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Generalized Sobolev-Shubin spaces, boundedness and Schatten class properties of Toeplitz operators

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Abstract: Let w and ω be two weight functions on $\mathbb{R}^{2d}$ and $1 \leq p,q \leq \infty$. Also let $M(p,q,\omega)(\mathbb{R}^d)$ denote the subspace of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ consisting of $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the Gabor transform $V_g f$ of f is in the weighted Lorentz space $L(p,q,\omega d\mu)(\mathbb{R}^{2d})$. In the present paper we define a space $Q_{g,w}^{M(p,q,\omega)}(\mathbb{R}^d)$ as counter image of $M(p,q,\omega)(\mathbb{R}^d)$ under Toeplitz operator with symbol w. We show that $Q_{g,w}^{M(p,q,\omega)}(\mathbb{R}^d)$ is a generalization of usual Sobolev-Shubin space $Q_s(\mathbb{R}^d)$. We also investigate the boundedness and Schatten-class properties of Toeplitz operators.

Key words: Sobolev-Shubin space, Gabor transform, modulation space, weighted Lorentz space, Toeplitz operators, Schatten-class

1. Introduction

Throughout this paper we denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space on $\mathbb{R}^d$ and by $\mathcal{S}'(\mathbb{R}^d)$ its topological dual. Let f be a measurable complex valued function on $\mathbb{R}^d$. The translation and modulation operators are defined as $T_x f(t) = f(t-x)$ and $M_w f(t) = e^{2\pi i w t} f(t)$ for $x, w \in \mathbb{R}^d$, respectively. The following canonical commutation relation holds:

$$T_x M_w = e^{-2\pi i x w} M_w T_x$$

between $T_x M_w$ and $M_w T_x$ operators which are called time-frequency shifts [14]. A weight function w on $\mathbb{R}^d$ is a non-negative, continuous and locally integrable function. w is called submultiplicative if $w(x+y) \leq w(x) w(y)$ for all $x, y \in \mathbb{R}^d$. Let v be a submultiplicative weight function on $\mathbb{R}^d$. A weight function w on $\mathbb{R}^d$ is $v-$moderate if $w(x+y) \leq v(x) w(y)$ for all $x, y \in \mathbb{R}^d$. If

$$w(x) \leq C v_s(x) = C\left(1 + |x|^2\right)^{\frac{s}{2}}$$

for some $C > 0$, $s \geq 0$ and $x \in \mathbb{R}^d$, then w is called polynomial growth. Let w_1 and w_2 be two weights. We say that $w_2 \preceq w_1$ if and only if there exists $c > 0$ such that $w_2(x) \leq c w_1(x)$ for all $x \in \mathbb{R}^d$. Two functions are called equivalent and we write $w_1 \approx w_2$, if $w_2 \preceq w_1$ and $w_1 \preceq w_2$.

Let w be a weight function on $\mathbb{R}^d$. Then the weighted L^p space is defined by

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$$L_w^p\left(\mathbb{R}^d\right) = \left\{f \mid fw \in L^p\left(\mathbb{R}^d\right)\right\},$$

for $1 \leq p \leq \infty$. $L_w^p\left(\mathbb{R}^d\right)$ is a Banach space under the norm $\|f\|_{p,w} = \|fw\|_p$. Moreover, if w is submultiplicative and $w \geq 1$ then $L_w^1\left(\mathbb{R}^d\right)$ is a Banach convolution algebra. It is called a Beurling algebra [9].

Let $f \in L^1\left(\mathbb{R}^d\right)$. Then the Fourier transform $\hat{f}$ (or $\mathcal{F}f$) of f is given by

$$\hat{f}(t) = \int\limits_{\mathbb{R}^d} f(x)\, e^{-2\pi i\langle x,t\rangle} dx,$$

where $\langle x, t\rangle = \sum\limits_{i=1}^{d} x_i t_i$ is the usual scalar product on $\mathbb{R}^d$.

Given any fix function $g \neq 0$, which is called the window function, the short-time Fourier transform (STFT) or Gabor transform of a function f with respect to g is given by

$$V_g f(x, w) = \int\limits_{\mathbb{R}^d} f(t)\, \overline{g(t-x)} e^{-2\pi i t w} dt,$$

for $x, w \in \mathbb{R}^d$, [8], [10], [21]. It is known that if $f, g \in L^2\left(\mathbb{R}^d\right)$ then $V_g f \in L^2\left(\mathbb{R}^{2d}\right)$ and $V_g f$ is uniformly continuous [14].

Fix a non-zero window $g \in \mathcal{S}\left(\mathbb{R}^d\right)$ and $1 \leq p, q \leq \infty$. Let w be a weight function of at most polynomial growth and v_s-moderate on $\mathbb{R}^{2d}$. That means

$$w(z_1 + z_2) \leq C\left(1 + |z_1|^2\right)^{\frac{s}{2}} w(z_2)$$

for all $z_1, z_2 \in \mathbb{R}^{2d}$ and for some $C > 0$, $s \geq 0$. Then the modulation space $M_w^{p,q}\left(\mathbb{R}^d\right)$ consists of all tempered distributions $f \in \mathcal{S}'\left(\mathbb{R}^d\right)$ such that $V_g f \in L_w^{p,q}\left(\mathbb{R}^{2d}\right)$ and $\|f\|_{M_w^{p,q}} = \|V_g f\|_{L_w^{p,q}}$ is finite, where $L_w^{p,q}\left(\mathbb{R}^{2d}\right)$ the weighted mixed-norm space. If $p = q$, and then we write $M_w^p\left(\mathbb{R}^d\right)$ instead of $M_w^{p,p}\left(\mathbb{R}^d\right)$ and if $w = 1$, then we have standard modulation space $M^{p,q}\left(\mathbb{R}^d\right)$. Moreover, if $p = q = 2$ and v_s is weight function in polynomial type, that means $v_s(z) = \langle z\rangle^s = \left(1 + |z|^2\right)^{\frac{s}{2}}$ for $z \in \mathbb{R}^{2d}$ and $s \in \mathbb{R}$, then we obtain the space $M_s^2\left(\mathbb{R}^d\right)$ [14].

Let $g \in \mathcal{S}\left(\mathbb{R}^d\right) \setminus \{0\}$ and $w(x, y)$ be a suitable weight function defined on the time-frequency plane $\mathbb{R}^{2d}$. The Toeplitz operator is given by the formula

$$\left(Tp_g(w)\, f_1, f_2\right) = \left(wV_g f_1, V_g f_2\right)$$

for all $f_1, f_2 \in \mathcal{S}\left(\mathbb{R}^d\right)$. This implies that

$$Tp_g(w)\, f = V_g^*\left(wV_g f\right),$$

where V_g^* is the adjoint for V_g. In this work we shall extend this definition to more general situations. The fundamental objects in the definition of Toeplitz operators are Gabor transforms. Hence time-frequency techniques are used for the analysis of Toeplitz operators. Also, Toeplitz operators are localization operators whose symbols $w(x,y)$ belong to suitable classes. Since the Gabor transform is injective, it is easy to show that the Toeplitz operator is injective.

For a given symbol $\sigma \in \mathcal{S}'\left(\mathbb{R}^{2d}\right)$, the pseudodifferential operator L_σ is defined to be

$$L_\sigma f = \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\xi, u)\, e^{-\pi i \xi u} T_{-u} M_\xi f du d\xi.$$

The map $\sigma \to L_\sigma$ is called the Weyl transform, σ and $\hat{\sigma}$ are called the Weyl symbol and the spreading function of the operator L_σ, respectively.

Let X be a separable Hilbert space, $B(X)$ be the space of bounded linear operators on X and $A \in B(X)$ be a compact operator. Then the linear operator $|A| : X \to X$ is positive and compact. Let $\{\varphi_k : k = 1, 2, ...\}$ be an orthonormal basis for X consisting of eigenvectors of $|A|$ and let $s_k(A)$ be the eigenvalue of $|A|$ corresponding to the eigenvector φ_k, $(k = 1, 2, \ldots)$.

A compact operator $A : X \to X$ is said to be in the Schatten-von Neumann class S_p, $p \in [1, \infty)$ if

$$\sum_{k=1}^{\infty} s_k(A)^p < \infty.$$

It can be shown that S_p is a Banach space with the norm

$$\|A\|_{S_p} = \left\{ \sum_{k=1}^{\infty} s_k(A)^p \right\}^{\frac{1}{p}}, \quad A \in S_p.$$

It is customary to call S_1 the trace class and S_2 the Hilbert-Schmidt class.

The Weyl transform $\sigma \to L_\sigma$ is a unitary map from $L^2\left(\mathbb{R}^{2d}\right)$ onto the algebra of Hilbert-Schmidt operators on $L^2\left(\mathbb{R}^d\right)$ under the Hilbert-Schmidt norm. This property is known as Pool's Theorem [14].

Let $g \in \mathcal{S}\left(\mathbb{R}^d\right) \setminus \{0\}$ be a window function and $w(x,y) = \left(1 + |x|^2 + |y|^2\right)^{\frac{s}{2}}$ for $s \in \mathbb{R}$. Then the usual Sobolev-Shubin space $Q_s\left(\mathbb{R}^d\right)$ is the set of all $f \in \mathcal{S}'\left(\mathbb{R}^d\right)$ such that

$$\|f\|_{Q_s} = \|Tp_g(w) f\|_{L^2} < \infty.$$

The usual Sobolev-Shubin space $Q_s\left(\mathbb{R}^d\right)$ coincides with $M_s^2\left(\mathbb{R}^d\right)$; see [2], [3], [4]. A generalization of usual Sobolev-Shubin space $Q_s\left(\mathbb{R}^d\right)$ is given in [2].

Let f be a complex-valued measurable function defined on the measure space $(G, wd\mu)$, where w is a weight function on G. For $y > 0$, we define

$$\lambda_f(y) = w\{x \in G \mid |f(x)| > y\} = \int_{\{x \in G \mid |f(x)| > y\}} w(x)\, d\mu(x).$$

The function $\lambda_f(y)$ is called the distribution function of f. The rearrangement of f is defined by

$$f^*(t) = \inf\{y > 0 \mid \lambda_f(y) \le t\} = \sup\{y > 0 \mid \lambda_f(y) > t\}$$

for $t > 0$. The average function of f is also defined by

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(t)\,dt.$$

Moreover, λ_f, f^* and f^{**} are nonincreasing and right continuous functions on $(0,\infty)$. The weighted Lorentz space $L(p,q,wd\mu)(G)$ is defined to be the vector space of all (equivalent classes) measurable functions f, such that $\|f\|^*_{p,q,w} < \infty$, where

$$\begin{aligned} \|f\|^*_{p,q,w} &= \left(\frac{q}{p}\int_0^\infty t^{\frac{q}{p}-1}\left[f^*(t)\right]^q dt\right)^{\frac{1}{q}}, \quad 0 < p, q < \infty, \\ \|f\|^*_{p,q,w} &= \sup_{t>0} t^{\frac{1}{p}} f^*(t), \qquad 0 < p \le q = \infty. \end{aligned}$$

It is known that $\left(L(p,q,wd\mu)(G), \|.\|_{p,q,w}\right)$ is a Banach space, where [5]:

$$\begin{aligned} \|f\|_{p,q,w} &= \left(\frac{q}{p}\int_0^\infty t^{\frac{q}{p}-1}\left[f^{**}(t)\right]^q dt\right)^{\frac{1}{q}}, \quad 1 < p < \infty, 1 \le q < \infty, \\ \|f\|_{p,q,w} &= \sup_{t>0} t^{\frac{1}{p}} f^{**}(t), \qquad 1 < p \le q = \infty. \end{aligned}$$

If $w = 1$, then weighted Lorentz space $L(p,q,wd\mu)(G)$ is the usual Lorentz space $L(p,q)(G)$ [13], [18], [19], [20].

Let $1 \le r, s \le \infty$ and a weight w be given. Fix a compact $Q \subset \mathbb{R}^d$ with nonempty interior. Then the Wiener amalgam space $W(L^r, L^s_w)(\mathbb{R}^d)$ with local component $L^r(\mathbb{R}^d)$ and global component $L^s_w(\mathbb{R}^d)$ is defined as the space of all measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that $f\chi_K \in L^r(\mathbb{R}^d)$ for each compact subset $K \subset \mathbb{R}^d$, for which the norm

$$\|f\|_{W(L^r,L^s_w)} = \|F_f\|_{s,w} = \left\|\|f\chi_{Q+x}\|_r\right\|_{s,w}$$

is finite, where χ_K is the characteristic function of K and

$$F_f(x) = \|f\chi_{Q+x}\|_r \in L^s_w(\mathbb{R}^d).$$

It is known that if $r_1 \ge r_2$ and $s_1 \le s_2$ then $W(L^{r_1}, L^{s_1}_w)(\mathbb{R}^d) \subset W(L^{r_2}, L^{s_2}_w)(\mathbb{R}^d)$. If w is moderate and $r = s$ then $W(L^r, L^r_w)(\mathbb{R}^d) = L^r_w(\mathbb{R}^d)$. If $w \approx C$, then $W(L^r, L^s_w)(\mathbb{R}^d)$ is the usual Wiener amalgam space $W(L^r, L^s)(\mathbb{R}^d)$, where C is a constant number [7], [11], [16], [17].

2. Generalized Sobolev-Shubin spaces

In this section we give another generalization of the usual Sobolev -Shubin space $Q_s(\mathbb{R}^d)$. First we mention a generalization of the usual modulation space $M^{p,q}(\mathbb{R}^d)$.

Let ω be a weight function on $\mathbb{R}^{2d}$ and $1 \leq p,q \leq \infty$. Fix a window function $g \in \mathcal{S}\left(\mathbb{R}^d\right) \setminus \{0\}$. Also let $M(p,q,\omega)\left(\mathbb{R}^d\right)$ denote the subspace of tempered distributions $\mathcal{S}'\left(\mathbb{R}^d\right)$ consisting of $f \in \mathcal{S}'\left(\mathbb{R}^d\right)$ such that the Gabor transform $V_g f$ of f is in the weighted Lorentz space $L(p,q,\omega d\mu)\left(\mathbb{R}^{2d}\right)$. We endow it with the norm $\|f\|_{M(p,q,\omega)} = \|V_g f\|_{p,q,\omega}$, where $\|.\|_{p,q,\omega}$ is the norm of the weighted Lorentz space. It is known that $M(p,q,\omega)\left(\mathbb{R}^d\right)$ is a Banach space and different windows yield equivalent norms [22]. If $\omega = 1$, then $M(p,q,\omega)\left(\mathbb{R}^d\right) = M(p,q)\left(\mathbb{R}^d\right)$ [12]. Also if $\omega = 1$ and $p = q$, then $M(p,q,\omega)\left(\mathbb{R}^d\right) = M(p,p)\left(\mathbb{R}^d\right) = M^{p,p}\left(\mathbb{R}^d\right) = M^p\left(\mathbb{R}^d\right)$. That means $M(p,q,\omega)\left(\mathbb{R}^d\right)$ is a generalization of the usual modulation space $M^p\left(\mathbb{R}^d\right)$. The space $M(p,q,\omega)\left(\mathbb{R}^d\right)$ is defined and studied in [22].

Definition 1 *Let w and ω be two weight functions on $\mathbb{R}^{2d}$ and $1 \leq p,q \leq \infty$. Fix a non-zero window $g \in \mathcal{S}\left(\mathbb{R}^d\right)$. Let us denote by $Q_{g,w}^{M(p,q,\omega)}\left(\mathbb{R}^d\right)$ the subspace of all tempered distributions $f \in \mathcal{S}'\left(\mathbb{R}^d\right)$ such that the Toeplitz transform $Tp_g(w)f$ of f is in the space $M(p,q,\omega)\left(\mathbb{R}^d\right)$. Since the Toeplitz operator is injective, it is easy to see that*

$$\|f\|_Q = \|Tp_g(w)f\|_{M(p,q,\omega)}$$

is a norm on the vector space $Q_{g,w}^{M(p,q,\omega)}\left(\mathbb{R}^d\right)$.

Proposition 2 *Let $1 < p < \infty$, $1 \leq q < \infty$ and ω be a weight function of polynomial type on $\mathbb{R}^{2d}$. Then the space $Q_{g,w}^{M(p,q,\omega)}\left(\mathbb{R}^d\right)$ is independent of the choice of the window function $g \in \mathcal{S}\left(\mathbb{R}^d\right)$.*

Proof It is known by Proposition 2.6 in [22] that $M(p,q,\omega)\left(\mathbb{R}^d\right)$ is independent of the choice of the window function $g \in \mathcal{S}\left(\mathbb{R}^d\right)$. Let $g,\ g_0 \in \mathcal{S}\left(\mathbb{R}^d\right)$. Take any $f \in Q_{g,w}^{M(p,q,\omega)}\left(\mathbb{R}^d\right)$. Since

$$C_1 \|Tp_g(w)f\|_{M(p,q,\omega)} \leq \|Tp_{g_0}(w)f\|_{M(p,q,\omega)} \leq C_2 \|Tp_g(w)f\|_{M(p,q,\omega)}$$

for some $C_1,\ C_2 > 0$, then $Q_{g,w}^{M(p,q,\omega)}\left(\mathbb{R}^d\right)$ is also independent of the choice of the windows. □

Remark 3 *Let $1 \leq p,q \leq \infty$. In [2] a space $Q_{(g,w)}^{p,q}\left(\mathbb{R}^d\right)$ is defined as counter image of standard modulation space $M^{p,q}\left(\mathbb{R}^d\right)$ under the Toeplitz operator with symbol w. It is proven in Theorem 3.5 and Corollary 3.6 in [2] that $Q_{(g,w)}^{p,q}\left(\mathbb{R}^d\right) = M_w^{p,q}\left(\mathbb{R}^d\right)$ for certain w, where $M_w^{p,q}\left(\mathbb{R}^d\right)$ is the weighted modulation space. This relation was extended in [15] to all polynomially moderate weights. Let us take $w(x,y) = \left(1 + |x|^2 + |y|^2\right)^{\frac{s}{2}}$ for $s \in \mathbb{R}$. It is known that $M_s^{2,2}\left(\mathbb{R}^d\right)$ coincides with the usual Sobolev-Shubin space $Q_s\left(\mathbb{R}^d\right)$, where $M_s^{2,2}\left(\mathbb{R}^d\right) = M_w^{2,2}\left(\mathbb{R}^d\right)$ [2], [4]. Thus $Q_{(g,w)}^{p,q}\left(\mathbb{R}^d\right)$ is a generalization of the usual Sobolev-Shubin space $Q_s\left(\mathbb{R}^d\right)$.*

Now we return to our space $Q_{g,w}^{M(p,q,\omega)}\left(\mathbb{R}^d\right)$. Since

$$M(p,q,\omega)\left(\mathbb{R}^d\right) = M(p,q)\left(\mathbb{R}^d\right) = M^{p,q}\left(\mathbb{R}^d\right)$$

for $p = q$, $\omega = 1$, and $Q_{(g,w)}^{p,q}\left(\mathbb{R}^d\right)$ is a generalization of the usual Sobolev-Shubin space $Q_s\left(\mathbb{R}^d\right)$, then by the above remarks $Q_{g,w}^{M(p,q,\omega)}\left(\mathbb{R}^d\right)$ is also an another generalization of usual Sobolev-Shubin space $Q_s\left(\mathbb{R}^d\right)$.

It is known that $M_w^{p,p}\left(\mathbb{R}^d\right)$ is a Banach space [14]. We know by Theorem 3.5 and Corollary 3.6 in [2] that $Q_{(g,w)}^{p,p}\left(\mathbb{R}^d\right) = M_w^{p,p}\left(\mathbb{R}^d\right)$ for certain w. We also know that $Q_{g,w}^{M(p,p,\omega)}\left(\mathbb{R}^d\right) = Q_{(g,w)}^{p,p}\left(\mathbb{R}^d\right)$ for $\omega = 1$. Then

$$Q_{g,w}^{M(p,p,\omega)}\left(\mathbb{R}^d\right) = Q_{g,w}^{M(p,p)}\left(\mathbb{R}^d\right) = Q_{(g,w)}^{p,p}\left(\mathbb{R}^d\right) = M_w^{p,p}\left(\mathbb{R}^d\right)$$

for certain w and for $\omega = 1$. Hence $(Q_{g,w}^{M(p,p,\omega)}\left(\mathbb{R}^d\right), \|.\|_Q)$ is a Banach space for certain w and for $\omega = 1$.

Theorem 4 *Let ω be a weight function of polynomial type on $\mathbb{R}^{2d}$.*

1. If w is a submultiplicative weight function, then the space $Q_{g,w}^{M(p,q,\omega)}\left(\mathbb{R}^d\right)$ is invariant under the time-frequency shifts.
2. If w is a bounded weight function and $1 < p < \infty$, $1 \leq q < \infty$, then the function $z = (z_1, z_2) \to \pi(z) f = M_{z_2} T_{z_1} f$ of $\mathbb{R}^{2d}$ into $Q_{g,w}^{M(p,q,\omega)}\left(\mathbb{R}^d\right)$ is continuous.

Proof 1. Let $f \in Q_{g,w}^{M(p,q,\omega)}\left(\mathbb{R}^d\right)$ and $z_1, z_2 \in \mathbb{R}^d$. Then we have $Tp_g(w) f = V_g^*(wV_g f) \in M(p,q,\omega)\left(\mathbb{R}^d\right)$ and $V_g V_g^*(wV_g f) \in L(p,q,\omega d\mu)\left(\mathbb{R}^{2d}\right)$. By using the equalities

$$\overline{V_\gamma(M_t T_u g)(x,y)} = e^{-2\pi i x(t-y)} V_g \gamma(u-x, t-y)$$

and

$$T_{(z_1,z_2)} V_g f(x,y) = e^{2\pi i(y-z_2)z_1} V_g(M_{z_2} T_{z_1} f)(x,y),$$

we write

$$\begin{aligned}
& \left|V_g V_g^*(wV_g \pi(z) f)(u,t)\right| \qquad (2.1)\\
= & \left|V_g V_g^*(wV_g(M_{z_2} T_{z_1} f))(u,t)\right| = \left|\langle wV_g(M_{z_2} T_{z_1} f), V_g(M_t T_u g)\rangle\right| \\
= & \left|\iint_{R^{2d}} w(x,y) V_g(M_{z_2} T_{z_1} f)(x,y) V_g g(u-x, t-y) e^{-2\pi i x(t-y)} dxdy\right| \\
= & \left|\iint_{R^{2d}} w(x,y) e^{-2\pi i(y-z_2)z_1} T_{(z_1,z_2)} V_g f(x,y) V_g g(u-x, t-y) e^{-2\pi i x(t-y)} dxdy\right| \\
\leq & \iint_{R^{2d}} |w(z_1+v_1, z_2+v_2)| \, |V_g f(v_1,v_2)| \, |V_g g((u-v_1)-z_1, (t-v_2)-z_2)| \, dv_1 dv_2 \\
\leq & \iint_{R^{2d}} w(z_1,z_2) \, |w(v_1,v_2)| \, |V_g f(v_1,v_2)| \, \left|T_{(z_1,z_2)} V_g g((u-v_1),(t-v_2))\right| dv_1 dv_2 \\
= & \, w(z_1,z_2) \left(|wV_g f| * \left|T_{(z_1,z_2)} V_g g\right|\right)(u,t).
\end{aligned}$$

As ω is a weight function of polynomial type and $V_g g \in \mathcal{S}\left(\mathbb{R}^{2d}\right)$, then $V_g g \in L_\omega^1\left(\mathbb{R}^{2d}\right)$. Also, by Proposition 3.1 in [5], $L(p,q,\omega d\mu)\left(\mathbb{R}^{2d}\right)$ is a Banach module over $L_\omega^1\left(\mathbb{R}^{2d}\right)$ and by Theorem 2.5 in [22], $wV_g f \in$

$L(p,q,\omega d\mu)\left(\mathbb{R}^{2d}\right)$. Then by (2.1) we obtain

$$\begin{aligned}
\|\pi(z) f\|_Q &= \|M_{z_2}T_{z_1}f\|_Q = \left\|V_g^*\left(wV_g\left(M_{z_2}T_{z_1}f\right)\right)\right\|_{M(p,q,\omega)} \\
&= \left\|V_gV_g^*\left(wV_g\left(M_{z_2}T_{z_1}f\right)\right)\right\|_{p,q,\omega} \\
&\leq w(z_1,z_2)\left\|\left|wV_gf\right| * \left|T_{(z_1,z_2)}V_gg\right|\right\|_{p,q,\omega} \\
&\leq w(z_1,z_2)\left\|wV_gf\right\|_{p,q,\omega}\left\|T_{(z_1,z_2)}V_gg\right\|_{1,\omega} \\
&\leq w(z_1,z_2)\,\omega(z_1,z_2)\left\|wV_gf\right\|_{p,q,\omega}\left\|V_gg\right\|_{1,\omega} < \infty.
\end{aligned}$$

Thus $Q_{g,w}^{M(p,q,\omega)}\left(\mathbb{R}^d\right)$ is invariant under the time-frequency shifts.

2. Let $f \in Q_{g,w}^{M(p,q,\omega)}\left(\mathbb{R}^d\right)$ and $z=(z_1,z_2)\in\mathbb{R}^{2d}$. By Theorem 2.5 in [22], we write

$$\begin{aligned}
\|\pi(z) f - f\|_Q &= \|M_{z_2}T_{z_1}f - f\|_Q \\
&= \left\|V_g^*\left(wV_g\left(M_{z_2}T_{z_1}f - f\right)\right)\right\|_{M(p,q,\omega)} \\
&\leq \left\|V_gg\right\|_{1,\omega}\left\|wV_g\left(M_{z_2}T_{z_1}f - f\right)\right\|_{p,q,\omega} \\
&\leq C\left\|V_gg\right\|_{1,\omega}\left\|V_g\left(M_{z_2}T_{z_1}f\right) - V_gf\right\|_{p,q,\omega},
\end{aligned} \tag{2.2}$$

where $C=\sup w(x,y)$. Using the equality

$$T_{(z_1,z_2)}V_gf(x,y) = e^{2\pi i(y-z_2)z_1}V_g\left(M_{z_2}T_{z_1}f\right)(x,y)$$

we have

$$\begin{aligned}
&\left\|V_g\left(M_{z_2}T_{z_1}f\right) - V_gf\right\|_{p,q,\omega} \\
= &\left\|e^{-2\pi i(y-z_2)z_1}T_{(z_1,z_2)}V_gf - V_gf\right\|_{p,q,\omega} \\
\leq &\left\|e^{-2\pi i(y-z_2)z_1}\left(T_{(z_1,z_2)}V_gf - V_gf\right)\right\|_{p,q,\omega} \\
&+\left\|\left(e^{-2\pi i(y-z_2)z_1} - 1\right)V_gf\right\|_{p,q,\omega} \\
= &\left\|\left(T_{(z_1,z_2)}V_gf - V_gf\right)(x,y)\right\|_{p,q,\omega} + \left\|\left(e^{-2\pi i(y-z_2)z_1} - 1\right)V_gf\right\|_{p,q,\omega}.
\end{aligned} \tag{2.3}$$

Since the translation operator is continuous from $\mathbb{R}^{2d}$ into $L(p,q,\omega d\mu)\left(\mathbb{R}^{2d}\right)$ by Proposition 2.2 in [5], then $\left\|T_{(z_1,z_2)}V_gf - V_gf\right\|_{p,q,\omega} \to 0$ as (z_1,z_2) tends to zero. Moreover, it is known that $\left\|\left(e^{-2\pi i(y-z_2)z_1} - 1\right)V_gf\right\|_{p,q,\omega}$ tends to zero as (z_1,z_2) tends to zero by the proof of Proposition 2.9 in [22]. Hence $\left\|V_g\left(M_{z_2}T_{z_1}f\right) - V_gf\right\|_{p,q,\omega} \to 0$ as (z_1,z_2) tends to zero. Finally by (2.2) and (2.3) we obtain

$$\|M_{z_2}T_{z_1}f - f\|_Q \leq C\left\|V_gg\right\|_{1,\omega}\left\|V_g\left(M_{z_2}T_{z_1}f\right) - V_gf\right\|_{p,q,\omega} \to 0$$

as (z_1,z_2) tends to zero. This completes the proof. □

Lemma 5 *Let* $1 \leq p, q < \infty$. *Assume that* ω *and* w *are two weight functions on* $\mathbb{R}^{2d}$.

1. If w is bounded, $M(p,q,\omega)(\mathbb{R}^d)$ is continuously embedded into $Q_{g,w}^{M(p,q,\omega)}(\mathbb{R}^d)$, i.e.

$$M(p,q,\omega)(\mathbb{R}^d) \hookrightarrow Q_{g,w}^{M(p,q,\omega)}(\mathbb{R}^d).$$

2. If $|\omega(z)| \leq C(1+|z|)^N$ for a fix $N \in \mathbb{N}$ and w is bounded then

$$\mathcal{S}(\mathbb{R}^d) \subset Q_{g,w}^{M(p,q,\omega)}(\mathbb{R}^d).$$

Proof It is known by Proposition 2.3 in [22] that $\mathcal{S}(\mathbb{R}^d) \subset M(p,q,\omega)(\mathbb{R}^d)$. Let $f \in M(p,q,\omega)(\mathbb{R}^d)$. Then $V_g f \in L(p,q,\omega d\mu)(\mathbb{R}^{2d})$. As w is bounded, by Theorem 2.5 in [22]

$$\begin{aligned} \|f\|_Q &= \left\|V_g^*(wV_g f)\right\|_{M(p,q,\omega)} \leq \|V_g g\|_{1,\omega} \|wV_g f\|_{p,q,\omega} \\ &\leq \sup_{(x,y)\in\mathbb{R}^{2d}} w(x,y) \|V_g g\|_{1,\omega} \|V_g f\|_{p,q,\omega} = K\|f\|_{M(p,q,\omega)} < \infty. \end{aligned} \tag{2.4}$$

This implies $f \in Q_{g,w}^{M(p,q,\omega)}(\mathbb{R}^d)$. Hence

$$M(p,q,\omega)(\mathbb{R}^d) \subset Q_{g,w}^{M(p,q,\omega)}(\mathbb{R}^d). \tag{2.5}$$

Also by (2.4) the unite map I of $M(p,q,\omega)(\mathbb{R}^d)$ into $Q_{g,w}^{M(p,q,\omega)}(\mathbb{R}^d)$ is continuous. That means $M(p,q,\omega)(\mathbb{R}^d) \hookrightarrow Q_{g,w}^{M(p,q,\omega)}(\mathbb{R}^d)$.

It is known by Proposition 2.3 in [22] that $\mathcal{S}(\mathbb{R}^d) \subset M(p,q,\omega)(\mathbb{R}^d)$. The proof of 2) is completed by (2.5). □

Proposition 6 *If* $1 \leq q_1 \leq q_2 \leq \infty$, *then* $Q_{g,w}^{M(p,q_1,\omega)}(\mathbb{R}^d) \subset Q_{g,w}^{M(p,q_2,\omega)}(\mathbb{R}^d)$.

Proof Since $1 \leq q_1 \leq q_2 \leq \infty$, then $L(p,q_1,\omega d\mu)(\mathbb{R}^d) \hookrightarrow L(p,q_2,\omega d\mu)(\mathbb{R}^d)$ by Proposition 2.5 in [5]. Hence $M(p,q_1,\omega)(\mathbb{R}^d) \hookrightarrow M(p,q_2,\omega)(\mathbb{R}^d)$. This implies $Q_{g,w}^{M(p,q_1,\omega)}(\mathbb{R}^d) \hookrightarrow Q_{g,w}^{M(p,q_2,\omega)}(\mathbb{R}^d)$. □

Proposition 7 *Let* w_1, w_2 *and* ω_1, ω_2 *be weight functions on* $\mathbb{R}^{2d}$. *If* $w_2 \preceq w_1$ *and* $\omega_2 \preceq \omega_1$ *then* $Q_{g,w_1}^{M(p,q,\omega_1)}(\mathbb{R}^d) \subset Q_{g,w_2}^{M(p,q,\omega_2)}(\mathbb{R}^d)$.

Proof Let $f \in Q_{g,w_1}^{M(p,q,\omega_1)}(\mathbb{R}^d)$. Then $f \in \mathcal{S}'(R^d)$ and $Tp_g(w_1)f = V_g^*(w_1 V_g f) \in M(p,q,\omega_1)(\mathbb{R}^d)$ and $w_1 V_g f \in L(p,q,\omega_1 d\mu)(\mathbb{R}^{2d})$. Since $w_2 \preceq w_1$, $\omega_2 \preceq \omega_1$ and weighted Lorentz space is a solid space, then by Proposition 2.14 in [22] we have

$$\|w_2 V_g f\|_{p,q,\omega_2} \leq C\|w_1 V_g f\|_{p,q,\omega_2} \leq C\|w_1 V_g f\|_{p,q,\omega_1} < \infty.$$

Thus $w_2 V_g f \in L(p,q,\omega_2 d\mu)(\mathbb{R}^{2d})$. By Theorem 2.5 in [22], we write $V_g^*(w_2 V_g f) = Tp_g(w_2)f \in M(p,q,\omega_2)(\mathbb{R}^d)$. Thus we obtain $f \in Q_{g,w_2}^{M(p,q,\omega_2)}(\mathbb{R}^d)$. That means $Q_{g,w_1}^{M(p,q,\omega_1)}(\mathbb{R}^d) \subset Q_{g,w_2}^{M(p,q,\omega_2)}(\mathbb{R}^d)$. □

3. Boundedness of Toeplitz operators

Theorem 8 *Let ω_1 and ω_2 be two weight functions of polynomial type on $\mathbb{R}^{2d}$ and $g \in \mathcal{S}\left(\mathbb{R}^d\right)\setminus\{0\}$ be a window function. Then*

a. If $p,q \in (1,\infty)$, $t' \in (1,\infty)$, $s \le t' \le r$ and $F \in W\left(L^r, L^s\right)$, then the Toeplitz operator

$$Tp_g(F) : M\left(tp, tq, \omega_1\right)\left(\mathbb{R}^d\right) \to M\left(\left(tp'\right)', \left(tq'\right)', \omega_2\right)\left(\mathbb{R}^d\right)$$

is bounded, where $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$ and $\frac{1}{t}+\frac{1}{t'}=1$. Moreover we have the norm estimate

$$\left\|Tp_g(F)\right\| \le \|F\|_{W(L^r,L^s)}.$$

b. If $t \in [1,\infty)$, $s \le t \le r$ and $F \in W\left(L^r, L^s\right)$, then the Toeplitz operator

$$Tp_g(F) : M\left(\infty, \infty, \omega_1\right)\left(\mathbb{R}^d\right) \to M\left(t, t, \omega_2\right)\left(\mathbb{R}^d\right)$$

is bounded. Moreover we have the norm estimate

$$\left\|Tp_g(F)\right\| \le \|F\|_{W(L^r,L^s)}.$$

c. If $t \in (1,\infty)$, $s \le t' \le r$ and $F \in W\left(L^r, L^s\right)$, then the Toeplitz operator

$$Tp_g(F) : M\left(t, t, \omega_1\right)\left(\mathbb{R}^d\right) \to M\left(1, 1, \omega_2\right)\left(\mathbb{R}^d\right)$$

is bounded, where $\frac{1}{t}+\frac{1}{t'}=1$. Also we have the norm estimate

$$\left\|Tp_g(F)\right\| \le \|F\|_{W(L^r,L^s)}.$$

Proof **a.** Let $t < \infty$, $f \in M\left(tp, tq, \omega_1\right)\left(\mathbb{R}^d\right)$ and $h \in M\left(tp', tq', \omega_2\right)\left(\mathbb{R}^d\right)$. Then $f \in M(tp,tq)\left(\mathbb{R}^d\right)$ and $h \in M\left(tp', tq'\right)\left(\mathbb{R}^d\right)$ and so we write $V_g f \in L(tp,tq)\left(\mathbb{R}^{2d}\right)$ and $V_g h \in L\left(tp', tq'\right)\left(\mathbb{R}^{2d}\right)$. Since $V_g f \in L(tp,tq)\left(\mathbb{R}^{2d}\right)$, then $\left\|V_g f\right\|_{tp,tq}^{*} < \infty$. By using the equality $\left(|f|^t\right)^{*} = \left(f^{*}\right)^t$ for $t \in (0,\infty)$ (see [6]), we obtain

$$\begin{aligned}
\left\|V_g f\right\|_{tp,tq}^{*} &= \left(\frac{tq}{tp}\int_0^\infty x^{\frac{tq}{tp}-1}\left(\left(V_g f\right)^{*}(x)\right)^{tq} dx\right)^{\frac{1}{tq}} \qquad (3.6)\\
&= \left(\frac{q}{p}\int_0^\infty x^{\frac{q}{p}-1}\left[\left(\left(V_g f\right)^{*}(x)\right)^{t}\right]^{q} dx\right)^{\frac{1}{tq}}\\
&= \left[\left(\frac{q}{p}\int_0^\infty x^{\frac{q}{p}-1}\left[\left(\left|V_g f\right|^{t}\right)^{*}(x)\right]^{q} dx\right)^{\frac{1}{q}}\right]^{\frac{1}{t}}\\
&= \left(\left\|\left|V_g f\right|^{t}\right\|_{p,q}^{*}\right)^{\frac{1}{t}}.
\end{aligned}$$

Thus we have $|V_g f|^t \in L(p,q)(\mathbb{R}^{2d})$. Similarly, we obtain $|V_g h|^t \in L(p',q')(\mathbb{R}^{2d})$. Hence, applying the Hölder inequality for Lorentz spaces and using (3.6), we write

$$\begin{aligned} \|V_g f \cdot V_g h\|_t^t &= \left\|(V_g f)^t (V_g h)^t\right\|_1 = \left\||V_g f|^t |V_g h|^t\right\|_1 \\ &\leq \left\||V_g f|^t\right\|_{p,q} \left\||V_g h|^t\right\|_{p',q'} \\ &= \|V_g f\|_{tp,tq}^t \|V_g h\|_{tp',tq'}^t \end{aligned}$$

and

$$\|V_g f \cdot V_g h\|_t \leq \|V_g f\|_{tp,tq} \|V_g h\|_{tp',tq'} . \tag{3.7}$$

Since $F \in W(L^r, L^s) \subset W\left(L^{t'}, L^{t'}\right) = L^{t'}(\mathbb{R}^{2d})$, then we have

$$\|F\|_{t'} \leq \|F\|_{W(L^r,L^s)} . \tag{3.8}$$

Moreover, using (3.7) and (3.8) and applying again Hölder inequality, we obtain

$$\begin{aligned} |\langle Tp_g(F) f, h\rangle| &= \left|\langle V_g^*(F V_g f), h\rangle\right| = |\langle F V_g f, V_g h\rangle| \\ &= \left|\iint_{\mathbb{R}^{2d}} F(x,y) V_g f(x,y) \overline{V_g h(x,y)} dxdy\right| \\ &\leq \iint_{\mathbb{R}^{2d}} |F(x,y)| \left|(V_g f \cdot V_g h)(x,y)\right| dxdy \\ &\leq \|F\|_{t'} \|V_g f \cdot V_g h\|_t \\ &\leq \|F\|_{t'} \|V_g f\|_{tp,tq} \|V_g h\|_{tp',tq'} \\ &\leq \|F\|_{W(L^r,L^s)} \|f\|_{M(tp,tq)} \|h\|_{M(tp',tq')} \\ &\leq \|F\|_{W(L^r,L^s)} \|f\|_{M(tp,tq,\omega_1)} \|h\|_{M(tp',tq',\omega_2)}. \end{aligned} \tag{3.9}$$

If $(tp')', (tq')' \neq \infty$, then $\left(M\left((tp')', (tq')', \omega_2\right)(\mathbb{R}^d)\right)^* = M(tp', tq', \omega_2)(\mathbb{R}^d)$ by Theorem 2.16 in [22]. Thus we obtain from (3.9) that

$$\begin{aligned} \|Tp_g(F) f\|_{M((tp')',(tq')',\omega_2)} &= \sup_{0\neq h\in M(tp',tq',\omega_2)} \frac{|\langle Tp_g(F) f, h\rangle|}{\|h\|_{M(tp',tq',\omega_2)}} \\ &\leq \|F\|_{W(L^r,L^s)} \|f\|_{M(tp,tq,\omega_1)} . \end{aligned}$$

Hence $Tp_g(F)$ is bounded. We also have

$$\|Tp_g(F)\| = \sup_{0\neq f\in M(tp,tq,\omega_1)} \frac{\|Tp_g(F) f\|_{M((tp')',(tq')',\omega_2)}}{\|f\|_{M(tp,tq,\omega_1)}} \leq \|F\|_{W(L^r,L^s)} .$$

b. Let us take any $f \in M(\infty,\infty,\omega_1)(\mathbb{R}^d)$ and $h \in M(t',t',\omega_2)(\mathbb{R}^d)$. Then $V_g f \in L(\infty,\infty,\omega_1 d\mu)(\mathbb{R}^{2d})$ and $V_g h \in L(t',t',\omega_2 d\mu)(\mathbb{R}^{2d})$, respectively. Since $L(\infty,\infty,\omega_1 d\mu)(\mathbb{R}^{2d}) \subset L^\infty(\mathbb{R}^{2d})$ and $L(t',t',\omega_2 d\mu)(\mathbb{R}^{2d}) \subset L^{t'}(\mathbb{R}^{2d})$, we have $V_g f \in L^\infty(\mathbb{R}^{2d})$ and $V_g h \in L^{t'}(\mathbb{R}^{2d})$. By (3.8) and Hölder inequality we have

$$\begin{aligned} |\langle Tp_g(F) f, h\rangle| &= \left| \iint_{\mathbb{R}^{2d}} F(x,y) V_g f(x,y) \overline{V_g h(x,y)} dxdy \right| \\ &\leq \|F V_g f\|_t \|V_g h\|_{t'} \leq \|F\|_t \|f\|_{M(\infty,\infty)} \|h\|_{M(t',t')} \\ &\leq \|F\|_{W(L^r,L^s)} \|f\|_{M(\infty,\infty)} \|h\|_{M(t',t')} \\ &\leq \|F\|_{W(L^r,L^s)} \|f\|_{M(\infty,\infty,\omega_1)} \|h\|_{M(t',t',\omega_2)} \end{aligned}$$

and

$$\|Tp_g(F) f\|_{M(t,t,\omega_2)} \leq \|F\|_{W(L^r,L^s)} \|f\|_{M(\infty,\infty,\omega_1)}. \tag{3.10}$$

Then $Tp_g(F)$ is bounded. By (3.10) we obtain $\|Tp_g(F)\| \leq \|F\|_{W(L^r,L^s)}$.

c. Let $f \in M(t,t,\omega_1)(\mathbb{R}^d)$ and $h \in M(\infty,\infty,\omega_2)(\mathbb{R}^d)$ be given. Then $V_g f \in L(t,t,\omega_1 d\mu)(\mathbb{R}^{2d}) \subset L^t(\mathbb{R}^{2d})$ and $V_g h \in L(\infty,\infty,\omega_2 d\mu)(\mathbb{R}^{2d}) \subset L^\infty(\mathbb{R}^{2d})$, respectively. Applying again the Hölder inequality and (3.8) we have

$$\begin{aligned} |\langle Tp_g(F) f, h\rangle| &\leq \|V_g h\|_\infty \iint_{\mathbb{R}^{2d}} |F(x,y)| |V_g f(x,y)| dxdy \\ &\leq \|F\|_{t'} \|V_g f\|_t \|V_g h\|_\infty \leq \|F\|_{W(L^r,L^s)} \|f\|_{M(t,t)} \|h\|_{M(\infty,\infty)} \\ &\leq \|F\|_{W(L^r,L^s)} \|f\|_{M(t,t,\omega_1)} \|h\|_{M(\infty,\infty,\omega_2)} \end{aligned}$$

and

$$\|Tp_g(F) f\|_{M(1,1,\omega_2)} \leq \|F\|_{W(L^r,L^s)} \|f\|_{M(t,t,\omega_1)}. \tag{3.11}$$

Hence $Tp_g(F)$ is bounded and from (3.11) we have $\|Tp_g(F)\| \leq \|F\|_{W(L^r,L^s)}$. This completes the proof.

□

Theorem 9 *Let ω be a moderate weight and $g \in \bigcap_{1\leq k,l<\infty} M(k,l,\omega)(\mathbb{R}^d)$. If $1 \leq s \leq r \leq \infty$ and $F \in W(L^r, L^s_\omega)$ then the Toeplitz operator*

$$Tp_g(F) : M(p,q,\omega)(\mathbb{R}^d) \to M(p,q,\omega)(\mathbb{R}^d)$$

is bounded. We have the norm estimate

$$\|Tp_g(F)\| \leq C \|F\|_{W(L^r,L^s_\omega)}$$

for some $C > 0$.

Proof Since $s \le r$, $W(L^r, L^s_\omega)(\mathbb{R}^{2d}) \subset W(L^s, L^s_\omega)(\mathbb{R}^{2d}) = L^s_\omega(\mathbb{R}^{2d})$ and

$$\|F\|_{s,\omega} \le \|F\|_{W(L^r, L^s_\omega)} \tag{3.12}$$

for all $F \in W(L^r, L^s_\omega)(\mathbb{R}^{2d})$. Let $B\left(M(p,q,\omega)(\mathbb{R}^d), M(p,q,\omega)(\mathbb{R}^d)\right)$ be the space of the bounded linear operators from $M(p,q,\omega)(\mathbb{R}^d)$ into $M(p,q,\omega)(\mathbb{R}^d)$.

Define an operator A from $L^1_\omega(\mathbb{R}^{2d})$ into $B\left(M(p,q,\omega)(\mathbb{R}^d), M(p,q,\omega)(\mathbb{R}^d)\right)$ by $A(F) = Tp_g(F)$. Take any $f \in M(p,q,\omega)(\mathbb{R}^d)$ and $h \in M(p',q',\omega)(\mathbb{R}^d)$, where $\frac{1}{p}+\frac{1}{p'} = 1$. Suppose that $F \in W\left(L^1, L^1_\omega\right)(\mathbb{R}^{2d}) = L^1_\omega(\mathbb{R}^{2d})$. Applying Hölder inequality we obtain

$$\begin{aligned}
|\langle A(F) f, h\rangle| &= |\langle Tp_g(F) f, h\rangle| = \left|\left\langle V_g^*(FV_g f), h\right\rangle\right| = |\langle FV_g f, V_g h\rangle| \\
&= \left|\iint_{\mathbb{R}^{2d}} F(x,y) V_g f(x,y) \overline{V_g h(x,y)} dxdy\right| \\
&\le \iint_{\mathbb{R}^{2d}} |F(x,y)| \; |V_g f(x,y)| \; |V_g h(x,y)| \, dxdy \\
&= \iint_{\mathbb{R}^{2d}} |F(x,y)| \; |\langle f, M_y T_x g\rangle| \; |\langle h, M_y T_x g\rangle| \, dxdy \\
&\le \iint_{\mathbb{R}^{2d}} |F(x,y)| \, \|f\|_{M(p,q,\omega)} \|M_y T_x g\|_{M(p',q',\omega)} \|h\|_{M(p',q',\omega)} \\
&\quad \|M_y T_x g\|_{M(p,q,\omega)} \, dxdy \\
&= \iint_{\mathbb{R}^{2d}} |F(x,y)| \, \|f\|_{M(p,q,\omega)} \, \omega^{\frac{1}{p'}}(x,y) \, \|g\|_{M(p',q',\omega)} \\
&\quad \|h\|_{M(p',q',\omega)} \, \omega^{\frac{1}{p}}(x,y) \, \|g\|_{M(p,q,\omega)} \, dxdy \\
&= \|f\|_{M(p,q,\omega)} \|g\|_{M(p',q',\omega)} \|h\|_{M(p',q',\omega)} \|g\|_{M(p,q,\omega)} \iint_{\mathbb{R}^{2d}} |F(x,y)| \\
&\quad \omega^{\frac{1}{p'}+\frac{1}{p}}(x,y) \, dxdy \\
&= \|f\|_{M(p,q,\omega)} \|g\|_{M(p',q',\omega)} \|h\|_{M(p',q',\omega)} \|g\|_{M(p,q,\omega)} \|F\|_{1,\omega}.
\end{aligned} \tag{3.13}$$

Thus by (3.13)

$$\begin{aligned}
\|A(F) f\|_{M(p,q,\omega)} &= \|Tp_g(F) f\|_{M(p,q,\omega)} = \sup_{0 \ne h \in M(p',q',\omega)} \frac{|\langle Tp_g(F) f, h\rangle|}{\|h\|_{M(p',q',\omega)}} \\
&\le \|g\|_{M(p',q',\omega)} \|g\|_{M(p,q,\omega)} \|f\|_{M(p,q,\omega)} \|F\|_{1,\omega}.
\end{aligned}$$

Hence

$$\begin{aligned} \|A(F)\| &= \|Tp_g(F)\| = \sup_{0\neq f\in M(p,q,\omega)} \frac{\|Tp_g(F)f\|_{M(p,q,\omega)}}{\|f\|_{M(p,q,\omega)}} \\ &\leq \|g\|_{M(p',q',\omega)} \|g\|_{M(p,q,\omega)} \|F\|_{1,\omega}. \end{aligned} \tag{3.14}$$

Finally the operator

$$A : L^1_\omega\left(\mathbb{R}^{2d}\right) \to B\left(M(p,q,\omega)\left(\mathbb{R}^d\right), M(p,q,\omega)\left(\mathbb{R}^d\right)\right) \tag{3.15}$$

is bounded.

Now define an operator A

$$A : L^\infty_\omega\left(\mathbb{R}^{2d}\right) = W\left(L^\infty, L^\infty_\omega\right)\left(\mathbb{R}^{2d}\right) \to B\left(M(p,q,\omega)\left(\mathbb{R}^d\right), M(p,q,\omega)\left(\mathbb{R}^d\right)\right)$$

by $A(F) = Tp_g(F)$. Take any $f \in M(p,q,\omega)\left(\mathbb{R}^d\right)$ and $h \in M(p',q',\omega)\left(\mathbb{R}^d\right)$. Then $V_g f \in L(p,q,\omega d\mu)\left(\mathbb{R}^{2d}\right)$, $V_g h \in L(p',q',\omega d\mu)\left(\mathbb{R}^{2d}\right)$. Again applying the Hölder inequality

$$\begin{aligned} |\langle A(F)f,h\rangle| &= |\langle Tp_g(F)f,h\rangle| = \left|\left\langle V_g^*(FV_gf), h\right\rangle\right| \\ &= |\langle FV_gf, V_gh\rangle| = \left|\iint_{\mathbb{R}^{2d}} F(x,y)\, V_gf(x,y)\, \overline{V_gh(x,y)}dxdy\right| \\ &\leq \iint_{\mathbb{R}^{2d}} |F(x,y)| \cdot |V_gf(x,y)| \cdot |V_gh(x,y)|\, dxdy \\ &\leq \|F\|_\infty \iint_{\mathbb{R}^{2d}} |V_gf(x,y)|\, |V_gh(x,y)|\, dxdy \\ &\leq \|F\|_\infty \|V_gf\|_{p,q,\omega} \|V_gh\|_{p',q',\omega} \\ &\leq \|F\|_{\infty,\omega} \|f\|_{M(p,q,\omega)} \|h\|_{M(p',q',\omega)}. \end{aligned} \tag{3.16}$$

By using (3.16) we have

$$\begin{aligned} \|A(F)f\|_{M(p,q,\omega)} &= \|Tp_g(F)f\|_{M(p,q,\omega)} \\ &= \sup_{0\neq h\in M(p',q',\omega)} \frac{|\langle Tp_g(F)f,h\rangle|}{\|h\|_{M(p',q',\omega)}} \\ &\leq \|F\|_{\infty,\omega} \|f\|_{M(p,q,\omega)}. \end{aligned} \tag{3.17}$$

Hence by (3.17)

$$\|A(F)\| = \|Tp_g(F)\| = \sup_{0\neq f\in M(p,q,\omega)} \frac{\|Tp_g(F)f\|_{M(p,q,\omega)}}{\|f\|_{M(p,q,\omega)}} \leq \|F\|_{\infty,\omega}.$$

That means the operator

$$A : L^\infty_\omega\left(\mathbb{R}^{2d}\right) \to B\left(M(p,q,\omega)\left(\mathbb{R}^d\right), M(p,q,\omega)\left(\mathbb{R}^d\right)\right) \tag{3.18}$$

is bounded. Combining (3.15) and (3.18) we obtain that

$$A : L^t_\omega\left(\mathbb{R}^{2d}\right) \to B\left(M\left(p,q,\omega\right)\left(\mathbb{R}^d\right), M\left(p,q,\omega\right)\left(\mathbb{R}^d\right)\right)$$

is bounded by interpolation theorem [[1], Theorem 5.5.1] for $1 \le t \le \infty$. That means the Toeplitz operator

$$Tp_g\left(F\right) : M\left(p,q,\omega\right)\left(\mathbb{R}^d\right) \to M\left(p,q,\omega\right)\left(\mathbb{R}^d\right)$$

is bounded for $1 \le t \le \infty$. Hence there exists $C > 0$ such that

$$\left\| A\left(F\right)\right\| = \left\| Tp_g\left(F\right)\right\| \le C\left\| F\right\|_{t,\omega}. \tag{3.19}$$

This implies that it is also true for $1 \le s \le \infty$. By (3.12) and (3.19) we have

$$\left\| A\left(F\right)\right\| = \left\| Tp_g\left(F\right)\right\| \le C\left\| F\right\|_{s,\omega} \le C\left\| f\right\|_{W\left(L^r, L^s_\omega\right)}.$$

□

Remark 10 *It is known by Proposition 2.3 in* [22] *that* $\mathcal{S}\left(\mathbb{R}^d\right) \subset M\left(k,l,\omega\right)\left(\mathbb{R}^d\right)$ *if* $\left|\omega\left(z\right)\right| \le C\left(1+\left|z\right|\right)^N$ *for a fix* $N \in \mathbb{N}$ *and* $1 \le k,l < \infty$. *Then* $\mathcal{S}\left(\mathbb{R}^d\right) \subset \bigcap\limits_{1\le k,l<\infty} M\left(k,l,\omega\right)\left(\mathbb{R}^d\right)$ *if* $\left|\omega\left(z\right)\right| \le C\left(1+\left|z\right|\right)^N$ *for a fix* $N \in \mathbb{N}$. *Hence, if* $g \in \mathcal{S}\left(\mathbb{R}^d\right)$, $1 \le s \le r \le \infty$ *and* $F \in W\left(L^r, L^s_\omega\right)\left(\mathbb{R}^d\right)$ *then the Toeplitz operator*

$$Tp_g\left(F\right) : M\left(p,q,\omega\right)\left(\mathbb{R}^d\right) \to M\left(p,q,\omega\right)\left(\mathbb{R}^d\right)$$

is bounded for $1 \le p,q < \infty$ *by Theorem 9.*

Proposition 11 *Let* $g \in \bigcap\limits_{1\le k,l<\infty} M\left(k,l,\omega\right)\left(\mathbb{R}^d\right)$. *If* $1 \le p,q < \infty$ *and* $F\omega^{\frac{1}{p}} \in L\left(p',q',\omega d\mu\right)\left(\mathbb{R}^{2d}\right)$ *then the Toeplitz operator*

$$Tp_g\left(F\right) : M\left(p,q,\omega\right)\left(\mathbb{R}^d\right) \to M\left(p,q,\omega\right)\left(\mathbb{R}^d\right)$$

is bounded, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof Suppose that $F\omega^{\frac{1}{p}} \in L\left(p',q',\omega d\mu\right)\left(\mathbb{R}^{2d}\right)$. Take any $f \in M\left(p,q,\omega\right)\left(\mathbb{R}^d\right)$ and $h \in M\left(p',q',\omega\right)\left(\mathbb{R}^d\right)$. Applying Hölder inequality we have by (3.13)

$$\begin{aligned} \left|\left\langle Tp_g\left(F\right)f, h\right\rangle\right| &\le \iint\limits_{\mathbb{R}^{2d}} \left|F\left(x,y\right)\right| \left|V_g f\left(x,y\right)\right| \left|\left\langle h, M_y T_x g\right\rangle\right| dxdy \qquad (3.20) \\ &\le \iint\limits_{\mathbb{R}^{2d}} \left|F\left(x,y\right)\right| \left|V_g f\left(x,y\right)\right| \left\|h\right\|_{M\left(p',q',\omega\right)} \left\|M_y T_x g\right\|_{M\left(p,q,\omega\right)} dxdy \\ &\le \iint\limits_{\mathbb{R}^{2d}} \left|F\left(x,y\right)\right| \left|V_g f\left(x,y\right)\right| \left\|h\right\|_{M\left(p',q',\omega\right)} \omega^{\frac{1}{p}}\left(x,y\right) \left\|g\right\|_{M\left(p,q,\omega\right)} dxdy \\ &= \left\|h\right\|_{M\left(p',q',\omega\right)} \left\|g\right\|_{M\left(p,q,\omega\right)} \iint\limits_{\mathbb{R}^{2d}} \left|F\left(x,y\right)\right| \omega^{\frac{1}{p}}\left(x,y\right) \left|V_g f\left(x,y\right)\right| dxdy \\ &\le \left\|h\right\|_{M\left(p',q',\omega\right)} \left\|g\right\|_{M\left(p,q,\omega\right)} \left\|f\right\|_{M\left(p,q,\omega\right)} \left\|F\omega^{\frac{1}{p}}\right\|_{p',q',\omega}. \end{aligned}$$

In analogy to (3.14), we have

$$\|Tp_g(F)\| \le \|g\|_{M(p,q,\omega)} \left\|F\omega^{\frac{1}{p}}\right\|_{p',q',\omega}. \tag{3.21}$$

Then the Toeplitz operator from $M(p,q,\omega)\left(\mathbb{R}^d\right)$ into $M(p,q,\omega)\left(\mathbb{R}^d\right)$ is bounded. This completes the proof. □

Remark 12 *It is known by Proposition 2.3 in* [22] *that* $\mathcal{S}\left(\mathbb{R}^d\right) \subset M(k,l,\omega)\left(\mathbb{R}^d\right)$ *if* $|\omega(z)| \le C(1+|z|)^N$ *for a fix* $N \in \mathbb{N}$ *and* $1 \le k,l < \infty$. *Thus if* $|\omega(z)| \le C(1+|z|)^N$ *for a fix* $N \in \mathbb{N}$, *then* $\mathcal{S}\left(\mathbb{R}^d\right) \subset \bigcap_{1\le k,l<\infty} M(k,l,\omega)\left(\mathbb{R}^d\right)$. *Hence if* $g \in \mathcal{S}\left(\mathbb{R}^d\right)$, $1 \le p,q < \infty$ *and* $F\omega^{\frac{1}{p}} \in L(p',q',\omega d\mu)\left(\mathbb{R}^{2d}\right)$, *then the Toeplitz operator*

$$Tp_g(F) : M(p,q,\omega)\left(\mathbb{R}^d\right) \to M(p,q,\omega)\left(\mathbb{R}^d\right)$$

is bounded by Proposition 11.

4. Hilbert-Schmidt and Schatten-class properties for symbols in $W(L^r, L^s)\left(\mathbb{R}^{2d}\right)$

Theorem 13 *Let* $1 \le p \le \infty$, $1 \le s \le r \le \infty$ *and* $g \in \mathcal{S}\left(\mathbb{R}^d\right)$. *If* $F \in W(L^r, L^s)\left(\mathbb{R}^d\right)$, *then* $Tp_g(F) : L^2\left(\mathbb{R}^d\right) \to L^2\left(\mathbb{R}^d\right)$ *is in the Schatten-class* S_p *and the inequality*

$$\|Tp_g(F)\|_{S_p} \le \|g\|_2^2 \|F\|_{W(L^r,L^s)}$$

holds.

Proof By Remark 10, the Toeplitz operator $Tp_g(F) : L^2\left(\mathbb{R}^d\right) \to L^2\left(\mathbb{R}^d\right)$ is bounded under these assumptions. We will show that $Tp_g(F)$ is in S_p. Let $p = r = s = 1$. For $z = (x,y) \in \mathbb{R}^{2d}$ we consider the rank one operator

$$\Lambda_z f = \langle f, M_y T_x g\rangle M_y T_x g, \quad f \in L^2\left(\mathbb{R}^d\right). \tag{4.22}$$

Then

$$\|\Lambda_z\|_{S_1} = \|g\|_2^2. \tag{4.23}$$

Hence the mapping $z \to \Lambda_z$ is continuous and the vector-valued integral

$$Tp_g(F) = \iint\limits_{\mathbb{R}^{2d}} F(x,y)\Lambda_z dxdy$$

is well defined. Also by (4.23) we write

$$\begin{aligned} \|Tp_g(F)\|_{S_1} &= \left\|\iint\limits_{\mathbb{R}^{2d}} F(x,y)\Lambda_z dxdy\right\|_{S_1} \le \\ &\le \iint\limits_{\mathbb{R}^{2d}} \|F(x,y)\Lambda_z\|_{S_1} dxdy = \iint\limits_{\mathbb{R}^{2d}} |F(x,y)| \|\Lambda_z\|_{S_1} dxdy \\ &= \|g\|_2^2 \iint\limits_{\mathbb{R}^{2d}} |F(x,y)| dxdy = \|g\|_2^2 \|F\|_1 . \end{aligned} \tag{4.24}$$

Now let $p=r=s=\infty$. Since $f\in L^2\left(\mathbb{R}^d\right)$ then by the proof of Theorem 8 we have

$$\begin{aligned} \left|\left\langle Tp_g(F)f,h\right\rangle\right| &= \left|\iint\limits_{\mathbb{R}^{2d}} F(x,y)\,V_gf(x,y)\,\overline{V_gh(x,y)}dxdy\right| \\ &\leq \|F\|_\infty \iint\limits_{\mathbb{R}^{2d}} |V_gf(x,y)|\,|V_gh(x,y)|\,dxdy \\ &\leq \|F\|_\infty \|V_gf\|_2 \|V_gh\|_2 = \|F\|_\infty \|f\|_2 \|h\|_2 \|g\|_2^2 . \end{aligned} \tag{4.25}$$

Hence

$$\|Tp_g(F)\|_\infty \leq \|F\|_\infty \|g\|_2^2 .$$

That means $Tp_g(F)$ is bounded on $L^2\left(\mathbb{R}^d\right)$. Since S_∞ denotes the algebra of all bounded operators on $L^2\left(\mathbb{R}^d\right)$, we have

$$\|Tp_g(F)\|_{S_\infty} \leq \|F\|_\infty \|g\|_2^2 .$$

Then by the interpolation theorem (see Theorem 2.11. in [23]), for $1\leq t,p\leq\infty$, $\left[L^1\left(\mathbb{R}^d\right),L^\infty\left(\mathbb{R}^d\right)\right]_\Theta = L^t\left(\mathbb{R}^d\right)$, $\left[S_1,S_\infty\right]_\Theta = S_p$ and $Tp_g(F)\in S_p$,

$$\|Tp_g(F)\|_{S_p} \leq \|F\|_t \|g\|_2^2 \tag{4.26}$$

for all $F\in L^t\left(\mathbb{R}^d\right)$. Hence $Tp_g(F)$ is in S_p.

Moreover, since $s\leq r$, there exists $1\leq t_0\leq\infty$ such that $s\leq t_0\leq r$. Hence $W(L^r,L^s)\left(\mathbb{R}^{2d}\right)\subset L^{t_0}\left(\mathbb{R}^{2d}\right)$ and

$$\|F\|_{t_0} \leq \|F\|_{W(L^r,L^s)} \tag{4.27}$$

for all $F\in W(L^r,L^s)\left(\mathbb{R}^{2d}\right)$. Finally by using (4.26) and (4.27), we obtain

$$\|Tp_g(F)\|_{S_p} \leq \|F\|_{t_0} \|g\|_2^2 \leq \|g\|_2^2 \|F\|_{W(L^r,L^s)}$$

for all $F\in W(L^r,L^s)$. This completes the proof. □

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