

A note on chaos in product maps

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Abstract: In this paper, we mainly discuss how chaos conditions on semi-flows carry over to their products. We show that if two semi-flows (or even one of them) are sensitive, so does their product. On the other side, the product of two topologically transitive semi-flows need not be topologically transitive. We then provide several sufficient conditions under which the product of two chaotic semi-flows is chaotic in the sense of Devaney. Also, stronger forms of sensitivity and transitivity for product systems are studied. In particular, we introduce the notion of ergodic sensitivity and prove that for any given two (not-necessarily continuous) maps $f : X \rightarrow X$ and $g : Y \rightarrow Y$ (resp. semi-flows $\psi : R^+ \times X \rightarrow X$ and $\phi : R^+ \times Y \rightarrow Y$) on the metric spaces X and Y , $f \times g$ (resp. $\psi \times \phi$) is ergodically sensitive if and only if f or g (resp. ψ or ϕ) is ergodically sensitive. Our results improve and extend some existing ones.

Key words: Chaos in the sense of Devaney, topological transitivity, sensitivity

1. Introduction

Let X and Y be two metric spaces. Denote $R^+ := [0, +\infty)$. We say that a map $\phi : R^+ \times X \rightarrow X$ is a semi-flow if it satisfies the following two properties:

- (1) $\phi(0, x) = x$ for any $x \in X$;
- (2) $\phi(t, (s, x)) = \phi(t + s, x)$ for any $t, s \in R^+$ and any $x \in X$.

Let $\phi : R^+ \times X \rightarrow X$ and $\psi : Y \rightarrow Y$ be two (not-necessarily continuous) semi-flows, which we assume to be chaotic in the sense of Devaney. It is natural to ask whether their product $\phi \times \psi : R^+ \times X \times Y \rightarrow X \times Y$ is chaotic (in the same sense), where the product $\phi \times \psi : R^+ \times X \times Y \rightarrow X \times Y$ is defined by $\phi \times \psi(t, (x, y)) = (\phi(t, x), \psi(t, y))$ for any $t \in R^+$ and any $(x, y) \in X \times Y$. In [5], for two chaotic maps (in the sense of Devaney), the authors proved by a counterexample that their product need not be chaotic. They then discussed the transfer of the sub-conditions of chaos, and finally gave some simple sufficient conditions under which the product of two given chaotic maps is chaotic. These conditions are satisfied for many known chaotic maps. In this paper we mainly discuss how chaos conditions on semi-flows carry over to their products. We show that if two semi-flows (or even one of them) are sensitive, so does their product. On the other side, the product of two topologically transitive semi-flows need not be topologically transitive. We then provide several sufficient conditions under which the product of two chaotic semi-flows is chaotic in the sense of Devaney. For continuous self-maps of compact metric spaces, in [13] the author initiated a preliminary study of stronger forms

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of sensitivity formulated in terms of large subsets of \mathbb{N} . Mainly he considered syndetic sensitivity and cofinite sensitivity and established the following results:

- (i) Any syndetically transitive, non-minimal map is syndetically sensitive. (This improves the result that sensitivity is redundant in Devaney’s definition of chaos.)
- (ii) Any sensitive map of $[0, 1]$ is cofinitely sensitive.
- (iii) Any sensitive subshift of finite type is cofinitely sensitive.
- (iv) Any syndetically transitive, infinite subshift is syndetically sensitive.
- (v) No Sturmian subshift is cofinitely sensitive.

Also, in the same paper he constructed a transitive, sensitive map which is not syndetically sensitive. In this paper, some stronger forms of sensitivity and transitivity for (not-necessarily continuous) product systems are studied. In particular, we introduce the notion of ergodic sensitivity, and it is shown that for any two not-necessarily continuous maps $f : X \rightarrow X$ and $g : Y \rightarrow Y$ (resp. semi-flows $\psi : R^+ \times X \rightarrow X$ and $\phi : R^+ \times Y \rightarrow Y$) on the metric spaces X and Y , $f \times g$ (resp. $\psi \times \phi$) is ergodically sensitive if and only if f (resp. ψ) or g (resp. ϕ) is ergodically sensitive. Consequently, we improve and extend the results of Değirmenci et al.

The organization of this paper is as follows. In Section 2, we recall some notations and basic concepts and introduce the concept of ergodic sensitivity. Main results are established in Section 3.

2. Preliminaries

First, we complete some notations and recall some concepts.

Let (X, d) be a metric space and $Z^+ = \{0, 1, 2, \dots\}$. A (not-necessarily continuous) map $f : X \rightarrow X$ (resp., a (not-necessarily continuous) semi-flow $\psi : R^+ \times X \rightarrow X$) is sensitive, if there exists $\varepsilon > 0$ such that for any $x \in X$ and any neighborhood U of x , there exist $x_0 \in U$ and an integer $n > 0$ (resp., a real number $t > 0$) such that $d(f^n(x), f^n(x_0)) > \varepsilon$ (resp., $d(\psi_t(x), \psi_t(x_0)) > \varepsilon$).

For $U, V \subset X$ and a map $f : X \rightarrow X$ (resp., a semi-flow $\psi : R^+ \times X \rightarrow X$), write

$$N_f(U, V) = \{n \in Z^+ : f^n(U) \cap V \neq \emptyset\}$$

and

$$N_\psi(U, V) = \{t \in R^+ : \psi(t, U) \cap V \neq \emptyset\}.$$

Obviously,

$$N_f(U, V) = \{n \in Z^+ : f^{-n}(V) \cap U \neq \emptyset\}$$

and

$$N_\psi(U, V) = \{t \in R^+ : \psi_t^{-1}(V) \cap U \neq \emptyset\}.$$

A (not-necessarily continuous) map $f : X \rightarrow X$ (resp., a (not-necessarily continuous) semi-flow $\psi : R^+ \times X \rightarrow X$) is topologically transitive if for any two non-empty open sets $U, V \subset X$, $N_f(U, V) \neq \emptyset$ (resp., $N_\psi(U, V) \neq \emptyset$).

A (not-necessarily continuous) map $f : X \rightarrow X$ (resp., a (not-necessarily continuous) semi-flow $\psi : R^+ \times X \rightarrow X$) is chaotic in the sense of Devaney, if it is sensitive, topologically transitive and, additionally,

the set of all periodic points of f (resp., ψ) is dense in X (see [7]). For an infinite metric space X and a continuous map $f : X \rightarrow X$ (resp., a continuous semi-flow $\psi : R^+ \times X \rightarrow X$), topological transitivity and denseness of periodic points imply sensitivity (see [3, 8]). In general, the topological transitivity is not equivalent to the existence of a dense orbit (see [6, 10]). Given two maps $f : X \rightarrow X$ and $g : Y \rightarrow Y$ (resp., two semi-flows $\psi : R^+ \times X \rightarrow X$ and $\phi : R^+ \times Y \rightarrow Y$) on metric spaces X and Y with metrics d_X and d_Y respectively, their product $f \times g : X \times Y \rightarrow X \times Y$ (resp., $\psi \times \phi : R^+ \times X \times Y \rightarrow X \times Y$) is defined by $(f \times g)(x, y) = (f(x), g(y))$ (resp., $(\psi \times \phi)(t, (x, y)) = (\psi(t, x), \phi(t, y))$) for any $(x, y) \in X \times Y$ and any $t \in R^+$, the product metric $d_{X \times Y}$ on $X \times Y$ is defined by $d_{X \times Y}((x, y), (x', y')) = d_X(x, x') + d_Y(y, y')$ for any $x \in X$ and any $y \in Y$.

A map $f : X \rightarrow X$ (resp., a semi-flow $\psi : R^+ \times X \rightarrow X$) is said to be topologically weakly mixing if $f \times f$ (resp., $\psi \times \psi$) is topologically transitive.

A map $f : X \rightarrow X$ (resp., a semi-flow $\psi : R^+ \times X \rightarrow X$) is said to be syndetically transitive if for any two nonempty open sets $U, V \subset X$, $N_f(U, V)$ (resp., $N_\psi(U, V)$) is a syndetic set, that is, there is an integer $M > 0$ (resp., a real number $M > 0$) such that $N_f(U, V) \cap \{n, n + 1, \dots, n + M\} \neq \emptyset$ (resp., $N_\psi(U, V) \cap [t, t + M] \neq \emptyset$) for any $n \in Z^+$ (resp., $t \in R^+$).

A map $f : X \rightarrow X$ (resp., a semi-flow $\psi : R^+ \times X \rightarrow X$) is said to be topologically mixing if for any two nonempty open sets $U, V \subset X$, $N_f(U, V)$ (resp., $N_\psi(U, V)$) is a cofinite set, that is, there is an integer $M > 0$ (resp., a real number $M > 0$) such that $N_f(U, V) \supset \{M, M + 1, \dots\}$ (resp., $N_\psi(U, V) \supset [M, +\infty)$).

For any $\delta > 0$ and any $U \subset X$, we define $N_f(U, \delta) = \{n \in Z^+ : d(f^n(x), f^n(y)) > \delta \text{ for some } x, y \in U\}$ and $N_\psi(U, \delta) = \{t \in R^+ : d(\psi(t, x), \psi(t, y)) > \delta \text{ for some } x, y \in U\}$.

Similarly, we give the following concepts (see [13]).

A map $f : X \rightarrow X$ (resp., a semi-flow $\psi : R^+ \times X \rightarrow X$) is said to be syndetically sensitive if there exists $\delta > 0$ such that for any nonempty open set $U \subset X$, $N_f(U, \delta)$ (resp., $N_\psi(U, \delta)$) is a syndetic set.

A map $f : X \rightarrow X$ (resp., a semi-flow $\psi : R^+ \times X \rightarrow X$) is said to be cofinitely sensitive if there exists $\delta > 0$ such that for any nonempty open set $U \subset X$, $N_f(U, \delta)$ (resp., $N_\psi(U, \delta)$) is a cofinite set.

A map $f : X \rightarrow X$ (resp., a semi-flow $\psi : R^+ \times X \rightarrow X$) is said to be multi-sensitive if there exists $\delta > 0$ such that for every integer $k > 0$ and any nonempty open sets $U_1, U_2, \dots, U_k \subset X$, $\bigcap_{1 \leq i \leq k} N_f(U_i, \delta) \neq \emptyset$ (resp.,

$$\bigcap_{1 \leq i \leq k} N_\psi(U_i, \delta) \neq \emptyset).$$

Let $S \subset Z^+$ (resp., $S \subset R^+$). Its upper density is defined by

$$\bar{d}(S) := \limsup_{k \rightarrow \infty} \frac{1}{k} |S \cap N_k|$$

(resp., $\limsup_{t \rightarrow \infty} \frac{1}{t} l(S \cap [0, t])$), where $l(S)$ is the Lebesgue measure of S [11].

Motivated by the idea in the definition of topological ergodicity introduced by Akin [1], we now introduce another stronger form of sensitivity as follows.

A map f (resp., a semi-flow φ) is called to be ergodically sensitive if there exists a positive constant $\delta > 0$ satisfying that $N_f(V, \delta)$ (resp., $N_\varphi(V, \delta)$) has a positive upper density for any nonempty open subset $V \subset X$.

We note that a map $f : X \rightarrow X$ (resp., a semi-flow $\psi : R^+ \times X \rightarrow X$) is sensitive if and only if there

exists $\delta > 0$ such that for any nonempty open set $U \subset X$, $N_f(U, \delta)$ (resp., $N_\psi(U, \delta)$) is nonempty.

3. Main results

In this section we improve and extend the results in [5].

By [2] or [9], one can easily prove the following theorem. For completeness, we now give a different proof here.

Theorem 3.1 *If a continuous tree map $f : T \rightarrow T$ is topologically transitive and is not topologically mixing, then $f \times f$ is not topologically transitive.*

Proof Assume on the contrary that the product map $f \times f$ is topologically transitive, that is, f is topologically weakly mixing.

Now we show that if a tree map f is topologically weak mixing, then it is topologically mixing. Let $E(T)$ and $O(T)$ denote the set of all ends of T and the set of all branching points of T , respectively. Let

$$T - O(T) = \bigcup_{j=1}^{i=t} I_j,$$

where I_j is a connected component of $T - O(T)$, for all $1 \leq j \leq t$. Assume that U and V are any connected open subsets of $T - O(T)$. Without loss of generality, we may assume that $U \subset I_k^o$ for some $1 \leq k \leq t$. Clearly, f is topologically transitive. By [15], we have $\overline{P(f)} = T$. For any $v, w \in U \cap P(f)$ with $v \neq w$, we suppose v is a periodic point of period m_1 and w is a periodic point of period m_2 . Therefore, we obtain that

$$O_f(v) \bigcup O_f(w) \subset T - E(T).$$

Suppose $u \in V$ is a periodic point of f with period m_3 and m is a common multiple of m_1, m_2 and m_3 . So, we get that

$$O_f(v) \bigcup O_f(w) \bigcup \{u\} \subset F(f^m).$$

Let $g = f^m$ and $W = \bigcup_{n=0}^{\infty} g^n(V)$. Obviously, W is a connected set. Since g is topologically transitive, by [4, 14] we have $\overline{W} = T$ and $W \supset I_k$. Therefore for every $x \in O_f(v) \bigcup O_f(w)$, there is a $s_x > 0$ such that $x \in g^{s_x}(V)$. Let

$$s = \max\{s_x : x \in O_f(v) \bigcup O_f(w)\}.$$

Since

$$O_f(v) \bigcup O_f(w) \subset F(g),$$

$$O_f(v) \bigcup O_f(w) \subset g^s(V) = f^{sl}(V).$$

It is easy to see that

$$f^n(V) \supset O_f(v) \bigcup O_f(w)$$

for all $n \geq sl$. Consequently, we have $f^n(V) \supset [v, w]$ for all $n \geq sl$. This shows that $f^n(V) \cap U \neq \emptyset$ for all $n \geq sl$. This is a contradiction. Thus, the proof is finished. \square

Remark 3.2 *Theorem 3.1 extends Example 1 in [5].*

Lemma 3.3 *Let X and Y be metric spaces with metrics d_X and d_Y , respectively, and let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be not-necessarily continuous maps. Then the following hold:*

- 1) *If f or g is syndetically sensitive, then $f \times g$ is syndetically sensitive.*
- 2) *If f or g is cofinitely sensitive, then $f \times g$ is cofinitely sensitive.*
- 3) *If f or g is multi-sensitive, then $f \times g$ is multi-sensitive.*
- 4) *$f \times g$ is ergodically sensitive if and only if f or g is ergodically sensitive.*

Proof Let $U \subset X$ and $V \subset Y$ be nonempty open sets. Then, for any $\delta > 0$, one can easily verify that $N_{f \times g}(U \times V, \delta) \supset N_f(U, \delta) \cup N_g(V, \delta)$. Therefore, parts 1), 2) and 3) of Lemma 3.3 are true.

From the above argument it is easy to see that, if f or g is ergodically sensitive, then so is the product map $f \times g$. Now we suppose that the product map $f \times g$ is ergodically sensitive and that both f and g are not ergodically sensitive. This means that for any given $\delta > 0$, there exists a certain open set $U \subset X$ with $\bar{d}(N_f(U, \frac{\delta}{3})) = 0$. Similarly, there exists a certain open set $V \subset Y$ with $\bar{d}(N_g(V, \frac{\delta}{3})) = 0$. It is easy to see that $N_{f \times g}(U \times V, \delta) \subset N_f(U, \frac{1}{3}\delta) \cup N_g(V, \frac{1}{3}\delta)$. This implies that

$$\bar{d}(N_{f \times g}(U \times V, \delta)) \leq \bar{d}(N_f(U, \frac{1}{3}\delta) \cup N_g(V, \frac{1}{3}\delta)) \leq \bar{d}(N_f(U, \frac{1}{3}\delta)) + \bar{d}(N_g(V, \frac{1}{3}\delta)) \leq 0.$$

It is a contradiction. So, the proof of part 4) is completed.

Thus, the entire proof is ended. □

Remark 3.4 *We do not know whether the following hold.*

- 1) *If $f \times g$ is syndetically sensitive, then f or g is syndetically sensitive.*
- 2) *If $f \times g$ is cofinitely sensitive, then f or g is cofinitely sensitive.*
- 3) *If $f \times g$ is multi-sensitive, then f or g is multi-sensitive.*

For a semi-flow on a metric space, we have the following lemma.

Lemma 3.5 *Let X and Y be metric spaces with metrics d_X and d_Y , respectively, and let $\psi : R^+ \times X \rightarrow X$ and $\phi : R^+ \times Y \rightarrow Y$ be not-necessarily continuous semi-flows. Then the following hold.*

- 1) *$\psi \times \phi : R^+ \times X \times Y \rightarrow X \times Y$ is sensitive if and only if $\psi : R^+ \times X \rightarrow X$ or $\phi : R^+ \times Y \rightarrow Y$ is sensitive.*
- 2) *$\psi \times \phi : R^+ \times X \times Y \rightarrow X \times Y$ is ergodically sensitive if and only if $\psi : R^+ \times X \rightarrow X$ or $\phi : R^+ \times Y \rightarrow Y$ is ergodically sensitive.*

Proof Let $U \subset X$ and $V \subset Y$ be nonempty open sets. Then, for any $\delta > 0$, one can easily verify that $N_{\psi \times \phi}(U \times V, \delta) \supset N_\psi(U, \delta) \cup N_\phi(V, \delta)$. Consequently, if $\psi \times \phi : R^+ \times X \times Y \rightarrow X \times Y$ is sensitive, then $\psi : R^+ \times X \rightarrow X$ or $\phi : R^+ \times Y \rightarrow Y$ is sensitive.

Assume that both ψ and ϕ are not sensitive. This means that for any $\varepsilon > 0$ there exists $x \in X$ such that for a certain open set $U \subset X$ with $x \in U$, $d_X(\psi_t(x), \psi_t(x')) \leq \frac{\varepsilon}{2}$ for any $x' \in U$ and any $t \in R^+$. Similarly, there is $y \in Y$ such that for a certain open set $V \subset Y$ with $y \in V$, $d_Y(\phi_t(y), \phi_t(y')) \leq \frac{\varepsilon}{2}$ for any $y' \in V$.

and any $t \in R^+$. This implies that $N_{\psi \times \phi}(U \times V, \varepsilon) = \emptyset$. Hence, $\psi \times \phi$ is not sensitive, which contradicts the hypothesis. Consequently, this ends the proof of part 1).

From the above argument it is easily seen that if ψ or ϕ is ergodically sensitive, then so does the product semi-flow $\psi \times \phi$. Now we suppose that the product semi-flow $\psi \times \phi$ is ergodically sensitive, and that both ψ and ϕ are not ergodically sensitive. This implies that for any given $\delta > 0$, there exists a certain open set $U \subset X$ with $\bar{d}(N_\psi(U, \frac{\delta}{3})) = 0$. Similarly, there exists a certain open set $V \subset Y$ with $\bar{d}(N_\phi(V, \frac{\delta}{3})) = 0$. It is easy to see that $N_{\psi \times \phi}(U \times V, \delta) \subset N_\psi(U, \frac{1}{3}\delta) \cup N_\phi(V, \frac{1}{3}\delta)$. This implies that

$$\bar{d}(N_{\psi \times \phi}(U \times V, \delta)) \leq \bar{d}(N_\psi(U, \frac{1}{3}\delta) \cup N_\phi(V, \frac{1}{3}\delta)) \leq \bar{d}(N_\psi(U, \frac{1}{3}\delta)) + \bar{d}(N_\phi(V, \frac{1}{3}\delta)) \leq 0.$$

It is a contradiction. So, the proof of part 2) is completed.

Thus, the entire proof is finished. □

Remark 3.6 Lemmas 3.3 and 3.5 improve and extend Lemma 1 in [5].

Remark 3.7 We do not know whether the following are true:

- 1) If $\psi \times \phi$ is syndetically sensitive, then ψ or ϕ is syndetically sensitive.
- 2) If $\psi \times \phi$ is cofinitely sensitive, then ψ or ϕ is cofinitely sensitive.
- 3) If $\psi \times \phi$ is multi-sensitive, then ψ or ϕ is multi-sensitive.

Remark 3.8 We know from Remark 4 in [5] that the map doubling the circle is locally eventually onto and hence topologically mixing. Clearly, this map is continuous. By Proposition 2 from [13] the map doubling the circle is cofinitely sensitive. So, Lemma 3.3 shows that the product map under Remark 6 in [5] is cofinitely sensitive. Consequently, this result improves the result of Remark 6 in [5], and it also shows that the converse of Proposition 2 from [13] is not true.

Lemma 3.9 Let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be (not-necessarily continuous) maps. Then the following hold.

- 1) If the product map $f \times g$ is syndetically transitive, then f and g are syndetically transitive.
- 2) If the product map $f \times g$ is topologically weakly mixing, then f and g are topologically weakly mixing.
- 3) If the product map $f \times g$ is topologically mixing, then f and g are topologically mixing.

Proof Let $U_1, U_2 \subset X$ and $V_1, V_2 \subset Y$ be non-empty open sets. Then the sets $U = U_1 \times Y$, $V = U_2 \times Y$, $P = X \times V_1$ and $Q = X \times V_2$ are open in $X \times Y$. Clearly, $N_{f \times g}(U, V) = N_f(U_1, U_2)$ and $N_{f \times g}(P, Q) = N_g(V_1, V_2)$. Therefore, by the definitions we know that Lemma 3.9 holds. Thus, the proof is finished. □

Similarly, for a semi-flow on a metric space, we have the following lemma.

Lemma 3.10 Let $\psi : R^+ \times X \rightarrow X$ and $\phi : R^+ \times Y \rightarrow Y$ be (not-necessarily continuous) semi-flows. Then the following hold.

- 1) If the product semi-flow $\psi \times \phi$ is topologically transitive, then ψ and ϕ are topologically transitive.
- 2) If the product semi-flow $\psi \times \phi$ is topologically weakly mixing, then ψ and ϕ are topologically weakly mixing.
- 3) If the product semi-flow $\psi \times \phi$ is topologically mixing, then ψ and ϕ are topologically weakly mixing.

Proof Let $U_1, U_2 \subset X$ and $V_1, V_2 \subset Y$ be non-empty open sets. Then the sets $U = U_1 \times Y$, $V = U_2 \times Y$, $P = X \times V_1$ and $Q = X \times V_2$ are open in $X \times Y$. Clearly, $N_{\psi \times \phi}(U, V) = N_\psi(U_1, U_2)$ and $N_{\psi \times \phi}(P, Q) = N_\phi(V_1, V_2)$. Therefore, by the definitions we know that Lemma 3.10 holds. Thus, the proof is finished. \square

Lemma 3.11 *Let $f : X \rightarrow X$ be a continuous map and $g : Y \rightarrow Y$ be a (not-necessarily continuous) map. If g is topologically mixing, then the following hold.*

- 1) *If f is topologically transitive, then so does the product map $f \times g$.*
- 2) *If f is syndetically transitive, then so does the product map $f \times g$.*
- 3) *If f is topologically weakly mixing, then so does the product map $f \times g$.*

Proof Given nonempty open sets $W_1, W_2 \subset X \times Y$, there exist nonempty open sets $U_1, U_2 \subset X$ and $V_1, V_2 \subset Y$ with $U_1 \times V_1 \subset W_1$ and $U_2 \times V_2 \subset W_2$. Obviously,

$$N_{f \times g}(U_1 \times V_1, U_2 \times V_2) = N_f(U_1, U_2) \cap N_g(V_1, V_2).$$

As g is topologically mixing, there is an integer $M > 0$ such that

$$N_g(V_1, V_2) \supset \{M, M + 1, \dots\}.$$

Since f is continuous, $f^{-M}(U_2)$ is a nonempty and open subset of X .

- (1) If f is topologically transitive, by hypothesis and the definition, there exists $l \in \mathbb{Z}^+$ such that

$$N_f(U_1, f^{-M}(U_2)) \neq \emptyset,$$

which implies that

$$N_f(U_1, U_2) \cap N_g(V_1, V_2) \neq \emptyset.$$

Consequently, the product map $f \times g$ is topologically transitive.

- (2) If f is syndetically transitive, by hypothesis and the definition, there exists an integer $l > 0$ such that

$$N_f(U_1, f^{-M}(U_2)) \cap \{n, n + 1, \dots, n + l\} \neq \emptyset$$

for every $n \in \mathbb{Z}^+$, which implies that

$$N_f(U_1, U_2) \cap N_g(V_1, V_2) \cap \{n, n + 1, \dots, n + l + M\} \neq \emptyset.$$

Consequently, the product map $f \times g$ is syndetically transitive.

- (3) Since the proof of part 3) is similar to that of part 1), it is omitted here.

Thus, the entire proof is completed. \square

Similarly, we obtain the following lemma.

Lemma 3.12 *Let $\psi : \mathbb{R}^+ \times X \rightarrow X$ be a continuous semi-flow and $\phi : \mathbb{R}^+ \times Y \rightarrow Y$ be a (not-necessarily continuous) semi-flow. If ϕ is topologically mixing, then the following hold.*

- 1) *If ψ is topologically transitive, then so does the product semi-flow $\psi \times \phi$.*

- 2) If ψ is syndetically transitive, then so does the product semi-flow $\psi \times \phi$.
- 3) If ψ is topologically weakly mixing, then so does the product semi-flow $\psi \times \phi$.
- 4) If ψ is topologically mixing, then so does the product semi-flow $\psi \times \phi$.

Proof For any nonempty open sets $W_1, W_2 \subset X \times Y$, there exist nonempty open sets $U_1, U_2 \subset X$ and $V_1, V_2 \subset Y$ with $U_1 \times V_1 \subset W_1$ and $U_2 \times V_2 \subset W_2$. Obviously,

$$N_{\psi \times \phi}(U_1 \times V_1, U_2 \times V_2) = N_\psi(U_1, U_2) \cap N_\phi(V_1, V_2).$$

As ϕ is topologically mixing, there is $M > 0$ such that

$$N_\phi(V_1, V_2) \supset [M, +\infty).$$

Since ψ is continuous, $\psi_M^{-1}(U_2)$ is a nonempty and open subset of X .

- (1) If ψ is topologically transitive, by hypothesis and the definition, there exists $l \in R^+$ such that

$$N_\psi(U_1, \psi_M^{-1}(U_2)) \neq \emptyset,$$

which implies that

$$N_\psi(U_1, U_2) \cap N_\phi(V_1, V_2) \neq \emptyset.$$

Consequently, the product semi-flow $\psi \times \phi$ is topologically transitive.

- (2) If ψ is syndetically transitive, by hypothesis and the definition, there exists $l > 0$ such that

$$N_\psi(U_1, \phi_M^{-1}(U_2)) \cap [t, t + l] \neq \emptyset$$

for every $t \in R^+$, which implies that

$$N_\psi(U_1, U_2) \cap N_\phi(V_1, V_2) \cap [t, t + l + M] \neq \emptyset.$$

Consequently, the product semi-flow $\psi \times \phi$ is syndetically transitive.

- (3) Since the proof of part 3) is similar to that of part 1), it is omitted here.

- (4) Since the proof of part 4) is similar to that of part 1), it is omitted here.

Thus, the entire proof is completed. □

By Lemma 3.11, one can easily prove the following theorem which is from [5].

Theorem 3.13 *Let $f : X \rightarrow X$ be a chaotic continuous map on the metric space X and $g : Y \rightarrow Y$ be a chaotic (not-necessarily continuous) map on the metric space Y . If g is topologically mixing, then $f \times g : X \times Y \rightarrow X \times Y$ is chaotic.*

Theorem 3.14 *Let $\psi : R^+ \times X \rightarrow X$ be a chaotic continuous semi-flow on the metric space X and $\phi : R^+ \times Y \rightarrow Y$ be a (not-necessarily continuous) semi-flow on the metric space Y . If ϕ is topologically mixing and the set of all periodic points of $\psi \times \phi$ is dense in $X \times Y$, then $\psi \times \phi : R^+ \times X \times Y \rightarrow X \times Y$ is chaotic.*

Proof By the definition and Lemmas 3.5 and 3.12, $\psi \times \phi : R^+ \times X \times Y \rightarrow X \times Y$ is chaotic. Thus, the proof is completed. □

Remark 3.15 *Theorem 3.14 extends Theorems 1 and 2 in [5] to semi-flows.*

Given a metric space X and a not-necessarily continuous map $f : X \rightarrow X$, we say that f has the Touhey property on X if given U and V , non-empty open subsets of X , there exist a periodic point $x \in U$ and an integer $k \geq 0$ such that $f^k(x) \in V$, that is, if every pair of non-empty open subsets of X shares a periodic orbit (see [5]). Similarly, we give its corresponding definition for a semi-flow.

Definition 3.16 *Given a metric space X and a not-necessarily continuous semi-flow $\psi : \mathbb{R}^+ \times X \rightarrow X$, we say that ψ has the Touhey property on X if given U and V , non-empty open subsets of X , there exist a periodic point $x \in U$ and an real number $t \geq 0$ such that $\psi(t, x) \in V$.*

Theorem 3.17 *Let X be a metric space and assume that $\psi : \mathbb{R}^+ \times X \rightarrow X$ is a continuous semi-flow with the Touhey property. Let $\phi : \mathbb{R}^+ \times Y \rightarrow Y$ be a not-necessarily continuous, chaotic and topologically mixing semi-flow on the metric space Y . If the set of all periodic points of $\psi \times \phi$ is dense in $X \times Y$, then $\psi \times \phi : \mathbb{R}^+ \times X \times Y \rightarrow X \times Y$ is chaotic.*

Proof By the definition and Lemmas 3.5 and 3.12, it is enough to show that $\psi \times \phi : \mathbb{R}^+ \times X \times Y \rightarrow X \times Y$ is topologically transitive. Let $U_1, U_2 \subset X$ and $V_1, V_2 \subset Y$ be nonempty open sets. Since ϕ is topologically mixing, there exists $t_0 > 0$ with $\phi(t, V_1) \cap V_2 \neq \emptyset$ for all $t \geq t_0$. By definition there exists a periodic point $x \in U_1$ whose orbit enters U_2 . Let L be the period of x . Then there exists L' with $0 \leq L' < L$ and $\psi(L', x) \in U_2$. This means that $\psi(nL + L', x) \in U_2$ for any integer $n > 0$. Choose $n > 0$ such that $L'' = nL + L' \geq t_0$. Therefore, there exists a point $y \in V_1$ with $\phi(L'', y) \in V_2$. Consequently, $(x, y) \in \psi \times \phi(L'', U_1 \times V_1) \cap (U_2 \times V_2)$, which implies that $\psi \times \phi : \mathbb{R}^+ \times X \times Y \rightarrow X \times Y$ is topologically transitive. Thus, the proof is completed. \square

Remark 3.18 *Theorem 3.17 extends Theorem 3 in [5] to semi-flows.*

Definition 3.19 *Let $f : X \rightarrow X$ (resp., $\psi : \mathbb{R}^+ \times X \rightarrow X$) be a (not-necessarily continuous) map (resp., semi-flow) on the topological space X . If for every nonempty open subset $U \subset X$ there exists an integer $n_0 > 0$ (resp., a real number $t_0 > 0$) such that for every $n \geq n_0$ (resp., $t \geq t_0$), $f^n(U) = X$ (resp., $\psi_t(U) = X$), then f (resp., ψ) is said to be locally eventually onto.*

Theorem 3.20 *Let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ (resp., $\psi : \mathbb{R}^+ \times X \rightarrow X$ and $\phi : \mathbb{R}^+ \times Y \rightarrow Y$) be two (not-necessarily continuous) locally eventually onto maps (resp., semi-flows) on the topological spaces X and Y respectively. Then the product map $f \times g$ (resp., the product semi-flow $\psi \times \phi$) is locally eventually onto.*

Proof By the definition, the proof is easy and is omitted. \square

Remark 3.21 *Theorem 3.20 shows that the products of any two of the maps under Remark 4 in [5] are locally eventually onto and hence topologically mixing.*

Now we present an example of a semi-flow exhibiting Devaney chaos.

For a continuous map $f : X \rightarrow X$ on a compact metric space X with metric d , we define an equivalence relation \mathcal{R} in the product space $[0, 1] \times X$ as follows.

For any $(t_1, x_1), (t_2, x_2) \in [0, 1] \times X$, $(t_1, x_1)\mathcal{R}(t_2, x_2)$ if and only if one of the following conditions:

- (1) $(t_1, x_1) = (t_2, x_2)$;
- (2) $t_1 = 1, t_2 = 0$ and $x_2 = f(x_1)$.

Write $Y = ([0, 1] \times X)/\mathcal{R}$. From [16] we know that the quotient space Y is a compact metric space. For a continuous map $f : X \rightarrow X$ on a compact metric space X with metric d , its suspended semi-flow $\varphi(f) : R^+ \times Y \rightarrow Y$ is defined by $\varphi(f)(r, [(t, x)]) = [t + r - n, f^n(x)]$ where $n \leq t + r < n + 1$, $r \in R^+$ and $[(t, x)] \in Y$.

Example 3.22 *Let $f : X \rightarrow X$ be a continuous map on a compact metric space X with a metric d . If f is chaotic in the sense of Devaney, then so does its suspended semi-flow $\varphi(f)$.*

Proof By the definitions and [16] it is easily seen that $x \in X$ is a periodic point of f if and only if $[(t, x)]$ is a periodic point of $\varphi(f)$ for any $0 \leq t < 1$, and that f is topologically transitive if and only if so does $\varphi(f)$. So, the set of all periodic points of f is dense in X if and only if the set of all periodic points of $\varphi(f)$ is dense in Y . Also, by [16] one can easily prove that f is sensitive if and only if so is $\varphi(f)$. By hypothesis and the definition, $\varphi(f)$ is chaotic in the sense of Devaney. Thus, the proof is complete. \square

Remark 3.23 *The results of Example 3.22 were first obtained and shown by Lianfa He and Zhenguo Zhang in [17].*

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References

- [1] Akin E.: The General Topology of Dynamical Systems, Providence. Amer. Math. Soc. Graduate Studies in Mathematics Vol.1, 1993.
- [2] Alseda L., del Ro M., Rodrguez J.: A survey on the relation between transitivity and dense periodicity for graph maps. J. Difference Equ. Appl. (3-4)9, 281-288 (2003).
- [3] Banks, J., Brooks, J. Chairs, G. Davis, G., Stacey, P.: On Devaney's Definition of Chaos. Amer. Math. Monthly 99, 332-334 (1992).
- [4] Block L., Coppel W. A.: Dynamics in one dimension. New York. Spring-Verlag. Lec. Notes in Math. 1513, 1992.
- [5] Bowen R., Walters P.: Expansive one-parameter flows. Journal of Differential Equations 12, 180-193 (1972).
- [6] Değirmenci N., Koçak Ş.: Chaos in product maps. Turk. J. Math. 34, 593-600 (2010).
- [7] Değirmenci N., Koçak Ş.: Existence of a Dense Orbit and Topological Transitivity: When Are They Equivalent? Acta Math. Hungar. (3)99, 185-187 (2003).
- [8] Devaney, R.: An Introduction to Chaotic Dynamical Systems. 2nd edition. Colorado. Addison-Wesley 1989.
- [9] Elaydi, S. N.: Discrete Chaos. London. Chapman and Hall 2000.

- [10] Haranczyk G., Kwietniak D., Oprocha P.: A note on transitivity, sensitivity and chaos for graph maps. *J. Difference Equ. Appl.* 17, 1549–1553 (2011).
- [11] Hasselblatt, B., Katok, A.: *A First Course in Dynamics*. New York. Cambridge University Press 2003.
- [12] He L., Yan X., Wang L.: Weak-mixing implies sensitive dependence. *J. Math. Anal. Appl.* 299, 300–304 (2004).
- [13] He L., Zhang Z.: Chaos in the semi-Flows and its inverse Limit systems. *Acta Math. Sci.* 17(Supplement), 46–51 (1997).
- [14] Kolyada, S., Snoha, S.: Some Aspects of Topological Transitivity-A Survey. *Grazer Math. Ber.* 334, 3–35 (1997).
- [15] Moothathu T. K. S.: Stronger forms of sensitivity for dynamical systems. *Nonlinearity* 20, 2115–2126 (2007).
- [16] Walter P.: *An introduction to ergodic theory*. New York. Springer-Verlag 1982.
- [17] Ye X.: The centre and the depth of a tree map. *Austral. Math. Soc.* 48, 347–350 (1993).