathbb{R})$ is the subspace of $L^2(\mathbb{R})$ consisting of functions such that

$$\|f\|_{H^s(\mathbb{R})}^2 = \int\limits_{-\infty}^{\infty} \left(1 + |\xi|^2\right)^s |\hat{f}(\xi)|^2 \, d\xi < \infty,$$

where $\hat{f}(\xi)$ is the Fourier transform of f and $s \geq 0$ (see [18]). The essential supremum of the absolute value of a function $|f|$ is usually denoted $\|f\|_{\infty}$, and this serves as the norm for L^∞-space.

$$\|f\|_{L^\infty} = \|f\|_{\infty} = \inf\left\{\alpha; |f(x)| \leq \alpha \ a.e \ on \ X\right\}.$$

The convolution of two functions is defined as

$$g * f(x) = \int_{-\infty}^{\infty} g(y)f(x - y)dy.$$

We define the space $C([0, T]; X)$, for any Banach space X (for instance H^s), as the Banach space of continuous maps $u(x, t) : [0, T] \to X$ with the norm

$$\|u\|_{C([0,T];X)} = \sup_{0 \leq t \leq T} \|u(\cdot, t)\|_X.$$

Finally, we define $C_b([0, \infty); H^s)$ to be the space of all functions $u(x, t)$ such that $u(\cdot, t)$ is a continuous and bounded function $t \to H^s$ for $t \in [0, \infty)$. For results on Sobolev spaces, the reader may consult [1, 5, 9].

In the following, we investigate well-posedness of (1.4). First, we state a local well-posedness result, and then look at two possibilities of extending the local existence to global-in-time existence. The local result is as follows.

Theorem 2.1 *If $u_0 = u(x, 0) \in H^1(\mathbb{R})$, $c(x) \in H^1(\mathbb{R})$ then there exists a t_0 which depends on u_0 and $c(x)$ such that the equation $u_t + c(x)u_x + uu_x - u_{xxt} = 0$, $x \in \mathbb{R}$ has a unique solution $u \in C([0, t_0]; H^1(\mathbb{R}))$.*

Proof The proof is based on the works of Albert and Bona [2] and Benjamin et al. [3]. They have already presented well-posedness theory for the BBM equation, but with constant $c(x)$. While we are interested in nonconstant coefficients, we provide the proof for this theorem for the interested reader. We rewrite the BBM equation (1.4) as

$$(1 - \partial_x^2)u_t = -[c(x)u_x + uu_x]. \tag{2.1}$$

The formal solution of (2.1) is

$$u_t = -\int\limits_{-\infty}^{\infty} G(x - \xi)\left(c(\xi)u_\xi + uu_\xi\right) d\xi,$$

where $G(x) = \frac{1}{2}e^{-|x|}$ is Green's function of the operator $(1-\partial_x^2)$. It is easy to check that $1-\partial_x^2 : H^1 \subset L^2 \to L^2$ is self-adjoint with respect to the L^2-inner product. Using integration by parts we rewrite the above equation as

$$u_t = \int\limits_{-\infty}^{\infty} K(x-\xi)\left(c(\xi)u + \tfrac{1}{2}u^2\right) d\xi + \int\limits_{-\infty}^{\infty} G(x-\xi)uc_\xi(\xi)\,d\xi,$$

where $K(x) = -sign(x)\frac{e^{-|x|}}{2}$ is the derivation of G. From the definition of Fourier transform, $\hat{G}(\xi) = \frac{1}{1+\xi^2}$ and $\hat{K}(\xi) = \frac{-i\xi}{1+\xi^2}$. One may show the mapping properties $\|K*u\|_{H^1} \le \|u\|_{L^2}$ and $\|G*u\|_{H^1} = \|u\|_{H^{-1}}$. Using convolution we write u_t as

$$u_t = K*(cu + \tfrac{1}{2}u^2) + G*(c_x u). \tag{2.2}$$

If we integrate (2.2) with respect to t, then we get

$$u(x,t) = u_0(x) + \int\limits_0^t \left[K*(cu + \tfrac{1}{2}u^2) + G*(c_x u) \right] d\tau. \tag{2.3}$$

Next we use a fixed point theorem to prove the existence of the solution in the sufficiently small interval. First we consider the map

$$A(U) = u_0 + \int\limits_0^t \left[K*(cU + \tfrac{1}{2}U^2) + G*(c_x U) \right] d\tau.$$

We have to prove that A is a contraction mapping in a closed ball $B_R \subset \mathcal{C} = C([0,t_0]; H^1(\mathbb{R}))$, where sufficiently small t_0. Now consider

$$
\begin{aligned}
\|AU(t)\|_{H^1} &\le \|u_0\|_{H^1} + \int\limits_0^t \left\| K*(cU + \tfrac{1}{2}U^2) + G*(c_x U) \right\|_{H^1} d\tau \\
&\le \|u_0\|_{H^1} + \int\limits_0^t \left\| cU + \tfrac{1}{2}U^2 \right\|_{L^2} d\tau + \int\limits_0^t \|c_x U\|_{H^{-1}} d\tau.
\end{aligned}
$$

And hence

$$
\begin{aligned}
\|AU_1(t) - AU_2(t)\|_{H^1} &\le \int\limits_0^t \left\| c(U_1 - U_2) + \tfrac{1}{2}(U_1^2 - U_2^2) \right\|_{L^2} d\tau \\
&\quad + \int\limits_0^t \|c_x(U_1 - U_2)\|_{H^{-1}} d\tau \\
&\le t\,\|U_1 - U_2\|_{L^2} \left[\left\| c + \tfrac{1}{2}(U_1 + U_2) \right\|_{\infty} + \|c_x\|_{L^2} \right] \\
&\le t\,\|U_1 - U_2\|_{H^1} \left[\|c\|_{\infty} + R + \|c_x\|_{L^2} \right].
\end{aligned}
$$

Now take the supremum $t \in [0, t_0]$ on both sides. We have

$$\|AU_1(t) - AU_2(t)\|_{H^1} \quad \leq \quad t_0 \left\{ \|c\|_\infty + R + \|c_x\|_{L^2} \right\} \|U_1 - U_2\|_{\mathcal{C}},$$

where we abbreviate $\|u\|_{\mathcal{C}} = \|u\|_{C([0,t_0];H^1)}$. From the above equality it can be confirmed that the operator A is a continuous mapping of the space $\mathcal{C}$ into itself. We can show that the mapping of the ball $\|U\|_{\mathcal{C}} \leq R$ satisfies Lipschitz condition with Lipschitz constant $\Theta < 1$.

We let $\|u_0\|_{H^1} \leq d$, then

$$\begin{aligned}
\|AU\|_{\mathcal{C}} &\leq \|AU - A0\|_{\mathcal{C}} + d \\
&\leq t_0 \left\{ \|c\|_\infty + R + \|c_x\|_{L^2} \right\} \|U\|_{\mathcal{C}} + d \\
&= \Theta R + d,
\end{aligned}$$

where $\Theta = t_0 \left\{ \|c\|_\infty + R + \|c_x\|_{L^2} \right\}$. If we choose $R = 2d$ and $t_0 = \frac{0.5}{\left\{ \|c\|_\infty + R + \|c_x\|_{L^2} \right\}}$, then $\Theta = 0.5 < 1$, and

$$\|AU(t)\|_{\mathcal{C}} \leq \frac{1}{2}R + \frac{R}{2} = R.$$

This shows that A is a contraction mapping on B_R. Therefore A has a unique fixed point u in the ball B_R. $\square$

3. Extension to global existence of solutions

In 1971, Benjamin et al. [3] proved global existence result for the case $c(x) = $ constant. Now for the case $c(x) \neq$ constant, we examine the global existence solution of (1.4). We can obtain two results depending on additional assumptions on $c(x)$. In the case where $c' \in L^\infty$, we obtain existence of solution on the interval $[0, T]$. In the case where $c' \leq 0$, we obtain solution on the interval $[0, \infty)$. Let us look at the first case. If $u \in C([0, t_0]; H^1)$, then from (2.2) we observe that $u_t \in C([0, t_0]; H^1)$. Since $u \in H^1(\mathbb{R})$, we have $u_x \in L^2(\mathbb{R})$ and $uu_x \in L^2(\mathbb{R})$. If $c(x) \in H^1(\mathbb{R})$, then $cu_x \in L^2(\mathbb{R})$. Now (2.1) shows that $u_{xxt} \in L^2(\mathbb{R})$, and therefore $u_{xt} \in H^1(\mathbb{R})$. Multiplying the equation (1.4) by u, we have

$$uu_t - uu_{xxt} + uc(x)u_x + u^2 u_x = 0.$$

We integrate the above equation between $x = -R$ to $x = R$ to obtain

$$\int\limits_{-R}^{R} uu_t \, dx - \int\limits_{-R}^{R} uu_{xxt} \, dx = -\int\limits_{-R}^{R} c(x)uu_x \, dx \quad - \quad \frac{1}{3} \int\limits_{-R}^{R} \frac{\partial}{\partial x}(u^3) \, dx$$

$$\int\limits_{-R}^{R} \frac{1}{2}\frac{d}{dt}\left[u^2\right] dx - \left[uu_{xt}\right]_{-R}^{R} + \int\limits_{-R}^{R} u_x u_{xt} \, dx \quad = \quad \frac{-1}{2} \int\limits_{-R}^{R} c(x)\frac{\partial}{\partial x}(u^2) \, dx$$

$$- \frac{1}{3}\left[u^3\right]_{-R}^{R}.$$

We use the fact that functions in $H^1(\mathbb{R})$ vanish at infinity. This fact can be proved by using Riemann-Lebesgue lemma. As we know that the terms u, u_t, u_x, u_{xt} are in $C([0, t_0]; L^2)$, use of the dominated convergence theorem

shows that as $R \to \infty$, we get

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}\left(u^2 + u_x^2\right)\,dx \quad = \quad -\frac{1}{2}\int_{-\infty}^{\infty}c(x)\frac{\partial}{\partial x}(u^2)\,dx.$$

Defining $E(u) = \frac{1}{2}\int_{-\infty}^{\infty}\left(u^2 + u_x^2\right)\,dx$, and observing that $E(u) = \|u(\cdot,t)\|_{H^1}^2$, we have

$$\frac{d}{dt}E(u) \quad = \quad \int_{-\infty}^{\infty}c'(x)u^2\,dx$$

$$\leq \quad \|c'\|_{\infty}\|u\|_{L^2}^2\,,$$

and hence

$$\frac{d}{dt}\|u(\cdot,t)\|_{H^1}^2 \leq \|c'\|_{\infty}\|u(\cdot,t)\|_{H^1}^2\,.$$

Now use Gronwall's inequality to show that

$$\|u(\cdot,t)\|_{H^1}^2 \leq e^{\|c'\|_{\infty}T}\|u_0\|_{H^1}^2\,,$$

for all $t \in [0,T]$. With this a priori estimate, it is possible to use the local existence proof repeatedly in order to get global existence. Indeed, using $2e^{\|c'\|_{\infty}T}\|u_0\|_{H^1}$ instead of R in the definition of the local time interval t_0, it can be guaranteed that the final time T is reached after applying the local existence result a finite number of times with the same small time step. Since the contraction mapping theorem is used, the solution is automatically unique in the ball B_R. One can also extend the uniqueness to $C([0,T];H^1(\mathbb{R}))$ and a detailed description can be found in [2]. Thus the following theorem has been reached.

Theorem 3.1 *If $u_0 = u(x,0) \in H^1(\mathbb{R})$, $c(x) \in H^1(\mathbb{R})$ and $c'(x) \in L^{\infty}$, then the equation $u_t + c(x)u_x + uu_x - u_{xxt} = 0$, $x \in \mathbb{R}$ has a unique solution $u \in C([0,T];H^1(\mathbb{R}))$ for any $T > 0$.*

Next consideration is given to the properties of bottom profile. That is, we give attention to the changes that occur in a solitary wave as it travels up a channel of decreasing depth. If we consider the decreasing height function $h(x)$, i.e. $h'(x) \leq 0$, then we can find some interesting new global well-posedness results. For the observation of global well-posedness, we consider again the energy integral $E(u)$. With the decreasing height profile, we also have $c'(x) \leq 0$ since $c(x) = \sqrt{gh(x)}$, and $c'(x) = \frac{\sqrt{g}}{2\sqrt{h(x)}}h'(x) \leq 0$. Thus we have

$$\frac{1}{2}\frac{d}{dt}E(u) = \int_{-\infty}^{\infty}c'(x)\frac{u^2}{2}\,dx \leq 0. \tag{3.1}$$

Therefore, we have $\frac{dE}{dt} \leq 0$ if $h'(x) \leq 0$. Then, the local argument can be used to provide existence of a solution on a time interval of arbitrary length. Accordingly we conclude the following theorem from the above reasoning.

Theorem 3.2 *If $u_0 = u(x,0) \in H^1(\mathbb{R})$, $c(x) \in H^1(\mathbb{R})$ and $c'(x) \leq 0$, then the equation $u_t + c(x)u_x + uu_x - u_{xxt} = 0$, $x \in \mathbb{R}$ has a solution $u \in C_b([0,\infty); H^1(\mathbb{R}))$.*

4. Numerical results

Now we use numerical observations to illustrate the results obtained in the previous section. We use a pseudo-spectral method coupled with a 4-stage Runge-Kutta time integration scheme to discretize our given BBM equation. The spectral method approximates the solution as linear combination of continuous functions that are generally non-zero over the domain of solution. If we apply, spectral method, we have excellent error properties in the form of an exponential convergence rate [10, 11, 19].

For numerical computations, we use the problem with periodic boundary conditions on the domain $x \in [0, L]$, where $L = 200$ was found sufficient for the computations shown in this work. The problem is translated to the interval $[0, 2\pi]$ using the scaling $u(bx, t) = v(x, t)$, where $b = \frac{L}{2\pi}$ (see [15]). Then (1.4) becomes

$$
\begin{aligned}
v_t + \frac{1}{b}\left[c(bx) + v(x,t)\right]v_x - \frac{1}{b^2}v_{xxt} &= 0, \quad x \in [0, 2\pi], t > 0 &\qquad (4.1) \\
v(x,0) &= u(bx, 0), \\
v(0,t) &= v(2\pi, t) \quad \text{for } t \geq 0.
\end{aligned}
$$

We first discretize the spatial domain $[0, 2\pi]$ into N evenly spaced grid points $x_j = \frac{2\pi j}{N}$, $j = 1, \ldots, N$. The function $v(x,t)$ is then expanded in a Fourier series. We use discrete Fourier transform to compute the corresponding Fourier coefficients of v, i.e.,

$$
\mathcal{F}(v(x,t), k) = \hat{v}(k,t) = \frac{2\pi}{N} \sum_{j=1}^{N} e^{-ikx_j} v(x_j, t), \quad k = -\frac{N}{2} + 1, \ldots, \frac{N}{2},
$$

where $i = \sqrt{-1}$. Similarly grid values of $v(x_j, t)$ can be found from the Fourier coefficients by the inverse discrete Fourier transform, as follows:

$$
\mathcal{F}^{-1}(\hat{v}, x_j) = v(x_j, t) = \frac{1}{2\pi} \sum_{k=-\frac{N}{2}+1}^{N/2} e^{ikx_j} \hat{v}(k,t).
$$

And the Fourier collocation derivative function can be represented as

$$
\mathcal{F}\left(\frac{\partial^l}{\partial x^l} v(x_j, t)\right) = (ik)^l \mathcal{F}v(x_j, t).
$$

Then the equation (4.1) becomes

$$
\begin{aligned}
b^2 \hat{v}_t + b\mathcal{F}\left(c(bx)v_x\right) + b\mathcal{F}\left(vv_x\right) + k^2 \hat{v}_t &= 0, t > 0 \\
\hat{v}(k, t=0) &= \mathcal{F}(v(x,0), k), \quad t = 0,
\end{aligned}
$$

where $\mathcal{F}$ is the Fourier transform operator. The Fourier collocation approximation is defined as follows:

$$\hat{v}_t = \frac{-b}{b^2 + k^2} \mathcal{F}\left[c(bx)\mathcal{F}^{-1}\left(ik\hat{v}\right)\right] \quad - \quad \frac{ikb}{2(b^2 + k^2)}\left[\mathcal{F}\left(\left[\mathcal{F}^{-1}\left(\hat{v}\right)\right]^2, k\right)\right],$$

$$k = -\frac{N}{2} + 1, ..\frac{N}{2}, \quad t > 0,$$

$$\hat{v}_t(k, t) = 0, \quad k = \frac{N}{2}, \quad t > 0,$$

$$\hat{v}(k, t = 0) = \mathcal{F}(v(x, 0), k) = \frac{2\pi}{N}\sum_{i=1}^{N}e^{-ikx_i}v(x_i, 0),$$

$$k = -\frac{N}{2} + 1, \ldots, \frac{N}{2}, \quad t = 0.$$

This is a system of N ordinary differential equations for the discrete Fourier coefficients $\hat{v}_N(k, t)$, for $k = -\frac{N}{2} + 1, \ldots, \frac{N}{2}$. We solve the system by using a fourth order explicit Runge-Kutta scheme with time step Δt.

We use discrete L^2- norm to test the convergence of the algorithm and the numerical implementation. The L^2- norm is defined by

$$\|v\|_{N,2}^2 = \frac{1}{N}\sum_{j=1}^{N}|v(x_j)|^2.$$

The relative L^2 - error is then defined to be

$$\frac{\|v - v_N\|_{N,2}}{\|v\|_{N,2}},$$

where $v_N(x_j)$ is the approximated numerical solution and $v(x_j)$ is the exact solution at a time T, for $j = 1, 2, \ldots, N$.

4.1. Verification of algorithm

For the case $c(x) = 1$, we consider the solitary wave solution $u(x, t) = \phi(x - Ct)$, here C is the speed of propagation of the solitary wave, then the long wave equation (1.4) becomes

$$C\phi'' + (1 - C)\phi + \frac{1}{2}\phi^2 = 0.$$

We can check that the particular solution of the above is

$$\phi(x) = 3(C - 1)\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{(C - 1)}{C}}x\right). \tag{4.2}$$

We use (4.2) as the exact form of the solitary waves with various values of C, both positive and negative, and we find the relative L^2 - error for various time step $\Delta t = 0.1/(2n)$, for $n = 1, 2, 3, \ldots$ and various $N = m \times 512$ for $m = 1, 2, 3, \ldots$.

When we consider the solitary wave speed C, we have the following results which are proved by Kalisch and Nguyen [12, 14, 16]. If $C < 1$, then the waves are strictly positive progressive, that is, waves which propagate to the right in the direction of increasing values of x without changing their profile over time. If $0 < C < 1$, then there are no solitary waves and if C is negative then the waves are strictly negative progressive waves propagating to the left (in the direction of decreasing values of x).

Table 1. BBM equation; error due to temporal discretization.

N	Δt	L^2 error	Ratio
2048	0.1000	7.61e-07	-
2048	0.0500	4.74e-08	16.05
2048	0.0250	2.96e-09	16.03
2048	0.0125	1.85e-10	16.01
2048	0.0063	1.15e-11	16.01
2048	0.0031	7.20e-13	16.01
2048	0.0016	4.00e-14	16.83

In the computations shown in Table 1 we approximate the solution from $t = 0$ to $t = 10$ with $L = 200$, and $N = 2048$. The time convergence of the scheme is apparent up to $\Delta t = 0.0016$. We see that the ratio mostly lies close to 16, which corresponds to a fourth-order convergence rate. That is, halving the time step results in a 16 times reduction of the error in the four-stage Runge Kutta method. This shows that we have reached the maximum precision for this value of Fourier modes, i.e., the spatial errors are dominating. Table 2 shows the spatial convergence error rate for a calculation with the time step $\Delta t = 0.001$. Here we observe exponential convergence before reaching the limit set by the size of the time step. We find similar results for all other trials. Hence Table 1 and Table 2 show that the implementation of the numerical algorithm is correct.

Table 2. BBM equation; error due to spatial discretization.

N	Δt	L^2 error	Ratio
512	0.001	1.9e-02	-
1024	0.001	9.07e-06	2159.97
2048	0.001	2.15e-12	5273.12
4096	0.001	2.00e-14	121.34
8192	0.001	1.99e-14	1.03

4.2. Solitary waves in a channel of decreasing depth

In order to verify the theoretical results presented in Section 3, we examine how solitary waves evolve on non-uniform depth. We simulate (1.4), the BBM equation with nonconstant coefficients. This simulation was done with the Fourier-collocation method coupled with a 4-stage Runge-Kutta time integration scheme as explained above.

We suppose that the depth of the water is slowly varying, and consider the bottom depth decreasing from h_0 to a small but finite h_1. This is shown in Figure 1. Suppose the amplitude of initial wave is a_0 and the amplitude of the wave at depicted bottom h_1 is a_1. Now we run the right-going solitary wave corresponding to the solution of the equation (1.4) for positive wave speed C. Since the analysis depends heavily on the bottom profile and the maximum height of a wave is an easy thing to measure, we first checked amplitude of the solitary

wave at depicted bottom and next we checked energy integral $E(u)$. The results of the solitary wave solution and the energy integral profile are shown in Figure 2.

Figure 2 shows that the energy E is a decreasing function of t (i.e. $\frac{dE}{dt} \leq 0$), so that amplitude of the solitary wave is also decreasing when we run the solitary wave on a height decreasing bottom profile. It shows the feature one would expect, the diminishing amplitude. We also note that the wavelength is less when we compare with initial wavelength. The numerical evidence supports our theoretical result (3.1) for the right going solitary wave. For left going solitary wave high i.e. solitary wave with negative wave velocity $C < 0$ (which is not actually the physical case [12]), we find that the energy is increasing function of t which is shown in Figure 3. Moreover, the wave also grows in amplitude.

Table 3. Amplitude variations of the solitary wave solutions of the equation $u_t + c(x)u_x + uu_x - u_{xxt} = 0$ for decreasing depth and positive wave speed.

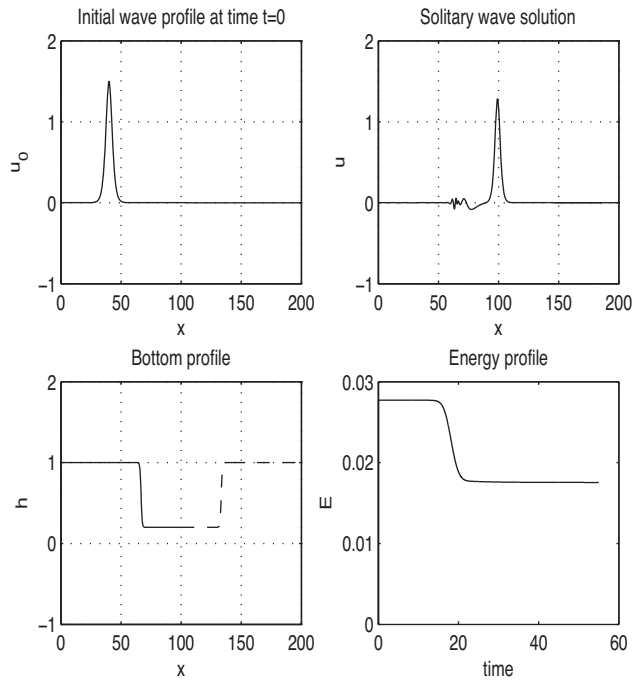| a_0	a_1/a_0				
	$\Delta h = 0.1$	0.3	0.5	0.7	0.9
0.150	0.997	0.988	0.966	0.911	0.733
0.300	0.994	0.974	0.936	0.868	0.700
0.450	0.991	0.965	0.925	0.854	0.706
0.600	0.989	0.962	0.922	0.856	0.721
0.750	0.989	0.962	0.922	0.860	0.739

Figure 2. Solitary wave solution for $u_t + c(x)u_x + uu_x - u_{xxt} = 0$ with positive wave speed. Solitary waves traveling in the direction of increasing values of x over an decreasing $h(x)$. We see that the energy is also decreasing. In bottom profile, solid lines are our physical domain.

Results were calculated for a number of initial wave amplitudes and different h_1. All the numerical results are presented here with $h_0 = 1$, $h_1 = (0.1)(n)h_0$ and $\Delta h = h_0 - h_1$ where $n = 1, 2, 3....9$. Results for

the variation in amplitudes of the waves are tabulated in Tables 3 and 4. Table 3 and Table 4 show that there is no systematic variation with the amplitude.
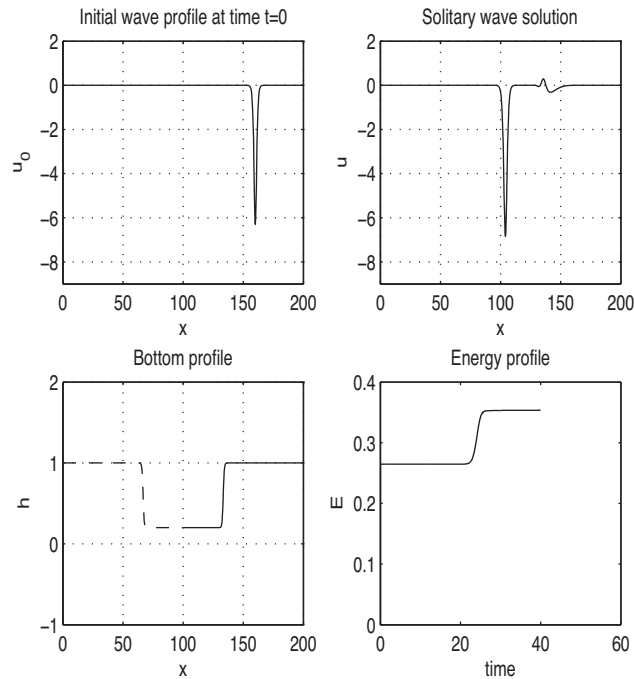

Figure 3. Solitary wave solution for $u_t + c(x)u_x + uu_x - u_{xxt} = 0$ with negative wave speed. Negative progressive waves propagate to left over a decreasing height profile. We observe that the energy is increasing. Our physical domain in the bottom profile is the right half of the numerical domain.

Table 4. Amplitude variations of the solitary wave solutions of the equation $u_t + c(x)u_x + uu_x - u_{xxt} = 0$ for for decreasing $h(x)$ and negative wave speed.

a_1	a_2/a_1				
	$\Delta h = 0.1$	0.3	0.5	0.7	0.9
3.1	1.190	1.274	1.338	1.388	1.447
3.3	1.108	1.184	1.246	1.301	1.356
3.5	1.055	1.134	1.189	1.235	1.288
3.7	1.042	1.104	1.151	1.198	1.248
3.9	1.031	1.084	1.133	1.177	1.223

4.3. Solitary waves in a channel of increasing depth

Now attention is given to the the solitary waves that run down a height increasing bottom profile. We investigate what happens to the energy E and amplitude of the solitary wave in this case. It was observed that after the wave has crossed into the region with increased depth, the amplitude of the right going solitary wave is amplified. Indeed, the result (3.1) now has the opposite sign, and the energy is increasing: $\frac{dE}{dt} \geq 0$. This is also shown in Figure 4. For the left going solitary wave the wave energy is decreasing, and we get a reduction in the amplitude and wavelength in the case $h'(x) \geq 0$. This behavior is shown in Figure 5.
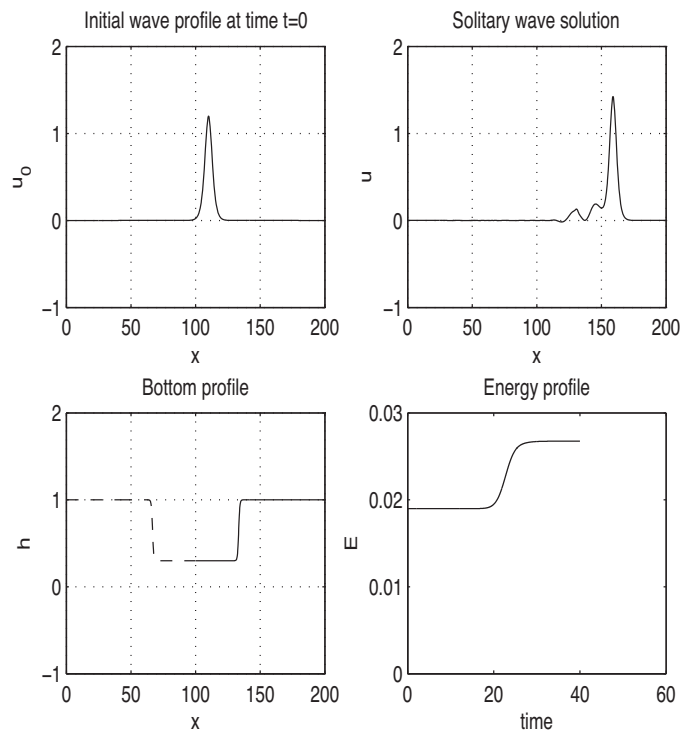
Figure 4. Solitary wave solution for $u_t + c(x)u_x + uu_x - u_{xxt} = 0$ with positive wave speed. Solitary waves traveling in the direction of increasing values of x over an increasing $h(x)$.
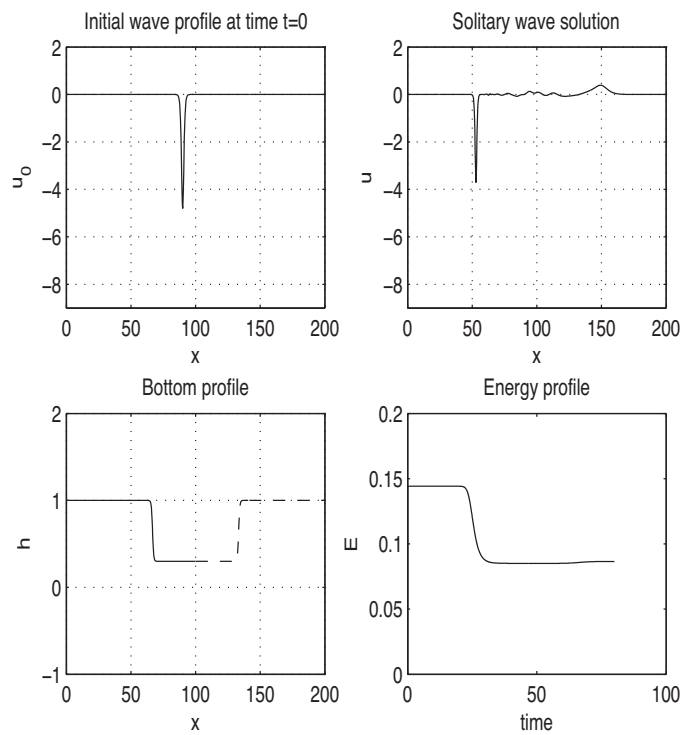

Figure 5. Solitary wave solution for $u_t + c(x)u_x + uu_x - u_{xxt} = 0$ with negative wave speed and increasing height bottom profile.

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