On integrability of Golden Riemannian structures

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Abstract: The main purpose of the present paper is to study the geometry of Riemannian manifolds endowed with Golden structures. We discuss the problem of integrability for Golden Riemannian structures by using a ϕ-operator which is applied to pure tensor fields. Also, the curvature properties for Golden Riemannian metrics and some properties of twin Golden Riemannian metrics are investigated. Finally, some examples are presented.

Key words: Golden structure, pure tensor, Riemannian manifold, twin metric

1. Introduction

Let M be a C^∞- manifold of finite dimension n. We denote by $\Im_s^r(M)$ the module over $F(M)$ of all C^∞-tensor fields of type (r, s) on M, i.e. of contravariant degree r and covariant degree s, where $F(M)$ is the algebra of C^∞-functions on M. Manifolds, tensor fields and connections are always assumed to be differentiable and of class C^∞.

Yano [25] introduced the notion of an f-structure which is a $(1, 1)$-tensor field of constant rank on M and satisfies the equality $f^3 + f = 0$. This notion is a generalization of almost complex and almost contact structures. In its turn, it has been generalized by Goldberg and Yano [2], who defined a polynomial structure of degree d which is a $(1, 1)$-tensor field f of constant rank on M and satisfies the equation $Q(f) = f^d + a_d f^{d-1} + ... + a_2 f + a_1 I = 0$, where $a_1, a_2, ..., a_d$ are real numbers and I is the identity tensor of type $(1, 1)$.

For a manifold M, let φ be a $(1, 1)$-tensor field on M. If the polynomial $X^2 - X - 1$ is the minimal polynomial for a structure φ satisfying $\varphi^2 - \varphi - I = 0$, then φ is called a golden structure on M and (M, φ) is a golden manifold [1, 4, 5]. This structure was inspired by the Golden Ratio, which was described by Johannes Kepler (1571–1630). The number $\eta = \frac{1+\sqrt{5}}{2} \approx 1.618...$, which is a solution of the equation $x^2 - x - 1 = 0$, is the golden ratio. We note that for golden structures, $\varphi \neq aI$, where $a \in R$. If $\varphi = aI$, $a = \frac{1+\sqrt{5}}{2}$, then its minimal polynomial is $X - a$. However, the minimal polynomial of the golden structure φ is $X^2 - X - 1$.

Let (M, g) be a Riemannian manifold endowed with the Golden structure φ such that [1, 4, 5]

$$g(\varphi X, Y) = g(X, \varphi Y), \tag{1.1}$$

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for all $X, Y \in \Im_0^1(M)$. If we substitute φX into X in (1.1), the equation (1.1) may also be written as

$$g(\varphi X, \varphi Y) = g(\varphi^2 X, Y) = g((\varphi + I)X, Y) = g(\varphi X, Y) + g(X, Y).$$

The Riemannian metric (1.1) is called φ-compatible and (M, φ, g) is named a Golden Riemannian manifold. Such Riemannian metrics are also referred to as pure metrics [6, 10-20].

Let φ be a $(1,1)$-tensor field on M, i.e. $\varphi \in \Im_1^1(M)$. A tensor field t of type (r, s) is called a pure tensor field with respect to φ if

$$t(\varphi X_1, X_2, ..., X_s; \overset{1}{\xi}, \overset{2}{\xi}, ..., \overset{r}{\xi}) = t(X_1, \varphi X_2, ..., X_s; \overset{1}{\xi}, \overset{2}{\xi}, ..., \overset{r}{\xi})$$

$$\vdots$$

$$= t(X_1, X_2, ..., \varphi X_s; \overset{1}{\xi}, \overset{2}{\xi}, ..., \overset{r}{\xi})$$

$$= t(X_1, X_2, ..., X_s; {}' \varphi \overset{1}{\xi}, \overset{2}{\xi}, ..., \overset{r}{\xi})$$

$$\vdots$$

$$= t(X_1, X_2, ..., X_s; \overset{1}{\xi}, \overset{2}{\xi}, ..., {}' \varphi \overset{r}{\xi})$$

for any $X_1, X_2, ..., X_s \in \Im_0^1(M)$ and $\overset{1}{\xi}, \overset{2}{\xi}, ..., \overset{r}{\xi} \in \Im_1^0(M)$, where ${}' \varphi$ is the adjoint operator of φ defined by

$$({}' \varphi \xi)(X) = \xi(\varphi X) = (\xi \circ \varphi)(X), X \in \Im_0^1(M), \xi \in \Im_1^0(M).$$

We define an operator

$$\phi_\varphi : \Im_s^0(M) \to \Im_{s+1}^0(M)$$

applied to the pure tensor field t of type (0,s) with respect to φ by [12, 15]

$$(\phi_\varphi t)(X, Y_1, ..., Y_s) = (\varphi X) t(Y_1, ..., Y_s) - X t(\varphi Y_1, ..., Y_s) \tag{1.2}$$

$$+ \sum_{\lambda=1}^{s} t(Y_1, ..., (L_{Y_\lambda} \varphi) X, ..., Y_s)$$

for any $X, Y_1, ..., Y_s \in \Im_0^1(M)$, where L_Y denotes the Lie differentiation with respect to Y.

Golden structure on a Riemannian manifod is important because this structure has relation with pure Riemannian metrics with respect to the structure. Pure metrics with respect to certain structures were studied by various authors (for example see [6–27], etc.). Since Riemannian golden and almost product structures are related to each other (see Theorem 2.3 in the present paper), the method of ϕ-operator used in the theory of almost product structures can be transferred to golden structures. Thus, in this paper a more difficult polynomial structure, $\varphi^2 - \varphi - I = 0$, is investigated by using the simple polynomial structure $\varphi^2 - I = 0$.

The paper is organized as follows. In section 2, a new sufficient condition of integrability for Golden Riemannian structures is given. Sections 3 and 4 deal with the some properties of twin Golden Riemannian metrics and the curvature properties of locally decomposable Golden Riemannian manifolds. Section 5 closes the paper with some examples of locally decomposable Golden Riemannian manifolds.

2. Integrable Golden Riemannian structures

Let M be a Golden manifold with a Golden structure φ. In order that the Golden structure φ is integrable, it is necessary and sufficient that it is possible to introduce a torsion-free affine connection ∇ with respect to which the structure tensor φ is covariantly constant. Also, we know that the integrability of φ is equivalent to the vanishing of the Nijenhuis tensor N_φ [1]. Now we shall introduce another possible sufficient condition of the integrability of Golden structures on Riemannian manifolds.

Theorem 2.1 *Let (M, φ, g) be a Golden Riemannian manifold. Then φ is integrable if $\phi_\varphi g = 0$.*

Proof As a direct consequence of (1.1) and $\nabla g = 0$, we obtain

$$g(X, (\nabla_Z \varphi)Y) = g((\nabla_Z \varphi)X, Y), \tag{2.3}$$

for all $X, Y \in \Im_0^1(M)$, where ∇ denotes the operator of the Riemannian covariant derivative with respect to g.

Using (2.3) and $[X, Y] = \nabla_X Y - \nabla_Y X$, we can transform (1.2) as follows:

$$(\phi_\varphi g)(X, Z_1, Z_2) = -g((\nabla_X \varphi)Z_1, Z_2) + g((\nabla_{Z_1} \varphi)X, Z_2) + g(Z_1, (\nabla_{Z_2} \varphi)X). \tag{2.4}$$

From this we have

$$(\phi_\varphi g)(Z_2, Z_1, X) = -g((\nabla_{Z_2} \varphi)Z_1, X) + g((\nabla_{Z_1} \varphi)Z_2, X) + g(Z_1, (\nabla_X \varphi)Z_2). \tag{2.5}$$

If we add (2.4) and (2.5), we find

$$(\phi_\varphi g)(X, Z_1, Z_2) + (\phi_\varphi g)(Z_2, Z_1, X) = 2g(X, (\nabla_{Z_1} \varphi)Z_2). \tag{2.6}$$

Putting $\phi_\varphi g = 0$ in (2.6), we find $\nabla \varphi = 0$. Thus, the proof is complete. □

Corollary 2.2 *Let (M, φ, g) be a Golden Riemannian manifold. The condition $\phi_\varphi g = 0$ is equivalent to $\nabla \varphi = 0$, where ∇ is the Levi-Civita connection of g.*

Let us recall almost product Riemannian manifolds. If an n-dimensional Riemannian manifold M, endowed with a positive definite Riemannian metric g, admits a non-trivial tensor field F of type (1.1) such that

$$F^2 = I$$

and

$$g(FX, Y) = g(X, FY)$$

for every vector field $X, Y \in \Im_0^1(M)$, then F is called an almost product structure and (M, F, g) is called an almost product Riemannian manifold. It is well known (see [3]) that a polynomial structure on a differentiable manifold M, defined by a tensor field of type $(1, 1)$, induces an almost product structure on M.

Theorem 2.3 *[1, 4, 5] If φ is a Golden structure on M, then*

$$F = \frac{1}{\sqrt{5}}(2\varphi - I) \tag{2.7}$$

is an almost product structure on M. Conversely, every almost product structure F on M induces two Golden Structures on M, given as follows:

$$\varphi_1 = \frac{1}{2}(I + \sqrt{5}F), \ \varphi_2 = \frac{1}{2}(I - \sqrt{5}F).$$

Proof Let $F^2 = I$, i.e. F is an almost product structure on a Riemannian manifold (M, g). Then each of the structures $\varphi_1 = \frac{1}{2}(I + \sqrt{5}F)$ and $\varphi_2 = \frac{1}{2}(I - \sqrt{5}F)$ obtained from the almost product structure F is a Golden structure. In fact,

$$\varphi_1^2 = \frac{I + 2\sqrt{5}F + 5I}{4} = \frac{\sqrt{5}F + 3I}{2} = \frac{1}{2}(3I + \sqrt{5}\frac{2\varphi_1 - I}{\sqrt{5}})$$

$$= \frac{1}{2}(2I + 2\varphi_1) = \varphi_1 + I.$$

Similarly, one can easily prove that $\varphi_2^2 - \varphi_2 - I = 0$.

Conversely, let φ be a Golden structure on a Riemannian manifold (M, g). Then the structure $F = \frac{1}{\sqrt{5}}(2\varphi - I)$ induced by the Golden structure φ is an almost product structure. In fact,

$$F^2 = \frac{4\varphi^2 - 4\varphi + I}{5} = \frac{4(\varphi^2 - \varphi) + I}{5} = \frac{5I}{5} = I.$$

$\square$

If a Riemannian metric g is pure with respect to a Golden structure φ, then the Riemannian metric g is pure with respect to the corresponding almost product structure F. A simple computation using the expression of the corresponding almost product structure via (2.7) gives

$$\phi_F g = \frac{2}{\sqrt{5}}\phi_\varphi g. \tag{2.8}$$

In [11], Salimov et al. proved that for an almost product Riemannian manifold with a pure metric g, if $\phi_F g = 0$, then the almost product structure F is integrable. Hence, by Theorem 2.1 and (2.8) we have

Proposition 2.4 *Let (M, φ, g) be a Golden Riemannian manifold and F its corresponding almost product structure. The golden structure φ is integrable if $\phi_F g = 0$.*

A Golden Riemannian manifold (M, φ, g) with an integrable Golden structure φ is called a locally Golden Riemannian manifold. If the metric g of the locally Golden Riemannian manifold has the form

$$ds^2 = g_{ab}(x^c)dx^a dx^b + g_{\bar{a}\bar{b}}(x^{\bar{c}})dx^{\bar{a}}dx^{\bar{b}}, \quad a, b, c = 1, ..., m, \quad \bar{a}, \bar{b}, \bar{c} = m + 1, ..., n$$

that is g_{ab} are functions of x^c only, $g_{a\bar{b}} = 0$ and $g_{\bar{a}\bar{b}}$ are functions of $x^{\bar{c}}$ only, then we call the manifold M a locally decomposable Golden Riemannian manifold. On the other hand, we say that the locally product Riemannian manifold with the corresponding product F is locally decomposable if and only if F is covariantly constant with respect to the Levi-Civita connection of g [26, p. 219–222], [28, p.418–420].

Since $\phi_F g = 0$ is equivalent to $\nabla F = 0$ [11], we have the following proposition.

Proposition 2.5 *Let (M, φ, g) be a Golden Riemannian manifold. The manifold M is a locally decomposable Golden Riemannian manifold if and only if $\phi_F g = 0$, where F is the corresponding almost product structure.*

3. Twin Golden Riemannian metrics

Let (M, φ, g) be a Golden Riemannian manifold. The twin Golden Riemannian metric is defined by

$$G(X, Y) = (g \circ \varphi)(X, Y) = g(\varphi X, Y) = g(X, \varphi Y)$$

for all vector fields X and Y on M. One can easily prove that G is a new pure Riemannian metric:

$$
\begin{aligned}
G(\varphi X, Y) &= (g \circ \varphi)(\varphi X, Y) = g(\varphi(\varphi X), Y) = g(\varphi^2 X, Y) \\
&= g(\varphi X, Y) + g(X, Y) = g(X, \varphi Y) + g(X, Y) \\
&= g(X, (\varphi + I)Y) = g(X, \varphi^2 Y) \\
&= (g \circ \varphi)(X, \varphi Y) = G(X, \varphi Y),
\end{aligned}
$$

which is called the twin metric of g. We shall now apply the ϕ_φ-operator to the metric G:

$$
\begin{aligned}
(\phi_\varphi G)(X, Y, Z) &= (\varphi X)(G(Y, Z)) - X(G(\varphi Y, Z)) \\
&+ G((L_Y \varphi)X, Z) + G(Y, (L_Z \varphi)X) \\
&= (L_{\varphi X} G - L_X(G \circ \varphi))(Y, Z) + G(Y, \varphi L_X Z) - G(\varphi Y, L_X Z) \\
&= (\phi_\varphi g)(X, \varphi Y, Z) + g(N_\varphi(X, Y), Z). \tag{3.9}
\end{aligned}
$$

Thus (3.9) implies the following.

Theorem 3.1 *In a Golden Riemannian manifold* (M, φ, g), *we have*

$$\phi_\varphi G = (\phi_\varphi g) \circ \varphi + g \circ (N_\varphi).$$

Corollary 3.2 *In a locally Golden Riemannian manifold* (M, φ, g), *the following conditions are equivalent:*

　　a) $\phi_\varphi g = 0$,

　　b) $\phi_\varphi G = 0$.

We denote by ${}^g \nabla$ the covariant differentiation of the Levi-Civita connection of the Golden Riemannian metric g. Then, we have

$${}^g \nabla G = ({}^g \nabla g) \circ \varphi + g \circ ({}^g \nabla \varphi) = g \circ ({}^g \nabla \varphi),$$

which implies ${}^g \nabla G = 0$ by virtue of Proposition 2.5 (${}^g \nabla F = 0$). Hence we have the following theorem.

Theorem 3.3 *Let* (M, φ, g) *be a locally decomposable Golden Riemannian manifold. Then the Levi-Civita connection of the Golden Riemannian metric* g *and the Levi-Civita connection of the twin Golden Riemannian metric* G *coincide to each other.*

4. Curvature properties of locally decomposable Golden Riemannian manifolds

If a pure tensor t satisfies $\phi_\psi t = 0$, then it is called a ϕ-tensor. If the $(1, 1)$-tensor ψ is a complex structure, then a ϕ-tensor is an analytic tensor. If ψ is a product structure, i.e. an almost product structure such that its Nijenhuis tensor vanishes, then a ϕ-tensor is a decomposable tensor [19].

Let R and S be the curvature tensors formed by the Golden Riemannian metric g and twin Golden Riemannian metric G, respectively; then for the locally decomposable Golden Riemannian manifold we have $R = S$ by means of Theorem 3.3.

Since the Riemannian curvature tensor R is pure with respect to both the Golden structure φ and the corresponding almost product structure F, we can apply the ϕ-operator to R. By similar devices (see proof of Theorem 2.1), we can prove that

$$(\phi_\varphi R)(X, Y_1, Y_2, Y_3, Y_4) = (\nabla_{\varphi X} R)(Y_1, Y_2, Y_3, Y_4) - (\nabla_X R)(\varphi Y_1, Y_2, Y_3, Y_4). \tag{4.10}$$

Taking account of the purity of R and applying Bianchi's 2nd identity to (4.10), we get

$$\begin{aligned}
(\phi_\varphi R)(X, Y_1, Y_2, Y_3, Y_4) &= g((\nabla_{\varphi X} R)(Y_1, Y_2, Y_3) - (\nabla_X R)(\varphi Y_1, Y_2, Y_3), Y_4) \\
&= g((\nabla_{\varphi X} R)(Y_1, Y_2, Y_3) - \varphi((\nabla_X R)(Y_1, Y_2, Y_3)), Y_4) \\
&= g(-(\nabla_{Y_1} R)(Y_2, \varphi X, Y_3) - (\nabla_{Y_2} R)(\varphi X, Y_1, Y_3) \\
&\quad - \varphi((\nabla_X R)(Y_1, Y_2, Y_3)), Y_4).
\end{aligned} \tag{4.11}$$

On the other hand, using $\nabla \varphi = 0$, we find

$$\begin{aligned}
(\nabla_{Y_2} R)(\varphi X, Y_1, Y_3) &= \nabla_{Y_2}(R(\varphi X, Y_1, Y_3)) - R(\nabla_{Y_2}(\varphi X), Y_1, Y_3) \\
&\quad - R(\varphi X, \nabla_{Y_2} Y_1, Y_3) - R(\varphi X, Y_1, \nabla_{Y_2} Y_3) \\
&= (\nabla_{Y_2} \varphi)(R(X, Y_1, Y_3)) + \varphi(\nabla_{Y_2} R(X, Y_1, Y_3)) \\
&\quad - R((\nabla_{Y_2} \varphi)X + \varphi(\nabla_{Y_2} X), Y_1, Y_3) \\
&\quad - R(\varphi X, \nabla_{Y_2} Y_1, Y_3) - R(\varphi X, Y_1, \nabla_{Y_2} Y_3) \\
&= \varphi(\nabla_{Y_2} R(X, Y_1, Y_3)) - \varphi(R(\nabla_{Y_2} X, Y_1, Y_3)) \\
&\quad - \varphi(R(X, \nabla_{Y_2} Y_1, Y_3)) - \varphi(R(X, Y_1, \nabla_{Y_2} Y_3)) \\
&= \varphi((\nabla_{Y_2} R)(X, Y_1, Y_3)).
\end{aligned} \tag{4.12}$$

Similarly

$$(\nabla_{Y_1} R)(Y_2, \varphi X, Y_3) = \varphi((\nabla_{Y_1} R)(Y_2, X, Y_3)). \tag{4.13}$$

Substituting (4.11) and (4.12) in (4.13) and using again Bianchi's 2nd identity, we obtain

$$\begin{aligned}
(\phi_\varphi R)(X, Y_1, Y_2, Y_3, Y_4) &= g(-\varphi((\nabla_{Y_1} R)(Y_2, X, Y_3)) - \varphi((\nabla_{Y_2} R)(X, Y_1, Y_3)) \\
&\quad - \varphi((\nabla_X R)(Y_1, Y_2, Y_3)), Y_4) \\
&= -g(\varphi(\sigma\{(\nabla_X R)(Y_1, Y_2)\}, Y_3)), Y_4) \\
&= 0,
\end{aligned}$$

where σ denotes the cyclic sum with respect to X, Y_1 and Y_2. Therefore we have

Theorem 4.1 *In a locally decomposable Golden Riemannian manifold, the Riemannian curvature tensor field is a ϕ-tensor field.*

By (1.2) and (2.7), we can find, in a similar way like (2.8),

$$\phi_F R = \frac{2}{\sqrt{5}} \phi_\varphi R, \tag{4.14}$$

where φ is the Golden structure and F is its corresponding almost product structure. From (4.14), we have this next proposition.

Proposition 4.2 *In a locally decomposable Golden Riemannian manifold, the Riemannian curvature tensor field is a decomposable tensor field.*

5. Examples

Example 1. Let (M, g) be an n-dimensional Riemannian manifold and $T(M)$ be its tangent bundle. Let ∇ be the Levi-Civita connection of g. Then the tangent space of $T(M)$ at any point $(x, u) \in T(M)$ splits into the horizontal and vertical subspaces with respect to ∇:

$$(T(M))_{(x,\,u)} = H_{(x,u)} \oplus V_{(x,\,u)}.$$

If $(x, u) \in T(M)$ is given, then for any vector $X \in \Im_0^1(M)$, there exists a unique vector $X^H \in H_{(x,u)}$ such that $\pi_* X^H = X$, where $\pi : T(M) \to M$ is the natural projection. We call X^H the horizontal lift of X to the point $(x, u) \in T(M)$. The vertical lift of a vector $X \in \Im_0^1(M)$ to $(x, u) \in T(M)$ is a vector $X^V \in V_{(x,u)}$ such that $X^V (df) = Xf$, for all functions f on M. Here we consider 1-forms df on M as functions on $T(M)$ (i.e. $(df)(x, u) = uf$).

The Sasaki metric on the tangent bundle $T(M)$ is defined by

$$^S g(^H X, {}^H Y) = {}^V (g(X, Y)), \tag{5.15}$$

$$^S g(^V X, {}^H Y) = {}^S g(^H X, {}^V Y) = 0, \tag{5.16}$$

$$^S g(^V X, {}^V Y) = {}^V (g(X, Y)), \tag{5.17}$$

for all $X, Y \in \Im_0^1(M)$ (see [29, p.155–175]. It is obvious that the Sasaki metric $^S g$ is contained in the class of the so-called g-natural metrics on the tangent bundle (recall that by a g-natural metric on tangent bundles we shall mean a metric which satisfies conditions (5.15) and (5.16)).

Now, let us introduce a Golden structure $\tilde{J}$ on $T(M)$, which implies $\tilde{J}^2 - \tilde{J} - I = 0$, defined by

$$\begin{cases} \tilde{J}(^H X) = \frac{1}{2}(^H X + \sqrt{5}\,{}^V X) \\ \tilde{J}(^V X) = \frac{1}{2}(^V X + \sqrt{5}\,{}^H X) \end{cases} \tag{5.18}$$

for all $X, Y \in \Im_0^1(M)$.

We put

$$A\left(\tilde{X}, \tilde{Y}\right) = {}^S g\left(\tilde{J}\tilde{X}, \tilde{Y}\right) - {}^S g\left(\tilde{X}, \tilde{J}\tilde{Y}\right)$$

for any $\tilde{X}, \tilde{Y} \in \Im_0^1(T(M))$. For all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form $^V X$, $^V Y$ or $^H X$, $^H Y$, from (5.15)–(5.17) and (5.18), we have $A\left(\tilde{X}, \tilde{Y}\right) = 0$, i.e. $^S g$ is pure with respect to the Golden structure $\tilde{J}$. Hence we have the following theorem:

Theorem 5.1 *Let (M, g) be a Riemannian manifold and let $T(M)$ be its tangent bundle equipped with the Sasaki metric $^S g$ and the Golden structure $\tilde{J}$ defined by (5.18). The triple $\left(T(M), \tilde{J}, {}^S g\right)$ is a Golden Riemannian manifold.*

Having determined both the Sasaki metric $^S g$ and the Golden structure $\tilde{J}$ and by using the fact that $^V X^V (g(Y, Z)) = 0$ and $^H X^V (g(Y, Z)) = {}^V (Xg(Y, Z))$, we calculate

$$(\phi_{\tilde{J}} {}^S g)(\tilde{X}, \tilde{Y}, \tilde{Z}) = (\tilde{J}\tilde{X})(^S g(\tilde{Y}, \tilde{Z})) - \tilde{X}(g(\tilde{J}\tilde{Y}, \tilde{Z})) + {}^S g((L_{\tilde{Y}}\tilde{J})\tilde{X}, \tilde{Z}) + {}^S g(\tilde{Y}, (L_{\tilde{Z}}\tilde{J})\tilde{X})$$

for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \Im_0^1(T(M))$. Then we get

$$(\phi_{\tilde{J}} {}^S g)(^V X, {}^V Y, {}^H Z) = \frac{\sqrt{5}}{2} {}^S g(^H (R(u, Y)X), {}^H Z),$$

$$(\phi_{\tilde{J}} {}^S g)(^V X, {}^V Y, {}^V Z) = 0,$$

$$(\phi_{\tilde{J}} {}^S g)(^V X, {}^H Y, {}^V Z) = \frac{\sqrt{5}}{2} {}^S g(^V (R(X, Y)u), {}^V Z),$$

$$(\phi_{\tilde{J}} {}^S g)(^V X, {}^H Y, {}^H Z) = 0,$$

$$(\phi_{\tilde{J}} {}^S g)(^H X, {}^V Y, {}^H Z) = 0,$$

$$(\phi_{\tilde{J}} {}^S g)(^H X, {}^V Y, {}^V Z) = 0,$$

$$(\phi_{\tilde{J}} {}^S g)(^H X, {}^H Y, {}^H Z) = \frac{\sqrt{5}}{2} {}^S g(^H (R(Y, X)u - R(u, Y)X), {}^H Z),$$

$$(\phi_{\tilde{J}} {}^S g)(^H X, {}^H Y, {}^V Z) = 0.$$

Therefore, from Proposition 2.5 and (2.8), we have this theorem.

Theorem 5.2 *Let (M, g) be a Riemannian manifold and let $T(M)$ be its tangent bundle equipped with the Sasaki metric $^S g$ and the Golden structure $\tilde{J}$ defined by (5.18). The triple $\left(T(M), \tilde{J}, {}^S g\right)$ is a locally decomposable Golden Riemannian manifold if and only if M is locally flat.*

Example 2. Let M be an n-dimensional differentiable Riemannian manifold of class C^∞ and with a Riemannian metric g, $^C T(M)$ its cotangent bundle, and π the natural projection $^C T(M) \to M$. If ω is a differentiable 1-form and X a vector field on M, $^V \omega$ denotes the vertical lift of ω and $^H X$ the horizontal lift of X to $^C T(M)$.

A Sasakian metric $^S g$ is defined on $^C T(M)$ by the three equations

$$^S g \left(^V \omega, {}^V \theta\right) = {}^V \left(g^{-1}(\omega, \theta)\right) = g^{-1}(\omega, \theta) \circ \pi, \tag{5.19}$$

$$^S g \left(^V \omega, {}^H Y\right) = 0, \tag{5.20}$$

$$^S g \left(^H X, {}^H Y\right) = {}^V (g(X, Y)) = g(X, Y) \circ \pi, \tag{5.21}$$

for any $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$. Since any tensor field of type $(0,2)$ on $^C T(M)$ is completely determined by its action on vector fields of type $^H X$ and $^V \omega$ (see [29, p.280]), it follows that $^S g$ is completely determined by the equations (5.19), (5.20) and (5.21). The Levi-Civita connection $^S \nabla$ of $^S g$ satisfies the following relations:

$$(i) \quad ^S \nabla_{V_\omega} {}^V \theta = 0,$$

$$(ii) \quad ^S \nabla_{V_\omega} {}^H Y = \frac{1}{2} {}^H \left(p \left(g^{-1} \circ R(\,, Y) \tilde{\omega}\right)\right),$$

$$(iii) \quad ^S \nabla_{H_X} {}^V \theta = {}^V (\nabla_X \theta) + \frac{1}{2} {}^H \left(p \left(g^{-1} \circ R(\,, X) \tilde{\theta}\right)\right), \tag{5.22}$$

$$(iv) \quad ^S \nabla_{H_X} {}^H Y = {}^H (\nabla_X Y) + \frac{1}{2} {}^V (pR(X, Y))$$

for all $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$, where $\tilde{\omega} = g^{-1} \circ \omega \in \Im_0^1(M)$, $R(\,, X) \tilde{\omega} \in \Im_1^1(M)$, $g^{-1} \circ R(\,, X) \tilde{\omega} \in \Im_0^2(M)$ [14].

We define a Golden structure $\tilde{\varphi}$ on $^C T(M)$ by

$$\begin{cases} \tilde{\varphi} {}^H X = \frac{1}{2}({}^H X + \sqrt{5} {}^V \tilde{X}), \\ \tilde{\varphi} {}^V \omega = \frac{1}{2}({}^V \omega + \sqrt{5} {}^H \tilde{\omega}) \end{cases} \tag{5.23}$$

for any $X \in \Im_0^1(M)$ and $\omega \in \Im_1^0(M)$, where $\tilde{X} = g \circ X \in \Im_1^0(M)$, $\tilde{\omega} = g^{-1} \circ \omega \in \Im_0^1(M)$. Also note that $^S g$ is pure with respect to $\tilde{\varphi}$. Then we have the next theorem.

Theorem 5.3 *Let (M, g) be a Riemannian manifold and let $^C T(M)$ be its cotangent bundle equipped with the Sasakian metric $^S g$ and the Golden structure $\tilde{\varphi}$ defined by (5.23). The triple $\left(^C T(M), \tilde{\varphi}, {}^S g\right)$ is a Golden Riemannian manifold.*

We now consider the covariant derivative of φ. Taking into account (i)-(iv) of (5.22) and (5.23), we obtain

$$\left(^S \nabla_{H_X} \tilde{\varphi}\right) \left(^H Y\right) = \frac{\sqrt{5}}{4} {}^H \left(pg^{-1} \circ (R(\,, X) Y - R(X, Y))\right)$$

$$\left(^S \nabla_{V_\omega} \tilde{\varphi}\right) \left(^H Y\right) = -\frac{\sqrt{5}}{4} {}^V (pR(\,, Y) \tilde{\omega})$$

$$\left(^S \nabla_{H_X} \tilde{\varphi}\right) \left(^V \theta\right) = \frac{\sqrt{5}}{4} {}^V \left(pR\left(X, \tilde{\theta}\right) - pR(\,, X) \tilde{\theta}\right) \tag{5.24}$$

$$\left(^S \nabla_{V_\omega} \tilde{\varphi}\right) \left(^V \theta\right) = \frac{\sqrt{5}}{4} {}^H \left(p \left(g^{-1} \circ R\left(\,, \tilde{\theta}\right) \tilde{\omega}\right)\right).$$

From (5.24) we have

Theorem 5.4 *The cotangent bundle of a Riemannian manifold, equipped with the Sasakian metric $^S g$ and the Golden structure $\tilde{\varphi}$ defined by (5.23), is a locally decomposable Golden Riemannian manifold if and only if the Riemannian manifold is flat.*

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