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L^p solutions of infinite time interval BSDEs and the corresponding g-expectations and g-martingales

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Abstract: In this paper we study the existence and uniqueness theorem for L^p $(1 < p < 2)$ solutions for a class of infinite time interval backward stochastic differential equations (BSDEs). Furthermore, we introduce generalized g-expectations and generalized g-martingales via the L^p solutions and prove the stability theorem of generalized g-expectations.

Key words: Backward stochastic differential equation (BSDE), comparison theorem, generalized g-expectation, generalized g-martingale

1. Introduction

The theory of backward stochastic differential equations (BSDEs) was developed by Pardoux and Peng [24], from which we know that there exists a unique adapted and square integrable solution to a BSDE of the type

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\mathrm{d}s - \int_t^T z_s \mathrm{d}W_s, \qquad t \in [0, T], \tag{1}$$

provided the function g (also called the generator) is Lipschitz in both variables y and z, and ξ and $(g(t, 0, 0))_{0 \le t \le T}$ are square integrable. Later, many researchers developed the theory of BSDEs and their applications in a series of papers (for example, see Briand et al. [3], Hu and Peng [16], Lepeltier and San Martin [19], Pardoux [22, 23], El Karoui et al. [13] and the references therein) under some other assumptions on the coefficients but for a fixed terminal time $T > 0$. Let us mention the contribution of Lepeltier and San Martin [19], which dealt with the quadratic of growth generator g in z and got the existence and uniqueness result in L^2. Let us mention also that when the generator g is Lipschitz continuous, a result of El Karoui et al. [13] provides for a solution when the data ξ and $\left\{(g(t, 0, 0)_{t \in [0,T]}\right\}$ are in L^p even for $p \in (1, 2)$. In 2003, Briand et al. [3] was devoted to the generalization of this result to the case of a monotone generator for BSDEs on a fixed time interval.

In 1997, Peng [27] introduced the notions of g-expectation and g-martingale via the L^2 solution of BSDE (1). Peng's g-expectation is a kind of nonlinear expectation, which can be considered as a nonlinear extension of the well-known Girsanov transformations. The original motivation for studying Peng's g-expectation comes from the theory of expected utility. Since the notion of Peng's g-expectation was introduced, many properties of Peng's g-expectation have been studied by Briand et al. [2], Chen [4], Chen and Wang [5], Chen and Epstein

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[6], Chen, Kulperger and Jiang [7, 8], Chen et al. [9], Coquet et al. [10], Hu [15], Jiang [20, 21], Rosazza Gianin [28] and in the references therein. In 2010, Hu and Chen [14] gave the extensions of Peng's g-expectations which are called generalized Peng's g-expectations, and investigated their related properties.

In this paper, we investigate generalized g-expectations and generalized g-martingales via L^p $(1<p<2)$ solutions of infinite time interval BSDEs. One difficulty of this problem is how to study the existence and uniqueness of BSDE (1) when $T\equiv\infty$ in L^p. In fact, such a problem in L^p $(1<p\leq 2)$ has been investigated by Briand et al. [3], Peng [26], Pardoux [22], Darling and Pardoux [11], Pardoux and Zhang [25] and other researchers under the assumption that terminal value $\xi=0$ or $E[e^{p\rho T}|\xi|^p]<\infty$ for some constant ρ and random terminal time T (i.e. T is a stopping time).

Let us mention the contribution of Briand et al. [3] which dealt with a monotone generator g in y and got the existence and uniqueness result in L^p $(1<p<2)$ on a random time interval. Furthermore, Briand et al. [3] strongly pointed out that their existence and uniqueness result covered the case of $T\equiv\infty$ (see the first paragraph of Section 5 and Remark 5.3 in [3]).

Let us mention also the contribution of Hu and Tessitore [17]. In 2007, Hu and Tessitore [17] studied the existence and uniqueness of mild solutions to a possibly degenerate elliptic partial differential equation $\mathcal{L}u(x)+\psi(x,u(x),\nabla u(x)G(x))-\lambda u(x)=0$ in Hilbert spaces. The main tool was existence, uniqueness and regular dependence on parameters of a bounded solution to a suitable BSDE with a random terminal time T.

In 2000, Chen and Wang [5] obtained the existence and uniqueness theorem for L^2 solutions of infinite time interval BSDEs when $T\equiv\infty$, by the martingale representation theorem and fixed point theorem. But in L^p $(1<p<2)$, there is no martingale representation theorem. In order to get rid of this difficulty, we give a new a priori estimate (Lemma 3.1). The main idea of this a priori estimate comes from Proposition 3.2 in Briand et al. [3]. Using this a priori estimate, we study the existence and uniqueness of L^p solutions to infinite time interval BSDEs. In fact, the difference between [3] and this paper is not the time horizon over which the problem is formulated but the assumptions on the function that appear in BSDE (1) (this paper's g and [3]'s f), in which λ and μ appearing in (H2) of [3] are constant, while our α and β are integrable Lipschitz functions on time t. These integrability conditions are introduced in [5]. In this paper, we also introduce generalized g-expectations and generalized g-martingales via L^p solutions of infinite time interval BSDEs. Furthermore, we give the stability theorem of generalized g-expectations.

This paper is organized as follows. In Section 2, we introduce some notations, assumptions and lemmas. In Section 3, we prove the existence and uniqueness theorem for L^p solutions of infinite time interval BSDEs. In Section 4, we introduce generalized g-expectations and generalized g-martingales via L^p solutions of infinite time interval BSDEs and prove the stability theorem of generalized g-expectations.

2. Preliminaries

In this section, we shall present some notations, assumptions and lemmas that are used in this paper.

Let $(\Omega,\mathcal{F},P)$ be a completed probability space, $(W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion defined on this space and $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration generated by Brownian motion $(W_t)_{t\geq 0}$, that is,

$$\mathcal{F}_t:=\sigma\{W_s;s\leq t\}\vee\mathcal{N},$$

where $\mathcal{N}$ is the set of all P-null subsets. Furthermore, we assume $\mathcal{F}:=\sigma\left(\bigcup_{t\geq 0}\mathcal{F}_t\right)$.

For simplicity, we just consider the case that $d = 1$, but our method can be easily extended to the other cases.

We consider the following spaces:

$L^p(\Omega, \mathcal{F}, P) := \{\xi : \xi$ is $\mathcal{F}$-measurable random variable such that $E[|\xi|^p] < \infty, p \geq 1\}$;

$\mathcal{L}(\Omega, \mathcal{F}, P) := \bigcup_{p>1} L^p(\Omega, \mathcal{F}, P)$;

$\mathcal{S}^p(\mathbb{R}) := \{V : V_t$ is $\mathcal{F}_t$-adapted process such that $E[\sup_{t\geq 0} |V_t|^p] < \infty, p \geq 1\}$;

$\mathcal{S}(\mathbb{R}) := \bigcup_{p>1} \mathcal{S}^p(\mathbb{R})$;

$\mathcal{L}^p(\mathbb{R}) := \{V : V_t$ is $\mathcal{F}_t$-adapted process such that $E[(\int_0^\infty |V_s|^2 \mathrm{d}s)^{\frac{p}{2}}] < \infty, p \geq 1\}$;

$\mathcal{L}(\mathbb{R}) := \bigcup_{p>1} \mathcal{L}^p(\mathbb{R})$.

In the sequel, we assume that $1 < p < 2$.

Consider the following infinite time interval BSDE:

$$Y_t = \xi + \int_t^\infty g(s, Y_s, Z_s)\mathrm{d}s + V_\infty - V_t - \int_t^\infty Z_s \mathrm{d}W_s. \tag{2}$$

Let

$$g : \Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$$

such that for any $(y, z) \in \mathbb{R} \times \mathbb{R}$, $g(\cdot, y, z)$ is $\mathcal{F}_t$-progressively measurable. We make the following assumptions:

(A.1) $E\left[\left(\int_0^\infty |g(t, 0, 0)|\mathrm{d}t\right)^2\right] < \infty$;

(A.2) There exists two positive non-random functions $\alpha(t)$ and $\beta(t)$, such that for all $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}$,

$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq \alpha(t)|y_1 - y_2| + \beta(t)|z_1 - z_2|,$$

where $\alpha(t)$ and $\beta(t)$ satisfy that $\int_0^\infty \alpha(t)\mathrm{d}t < \infty$, $\int_0^\infty \beta(t)\mathrm{d}t < \infty$, $\int_0^\infty \beta^2(t)\mathrm{d}t < \infty$;

(A.3) There exists some constant $T \in [0, \infty)$ such that

$$E\left[\left(\int_0^T |g(t, 0, 0)|\mathrm{d}t\right)^p\right] < \infty,$$

$$E\left[\left(\int_T^\infty |g(t, 0, 0)|\mathrm{d}t\right)^2\right] < \infty.$$

(A.4) $(V_t)_{t\geq 0}$ is an RCLL process (i.e. $(V_t)_{t\geq 0}$ has sample paths which are right continuous with left limits) with $(V_t)_{t\geq 0} \in \mathcal{S}^2(\mathbb{R})$.

The following lemmas are very useful in this paper.

Lemma 2.1 *Let $\{K_t\}_{t\geq 0}$ and $\{H_t\}_{t\geq 0}$ be two progressively measurable processes with values in $\mathbb{R}$ such that P-a.s.,*

$$\int_0^\infty (|K_t| + |H_t|^2)\mathrm{d}t < +\infty.$$

We consider the $\mathbb{R}$*-valued semi-martingale* $\{X_t\}_{t\geq 0}$ *defined by*

$$X_t = X_0 + \int_0^t K_s \mathrm{d}s + \int_0^t H_s \mathrm{d}W_s, \qquad 0 \leq t \leq \infty.$$

Then, for any $p \geq 1$*, we have*

$$\begin{array}{rcl} |X_\infty|^p & \geq & |X_t|^p + p\int_t^\infty |X_s|^{p-1}\frac{X_s}{|X_s|}1_{(X_s\neq 0)}K_s\mathrm{d}s + p\int_t^\infty |X_s|^{p-1}\frac{X_s}{|X_s|}1_{(X_s\neq 0)}H_s\mathrm{d}W_s \\ & + & c(p)\int_t^\infty |X_s|^{p-2}1_{(X_s\neq 0)}|H_s|^2\mathrm{d}s, \end{array} \tag{3}$$

where $c(p) = \frac{p[1\wedge(p-1)]}{2}$, $1\wedge(p-1) := \min\{1,(p-1)\}$.

The proof of Lemma 2.1 is very similar to that of Lemma 2.2 in [3]. It is almost verbatim adapted from [3]. Now we briefly give the idea of the proof of Lemma 2.1. Since the function $x \mapsto |x|^p$ *is not smooth enough (for* $p \in [1,2)$*) to apply Itô's formula, we use an approximation. Let* $\varepsilon > 0$ *and let us consider the function* $u_\varepsilon(x) := (|x|^2+\varepsilon^2)^{\frac{1}{2}}$*. Obviously, it is a smooth function. Itô's formula leads to the following equality:*

$$\begin{array}{rcl} u_\varepsilon^p(X_\infty) & = & u_\varepsilon^p(X_t) + p\int_t^\infty u_\varepsilon^{p-2}(X_s)X_sK_s\mathrm{d}s + p\int_t^\infty u_\varepsilon^{p-2}(X_s)X_sH_s\mathrm{d}W_s \\ & + & \frac{1}{2}p\int_t^\infty \left[u_\varepsilon^{p-2}(X_s) + (p-2)u_\varepsilon^{p-4}(X_s)X_s^2\right]H_s^2\mathrm{d}s. \end{array} \tag{4}$$

Letting $\varepsilon \to 0$ *in (4) and applying convergence, we can obtain (3).*

Lemma 2.2 *If* (Y,Z) *is a solution of the following BSDE:*

$$Y_t = \xi + \int_t^\infty g(s,Y_s,Z_s)\mathrm{d}s - \int_t^\infty Z_s\mathrm{d}W_s, \qquad 0 \leq t \leq \infty, \tag{5}$$

then we have

$$\begin{array}{rl} & |Y_t|^p + \frac{p(p-1)}{2}\int_t^\infty |Y_s|^{p-2}1_{(Y_s\neq 0)}|Z_s|^2\mathrm{d}s \\ \leq & |\xi|^p + p\int_t^\infty |Y_s|^{p-1}\frac{Y_s}{|Y_s|}1_{(Y_s\neq 0)}g(s,Y_s,Z_s)\mathrm{d}s \\ - & p\int_t^\infty |Y_s|^{p-1}\frac{Y_s}{|Y_s|}1_{(Y_s\neq 0)}Z_s\mathrm{d}W_s. \end{array} \tag{6}$$

Proof Noting that

$$Y_t = Y_0 - \int_0^t g(s,Y_s,Z_s)\mathrm{d}s + \int_0^t Z_s\mathrm{d}W_s, \quad 0 \leq t \leq \infty,$$

then, together with (3), we obtain (6). □

3. Existence and uniqueness

In this section, we prove the existence and uniqueness theorem for L^p solutions of infinite time interval BSDEs which generalizes the result of [5] and give the corresponding comparison theorem.

Theorem 3.1 *Under assumptions (A.2)–(A.4), if* $\xi \in L^p(\Omega,\mathcal{F},P)$*, then BSDE (2) has a unique solution* $(Y,Z) \in \mathcal{S}^p(\mathbb{R}) \times \mathcal{L}^p(\mathbb{R})$*.*

In order to prove Theorem 3.1, we give an a priori estimate.

Lemma 3.1 *Suppose that (A.2) holds for g. Furthermore, each ϕ_i satisfies that*

$$E\left[\left(\int_0^\infty |\phi_i(s)|\mathrm{d}s\right)^p\right] < \infty.$$

Let $\xi_i \in L^p(\Omega, \mathcal{F}, P)$, $(Y^i, Z^i) \in \mathcal{S}^p(\mathbb{R}) \times \mathcal{L}^p(\mathbb{R})$ satisfy the following BSDEs:

$$Y_t^i = \xi_i + \int_t^\infty \left[g\left(s, Y_s^i, Z_s^i\right) + \phi_i(s)\right] \mathrm{d}s - \int_t^\infty Z_s^i \mathrm{d}W_s, \quad i = 1, 2.$$

Then

$$\begin{aligned} & E\left[\sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^p + \left(\int_0^\infty \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \\ \leq\ & C_p E\left[|\xi_1 - \xi_2|^p + \left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^p\right], \end{aligned}$$

where C_p is a positive constant depending only on p.

Proof It is easy to check that

$$\int_0^\infty \left(\left|g\left(s, Y_s^1, Z_s^1\right) - g\left(s, Y_s^2, Z_s^2\right) + \phi_1(s) - \phi_2(s)\right| + \left|Z_s^1 - Z_s^2\right|^2\right) \mathrm{d}s < \infty,$$

so applying Itô's formula to $\left(Y_s^1 - Y_s^2\right)^2$, we have

$$\begin{aligned} & \left|Y_0^1 - Y_0^2\right|^2 + \int_0^\infty \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s \\ = \ & |\xi_1 - \xi_2|^2 + 2\int_0^\infty \left(Y_s^1 - Y_s^2\right)\left(g\left(s, Y_s^1, Z_s^1\right) - g\left(s, Y_s^2, Z_s^2\right) + \phi_1(s) - \phi_2(s)\right) \mathrm{d}s \\ - \ & 2\int_0^\infty \left(Y_s^1 - Y_s^2\right)\left(Z_s^1 - Z_s^2\right) \mathrm{d}W_s. \end{aligned}$$

From the Lipschitz assumption (A.2) on g, we have

$$\begin{aligned} & 2\left(Y_s^1 - Y_s^2\right)\left(g\left(s, Y_s^1, Z_s^1\right) - g\left(s, Y_s^2, Z_s^2\right)\right) \\ \leq\ & 2\alpha(s)\left|Y_s^1 - Y_s^2\right|^2 + 2\beta(s)\left|Y_s^1 - Y_s^2\right|\left|Z_s^1 - Z_s^2\right| \\ \leq\ & 2\alpha(s)\left|Y_s^1 - Y_s^2\right|^2 + 2\beta^2(s)\left|Y_s^1 - Y_s^2\right|^2 + \tfrac{1}{2}\left|Z_s^1 - Z_s^2\right|^2 \\ \leq\ & 2\left(\alpha(s) + \beta^2(s)\right)\sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^2 + \tfrac{1}{2}\left|Z_s^1 - Z_s^2\right|^2. \end{aligned}$$

It follows that

$$\begin{aligned} & \tfrac{1}{2}\int_0^\infty \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s \\ \leq\ & \left[1 + 2\left(\int_0^\infty \alpha(s)\mathrm{d}s + \int_0^\infty \beta^2(s)\mathrm{d}s\right)\right]\sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^2 \\ +\ & 2\int_0^\infty \left|Y_s^1 - Y_s^2\right||\phi_1(s) - \phi_2(s)|\mathrm{d}s + 2\left|\int_0^\infty \left(Y_s^1 - Y_s^2\right)\left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s\right|. \end{aligned}$$

Since $2\int_0^\infty \left|Y_s^1 - Y_s^2\right||\phi_1(s) - \phi_2(s)|\mathrm{d}s \leq \sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^2 + \left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^2$, we have

$$\begin{aligned} & \int_0^\infty \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s \\ \leq\ & 4\left(\left[1 + \left(\int_0^\infty \alpha(s)\mathrm{d}s + \int_0^\infty \beta^2(s)\mathrm{d}s\right)\right]\sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^2\right) \\ +\ & 4\left(\left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^2 + \left|\int_0^\infty \left(Y_s^1 - Y_s^2\right)\left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s\right|\right). \end{aligned}$$

Using the fact that if $b,\ a_i \geq 0$ and $b \leq \sum_{i=1}^{n} a_i$, then $b^p \leq \sum_{i=1}^{n} a_i^p$ for any $p \in (0,1)$ (see, e.g., Kuang [18, page 132]), we have

$$\begin{aligned}\left(\int_0^\infty \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s\right)^{\frac{p}{2}} &\leq c_p \left(\sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^p + \left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^p\right) \\ &+ c_p \left(\left|\int_0^\infty (Y_s^1 - Y_s^2)(Z_s^1 - Z_s^2)\mathrm{d}W_s\right|^{\frac{p}{2}}\right),\end{aligned} \tag{7}$$

where c_p is a positive constant depending only on p. By the Burkholder-Davis-Gundy inequality (see, e.g., Barlow et al. [1, Table 4.1 page 162]), we get

$$\begin{aligned}c_p E\left[\left|\int_0^\infty \left(Y_s^1 - Y_s^2\right)\left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s\right|^{\frac{p}{2}}\right] &\leq d_p E\left[\left(\int_0^\infty \left|Y_s^1 - Y_s^2\right|^2 \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s\right)^{\frac{p}{4}}\right] \\ &\leq d_p E\left[\sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^{\frac{p}{2}} \left(\int_0^\infty \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s\right)^{\frac{p}{4}}\right]\end{aligned}$$

and thus

$$\begin{aligned}c_p E\left[\left|\int_0^\infty \left(Y_s^1 - Y_s^2\right)\left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s\right|^{\frac{p}{2}}\right] &\leq \tfrac{1}{2} E\left[\left(\int_0^\infty \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \\ &+ \tfrac{d_p^2}{2} E\left[\sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^p\right],\end{aligned} \tag{8}$$

where d_p is a positive constant depending only on p. From (7) and (8), we have

$$E\left[\left(\int_0^\infty \left|Z_s^1 - Z_s^2\right|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \leq CE\left[\sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^p + \left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^p\right], \tag{9}$$

where C is a positive constant depending only on p.

Now, we prove that

$$E\left[\sup_{s\geq 0}\left|Y_s^1 - Y_s^2\right|^p\right] \leq C' E\left[|\xi_1 - \xi_2|^p + \left(\int_0^\infty |\phi_1(s) - \phi_2(s)|\mathrm{d}s\right)^p\right], \tag{10}$$

where C' is a positive constant depending only on p. The proof of (10) is similar to that of Proposition 3.2 of Briand et al. [3]. Let us fix $\theta(t) := \alpha(t) + \frac{\beta^2(t)}{p-1}$ and define $\overline{\xi} := \mathrm{e}^{\int_0^\infty \theta(s)\mathrm{d}s}\xi$, $\overline{Y}_t^i := \mathrm{e}^{\int_0^t \theta(s)\mathrm{d}s} Y_t^i$, $\overline{Z}_t^i := \mathrm{e}^{\int_0^t \theta(s)\mathrm{d}s} Z_t^i$, $i = 1, 2$, which solve the following BSDEs, respectively:

$$\overline{Y}_t^i = \overline{\xi}_i + \int_t^\infty \left[\overline{g}\left(s, \overline{Y}_s^i, \overline{Z}_s^i\right) + \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r}\phi_i(s)\right]\mathrm{d}s - \int_t^\infty \overline{Z}_s^i \mathrm{d}W_s, \quad i = 1, 2,$$

where $\overline{g}(t,y,z) := \mathrm{e}^{\int_0^t \theta(s)\mathrm{d}s} g\left(t, \mathrm{e}^{-\int_0^t \theta(s)\mathrm{d}s} y, \mathrm{e}^{-\int_0^t \theta(s)\mathrm{d}s} z\right) - \theta(t)y$.

By Lemma 2.2, we can get the inequality

$$\begin{aligned}&\left|\overline{Y}_t^1 - \overline{Y}_t^2\right|^p + \tfrac{p(p-1)}{2}\int_t^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-2} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)} \left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2 \mathrm{d}s \\ \leq\ &\left|\overline{\xi}_1 - \overline{\xi}_2\right|^p + p\int_t^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \tfrac{\overline{Y}_s^1 - \overline{Y}_s^2}{\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)} \left(\overline{g}\left(s, \overline{Y}_s^1, \overline{Z}_s^1\right) - \overline{g}\left(s, \overline{Y}_s^2, \overline{Z}_s^2\right)\right)\mathrm{d}s \\ +\ & p\int_t^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r} |\phi_1(s) - \phi_2(s)|\mathrm{d}s \\ -\ & p\int_t^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \tfrac{\overline{Y}_s^1 - \overline{Y}_s^2}{\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)} \left(\overline{Z}_s^1 - \overline{Z}_s^2\right)\mathrm{d}W_s.\end{aligned} \tag{11}$$

From the Lipschitz assumption (A.2) on g and with the help of

$$\theta(t) := \alpha(t) + \frac{\beta^2(t)}{p-1},$$

$$\overline{Y}_t^i := \mathrm{e}^{\int_0^t \theta(s)\mathrm{d}s} Y_t^i, \quad \overline{Z}_t^i := \mathrm{e}^{\int_0^t \theta(s)\mathrm{d}s} Z_t^i, \quad i = 1, 2$$

and

$$\overline{g}(t, y, z) := \mathrm{e}^{\int_0^t \theta(s)\mathrm{d}s} g\left(t, \mathrm{e}^{-\int_0^t \theta(s)\mathrm{d}s} y, \mathrm{e}^{-\int_0^t \theta(s)\mathrm{d}s} z\right) - \theta(t) y,$$

we have

$$\begin{aligned}
& p\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \tfrac{\overline{Y}_s^1 - \overline{Y}_s^2}{\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)} \left(\overline{g}\left(s, \overline{Y}_s^1, \overline{Z}_s^1\right) - \overline{g}\left(s, \overline{Y}_s^2, \overline{Z}_s^2\right)\right) \\
= \; & p\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \tfrac{\overline{Y}_s^1 - \overline{Y}_s^2}{\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r} \left(g\left(s, Y_s^1, Z_s^1\right) - g\left(s, Y_s^2, Z_s^2\right)\right) \\
- \; & p\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \tfrac{\overline{Y}_s^1 - \overline{Y}_s^2}{\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)} \theta(s) \left(\overline{Y}_s^1 - \overline{Y}_s^2\right) \\
\leq \; & p\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r} \left|g\left(s, Y_s^1, Z_s^1\right) - g\left(s, Y_s^2, Z_s^2\right)\right| \\
- \; & p\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \tfrac{\overline{Y}_s^1 - \overline{Y}_s^2}{\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)} \theta(s) \left(\overline{Y}_s^1 - \overline{Y}_s^2\right) \\
= \; & p\alpha(s)\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r} \left|Y_s^1 - Y_s^2\right| \\
+ \; & p\beta(s)\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r} \left|Z_s^1 - Z_s^2\right| - p\theta(s)\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^p \\
= \; & p\alpha(s)\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^p + p\beta(s)\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1}\left|\overline{Z}_s^1 - \overline{Z}_s^2\right| - p\theta(s)\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^p \\
= \; & p\beta(s)\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1}\left|\overline{Z}_s^1 - \overline{Z}_s^2\right| - \tfrac{p\beta^2(s)}{p-1}\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^p.
\end{aligned} \tag{12}$$

Noting that

$$\begin{aligned}
& p\beta(s)\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1}\left|\overline{Z}_s^1 - \overline{Z}_s^2\right| \\
= \; & p\beta(s)\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{\frac{p}{2}}\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{\frac{p}{2}-1} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)}\left|\overline{Z}_s^1 - \overline{Z}_s^2\right| \\
\leq \; & \tfrac{p\beta^2(s)}{p-1}\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^p + \tfrac{p(p-1)}{4}\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-2} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)}\left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2,
\end{aligned}$$

(where the inequality comes from the fact that if $a,\ b \geq 0$, then $ab \leq a^2 + \frac{b^2}{4}$), we have

$$\begin{aligned}
& p\int_t^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \tfrac{\overline{Y}_s^1 - \overline{Y}_s^2}{\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)} \left(\overline{g}\left(s, \overline{Y}_s^1, \overline{Z}_s^1\right) - \overline{g}\left(s, \overline{Y}_s^2, \overline{Z}_s^2\right)\right) \mathrm{d}s \\
\leq \; & \tfrac{p(p-1)}{4}\int_t^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-2} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)}\left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2 \mathrm{d}s.
\end{aligned} \tag{13}$$

Thus from (11) and (13), we obtain the following inequality:

$$\begin{aligned}
& \left|\overline{Y}_t^1 - \overline{Y}_t^2\right|^p + \tfrac{p(p-1)}{4}\int_t^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-2} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)}\left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2 \mathrm{d}s \\
\leq \; & \left|\overline{\xi}_1 - \overline{\xi}_2\right|^p + p\int_t^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r} |\phi_1(s) - \phi_2(s)| \mathrm{d}s \\
- \; & p\int_t^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \tfrac{\overline{Y}_s^1 - \overline{Y}_s^2}{\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)} \left(\overline{Z}_s^1 - \overline{Z}_s^2\right) \mathrm{d}W_s.
\end{aligned} \tag{14}$$

Denote

$$M_t := \int_0^t \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \frac{\overline{Y}_s^1 - \overline{Y}_s^2}{\left|\overline{Y}_s^1 - \overline{Y}_s^2\right|} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)} \left(\overline{Z}_s^1 - \overline{Z}_s^2\right) \mathrm{d}W_s.$$

By the Burkholder-Davis-Gundy inequality (for example, see Sect. 3 of Chap. VII of Dellacherie and Meyer [12]) and Young's inequality (i.e. $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, $a \geq 0$, $b \geq 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, see, e.g., Kuang [18, page 136]), we have

$$\begin{aligned} E[|M_t|] &\leq E\left[\left(\int_0^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{2p-2} \left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right] \\ &\leq E\left[\sup_{s\geq 0} \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \left(\int_0^\infty \left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right] \\ &\leq \tfrac{p-1}{p} E\left[\sup_{s\geq 0} \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^p\right] + \tfrac{1}{p} E\left[\left(\int_0^\infty \left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \\ &< \infty. \end{aligned}$$

It then follows that $\{M_t\}_{t\geq 0}$ is a martingale. For notational simplification, let

$$X := \int_0^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-2} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)} \left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2 \mathrm{d}s.$$

Coming back to inequality (14), we get both

$$\frac{p(p-1)}{4} E[X] \leq E\left[\left|\overline{\xi}_1 - \overline{\xi}_2\right|^p\right] + pE\left[\int_0^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r} |\phi_1(s) - \phi_2(s)| \mathrm{d}s\right] \tag{15}$$

and

$$\begin{aligned} & E\left[\sup_{s\geq 0} \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^p\right] \\ \leq\ & E\left[\left|\overline{\xi}_1 - \overline{\xi}_2\right|^p + p\int_0^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r} |\phi_1(s) - \phi_2(s)| \mathrm{d}s\right] \\ +\ & D_p E[|M_\infty|], \end{aligned} \tag{16}$$

where D_p is a positive constant depending only on p. Applying the Burkholder-Davis-Gundy inequality (for example, see Sect. 3 of Chap. VII of Dellacherie and Meyer [12]) again, we have

$$\begin{aligned} D_p E[|M_\infty|] &\leq D_p E\left[\left(\int_0^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{2p-2} \left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right] \\ &\leq D_p E\left[\sup_{s\geq 0} \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{\frac{p}{2}} \left(\int_0^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-2} 1_{\left(\overline{Y}_s^1 - \overline{Y}_s^2 \neq 0\right)} \left|\overline{Z}_s^1 - \overline{Z}_s^2\right|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right] \\ &\leq \tfrac{1}{2} E\left[\sup_{s\geq 0} \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^p\right] + \tfrac{D_p^2}{2} E[X]. \end{aligned}$$

It then follows from (15) and (16) that

$$E\left[\sup_{s\geq 0} \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^p\right] \leq K_p E\left[\left|\overline{\xi}_1 - \overline{\xi}_2\right|^p + p\int_0^\infty \left|\overline{Y}_s^1 - \overline{Y}_s^2\right|^{p-1} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r} |\phi_1(s) - \phi_2(s)| \mathrm{d}s\right], \tag{17}$$

where K_p is a positive constant depending only on p. Applying once again Young's inequality, we get

$$\begin{aligned} & pK_pE\left[\int_0^\infty \left|\overline{Y}_s^1-\overline{Y}_s^2\right|^{p-1} \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r}|\phi_1(s)-\phi_2(s)|\mathrm{d}s\right] \\ \leq\ & pK_pE\left[\sup_{s\geq 0}\left|\overline{Y}_s^1-\overline{Y}_s^2\right|^{p-1} \int_0^\infty \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r}|\phi_1(s)-\phi_2(s)|\mathrm{d}s\right] \\ \leq\ & \tfrac{1}{2}E\left[\sup_{s\geq 0}\left|\overline{Y}_s^1-\overline{Y}_s^2\right|^{p}\right] + M_pE\left[\left(\int_0^\infty \mathrm{e}^{\int_0^s \theta(r)\mathrm{d}r}|\phi_1(s)-\phi_2(s)|\mathrm{d}s\right)^p\right] \\ \leq\ & \tfrac{1}{2}E\left[\sup_{s\geq 0}\left|\overline{Y}_s^1-\overline{Y}_s^2\right|^{p}\right] + M_p\left(\mathrm{e}^{\int_0^\infty \theta(s)\mathrm{d}s}\right)^p E\left[\left(\int_0^\infty |\phi_1(s)-\phi_2(s)|\mathrm{d}s\right)^p\right], \end{aligned}$$

where M_p is a positive constant depending only on p. From this, we deduce that

$$E\left[\sup_{s\geq 0}\left|\overline{Y}_s^1-\overline{Y}_s^2\right|^{p}\right] \leq C^{'}E\left[\left|\overline{\xi}_1-\overline{\xi}_2\right|^p + \left(\int_0^\infty |\phi_1(s)-\phi_2(s)|\mathrm{d}s\right)^p\right], \tag{18}$$

where $C^{'}$ is a positive constant depending only on p.

Combining (9) with (18), we get

$$\begin{aligned} & E\left[\sup_{s\geq 0}\left|Y_s^1-Y_s^2\right|^p + \left(\int_0^\infty \left|Z_s^1-Z_s^2\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right] \\ \leq\ & C_pE\left[|\xi_1-\xi_2|^p + \left(\int_0^\infty |\phi_1(s)-\phi_2(s)|\mathrm{d}s\right)^p\right], \end{aligned}$$

where C_p is a positive constant depending only on p. The proof of Lemma 3.1 is complete. □

Lemma 3.2 *([5]) Let $\xi\in L^2(\Omega,\mathcal{F},P)$ be given. Suppose that (A.1) and (A.2) hold for g, then BSDE*

$$Y_t = \xi + \int_t^\infty g\left(s,Y_s,Z_s\right)\mathrm{d}s - \int_t^\infty Z_s\mathrm{d}W_s \tag{19}$$

has a unique solution $(Y,Z)\in\mathcal{S}^2(\mathbb{R})\times\mathcal{L}^2(\mathbb{R})$.

Proof of Theorem 3.1. We prove this theorem in two steps.

Step 1. We prove the existence and uniqueness to BSDE (19). Let $\xi^n:=(\xi\wedge n)\vee(-n)$ and $g_n(t,y,z):=g(t,y,z)-g(t,0,0)+f_n(g(t,0,0))$, where $f_n(g(t,0,0)):=\frac{g(t,0,0)n}{|g(t,0,0)|\vee n}$, if $t\leq T$; $f_n(g(t,0,0))=g(t,0,0)$, if $t>T$. It is easy to check that for each n, the function g_n satisfies (A.1) and (A.2). Then by Lemma 3.2, BSDE

$$Y_t^n = \xi^n + \int_t^\infty g_n\left(s,Y_s^n,Z_s^n\right)\mathrm{d}s - \int_t^\infty Z_s^n\mathrm{d}W_s$$

has a unique solution $(Y^n,Z^n)\in\mathcal{S}^2(\mathbb{R})\times\mathcal{L}^2(\mathbb{R})$. Using Lemma 3.1, we have

$$\begin{aligned} & E\left[\sup_{t\geq 0}\left|Y_t^{n+m}-Y_t^n\right|^p + \left(\int_0^\infty \left|Z_s^{n+m}-Z_s^n\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right] \\ \leq\ & C_pE\left[|\xi^{n+m}-\xi^n|^p + \left(\int_0^\infty |f_{n+m}(g(s,0,0))-f_n(g(s,0,0))|\,\mathrm{d}s\right)^p\right]. \end{aligned}$$

The right-hand side of the above inequality clearly tends to 0, as $n \to \infty$, uniformly in m, so we have a Cauchy sequence and the limit is a solution to BSDE (19). Let us consider (Y, Z) and $(Y^{'}, Z^{'})$ to be two solutions to BSDE (19). Using Lemma 3.1 again, we get immediately $(Y, Z) = (Y^{'}, Z^{'})$.

Step 2. Let $\hat{\xi} := \xi + V_\infty$ and $\hat{Y}_t := Y_t + V_t$, then BSDE (2) can be rewritten as

$$\hat{Y}_t = \hat{\xi} + \int_t^\infty \hat{g}\left(s, \hat{Y}_s, Z_s\right) \mathrm{d}s - \int_t^\infty Z_s \mathrm{d}W_s, \tag{20}$$

where $\hat{g}(t, y, z) := g(t, y - V_t, z)$. It is easy to check that $\hat{g}(t, y, z)$ satisfies (A.2), (A.3) and $\hat{\xi} \in L^p(\Omega, \mathcal{F}, P)$. By Step 1, there exists a unique pair $\left(\hat{Y}, Z\right)$ of adapted processes in $\mathcal{S}^p(\mathbb{R}) \times \mathcal{L}^p(\mathbb{R})$ solving BSDE (20). Using the fact $|Y_t|^p \leq 2^p(|\hat{Y}_t|^p + |V_t|^p)$, we have $(Y, Z) \in \mathcal{S}^p(\mathbb{R}) \times \mathcal{L}^p(\mathbb{R})$. The proof of Theorem 3.1 is complete.

Remark 3.1 If $g(t, 0, 0) \equiv 0$, then by Theorem 3.1, we have: Under assumptions (A.2) and (A.4), for each given $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$, BSDE (2) has a unique solution $(Y, Z) \in \mathcal{S}(\mathbb{R}) \times \mathcal{L}(\mathbb{R})$.

Example 3.1 *Suppose that $1 < p < 2$. Consider the BSDE:*

$$Y_t = \exp\left(\frac{W_1^2}{2p} - W_1\right) 1_{(W_1 \geq p)} + \int_t^\infty \frac{1}{(1+s)^2}(Y_s + Z_s)\mathrm{d}s - \int_t^\infty Z_s \mathrm{d}W_s. \tag{21}$$

For notational simplification, let $\xi := \exp\left(\frac{W_1^2}{2p} - W_1\right) 1_{(W_1 \geq p)}$, $g(t, y, z) := \frac{1}{(1+t)^2}(y + z)$, $\alpha(t) := \frac{1}{(1+t)^2}$, $\beta(t) := \frac{1}{(1+t)^2}$. Obviously, g satisfies (A.2) and (A.3). On the other hand,

$$E[|\xi|^p] = \int_p^\infty \exp\left(\frac{x^2}{2} - px\right) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2} \mathrm{d}x = \frac{1}{\sqrt{2\pi}p} \mathrm{e}^{-p^2} < \infty,$$

and

$$E[|\xi|^2] = \infty.$$

Hence, $\xi \in L^p(\Omega, \mathcal{F}, P)$, $\xi \notin L^2(\Omega, \mathcal{F}, P)$. But by Theorem 3.1, we have: BSDE (21) has a unique solution $(Y, Z) \in \mathcal{S}^p(\mathbb{R}) \times \mathcal{L}^p(\mathbb{R})$.

The following comparison theorem is very useful. Since the proof is very similar to that of Theorem 2.2 in [13], we omit it.

Theorem 3.2 *(Comparison Theorem) We make the same assumptions as in Theorem 3.1. Let $\left(\overline{Y}, \overline{Z}\right)$ be the solution of the BSDE*

$$\overline{Y}_t = \overline{\xi} + \int_t^\infty \overline{g}\left(s, \overline{Y}_s, \overline{Z}_s\right) \mathrm{d}s + \overline{V}_\infty - \overline{V}_t - \int_t^\infty \overline{Z}_s \mathrm{d}W_s,$$

where $\overline{g}(t, y, z)$ satisfies (A.2) and (A.3), $\overline{V}_t$ satisfies (A.4) and $\overline{\xi} \in L^p(\Omega, \mathcal{F}, P)$. If we suppose that

$$\hat{\xi} := \xi - \overline{\xi} \geq 0, \quad \hat{g}_t := g\left(t, \overline{Y}_t, \overline{Z}_t\right) - \overline{g}\left(t, \overline{Y}_t, \overline{Z}_t\right) \geq 0, \quad a.s.,$$
$$\hat{V}_t := V_t - \overline{V}_t \quad \text{is an RCLL increasing process,}$$

then

$$Y_t \geq \overline{Y}_t, \quad a.s., \quad \forall t \in [0, \infty).$$

Moreover, if $P\left(\hat{\xi} > 0\right) > 0$*, then* $P\left(Y_t > \overline{Y}_t\right) > 0$*, for all* $t \geq 0$*. In particular,* $Y_0 > \overline{Y}_0$*.*

4. Generalized g-expectation and generalized g-martingale

In this section, we make an additional assumption on the function g:

(A.5) $g(\cdot, y, 0) \equiv 0$, $\forall y \in \mathbb{R}$.

For any given g, the solution (Y, Z) of BSDE (19) depends on terminal value ξ. Referring to Definition 36.1 in [27] or Definition 3.1 in [14], now we introduce the definitions of generalized g-expectation and generalized conditional g-expectation via the solution of BSDE (19).

Definition 4.1 *(Generalized g-expectation) Suppose g satisfies (A.2) and (A.5). For any $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$, let (Y, Z) be the solution of BSDE (19). Consider the mapping $\mathcal{E}_g[\cdot] : \mathcal{L}(\Omega, \mathcal{F}, P) \mapsto \mathbb{R}$ denoted by $\mathcal{E}_g[\xi] := Y_0$. We call $\mathcal{E}_g[\xi]$ generalized g-expectation of ξ.*

Definition 4.2 *(Generalized conditional g-expectation) Suppose g satisfies (A.2) and (A.5). Generalized conditional g-expectation of ξ with respect to $\mathcal{F}_t$ is defined by*

$$\mathcal{E}_g[\xi | \mathcal{F}_t] := Y_t.$$

Generalized g-expectation has the following property.

Proposition 4.1 *$\mathcal{E}_g[\xi | \mathcal{F}_t]$ is the unique random variable η in $\mathcal{L}(\Omega, \mathcal{F}_t, P)$ such that*

$$\mathcal{E}_g[1_A \xi] = \mathcal{E}_g[1_A \eta], \qquad \forall A \in \mathcal{F}_t.$$

By Theorem 3.2 and (A.5), we can prove Proposition 4.1 by using the same method as that of Proposition 36.4 in [27], so we omit the proof.

The following proposition will tell us that generalized conditional g-expectations that we introduced meet some basic properties of Peng's conditional g-expectations.

Proposition 4.2 *Suppose ξ, ξ_1, $\xi_2 \in \mathcal{L}(\Omega, \mathcal{F}, P)$, then*

(i) *If ξ is $\mathcal{F}_t$-measurable, then $\mathcal{E}_g[\xi | \mathcal{F}_t] = \xi$;*

(ii) *For all stopping times τ and σ, $\mathcal{E}_g[\mathcal{E}_g[\xi | \mathcal{F}_\tau] | \mathcal{F}_\sigma] = \mathcal{E}_g[\xi | \mathcal{F}_{\tau \wedge \sigma}]$;*

(iii) *If $\xi_1 \geq \xi_2$ a.s., then $\mathcal{E}_g[\xi_1 | \mathcal{F}_t] \geq \mathcal{E}_g[\xi_2 | \mathcal{F}_t]$; if, moreover, $P(\xi_1 > \xi_2) > 0$, then*

$$P\left(\mathcal{E}_g[\xi_1 | \mathcal{F}_t] > \mathcal{E}_g[\xi_2 | \mathcal{F}_t]\right) > 0;$$

(iv) *For each $B \in \mathcal{F}_t$, $\mathcal{E}_g[1_B \xi | \mathcal{F}_t] = 1_B \mathcal{E}_g[\xi | \mathcal{F}_t]$;*

(v) *If g does not depend on y, then for any $(\xi, \eta) \in \mathcal{L}(\Omega, \mathcal{F}, P) \times \mathcal{L}(\Omega, \mathcal{F}_t, P)$,*

$$\mathcal{E}_g[\xi + \eta | \mathcal{F}_t] = \mathcal{E}_g[\xi | \mathcal{F}_t] + \eta.$$

By Theorem 3.2 and using the similar arguments as that of Lemma 36.6 in [27] and Lemma 4.2 in [2], we can prove Proposition 4.2.

Now we shall prove the stability theorem of generalized g-expectations.

Theorem 4.1 *(Stability Theorem) Suppose g satisfies (A.2) and (A.5). For $\xi, \eta_n \in \mathcal{L}(\Omega, \mathcal{F}, P)$, where $n = 1, 2, \cdots$, if $E[|\xi - \eta_n|^p|\mathcal{F}_t] \to 0$, a.s., $t \in [0, \infty)$, then*

$$\lim_{n\to\infty} \mathcal{E}_g[\eta_n|\mathcal{F}_t] = \mathcal{E}_g[\xi|\mathcal{F}_t], \quad a.s., \ \ t \in [0, \infty).$$

Proof From Theorem 3.1, we know that

$$\begin{array}{l} \mathcal{E}_g[\eta_n|\mathcal{F}_t] = \eta_n + \int_t^\infty g\left(s, \mathcal{E}_g[\eta_n|\mathcal{F}_s], Z_s^n\right) \mathrm{d}s - \int_t^\infty Z_s^n \mathrm{d}W_s, \quad n = 1, 2, \cdots, \\ \mathcal{E}_g[\xi|\mathcal{F}_t] = \xi + \int_t^\infty g\left(s, \mathcal{E}_g[\xi|\mathcal{F}_s], Z_s\right) \mathrm{d}s - \int_t^\infty Z_s \mathrm{d}W_s. \end{array}$$

Then

$$\begin{array}{rcl} \mathcal{E}_g[\xi|\mathcal{F}_t] - \mathcal{E}_g[\eta_n|\mathcal{F}_t] & = & \xi - \eta_n + \int_t^\infty \left[a_s\left(\mathcal{E}_g[\xi|\mathcal{F}_s] - \mathcal{E}_g[\eta_n|\mathcal{F}_s]\right) + b_s\left(Z_s - Z_s^n\right)\right] \mathrm{d}s \\ & - & \int_t^\infty (Z_s - Z_s^n) \mathrm{d}W_s, \end{array} \tag{22}$$

where

$$\begin{array}{l} a_s := \frac{g(s, \mathcal{E}_g[\xi|\mathcal{F}_s], Z_s) - g(s, \mathcal{E}_g[\eta_n|\mathcal{F}_s], Z_s)}{\mathcal{E}_g[\xi|\mathcal{F}_s] - \mathcal{E}_g[\eta_n|\mathcal{F}_s]} 1_{(\mathcal{E}_g[\xi|\mathcal{F}_s] - \mathcal{E}_g[\eta_n|\mathcal{F}_s] \neq 0)}, \\ b_s := \frac{g(s, \mathcal{E}_g[\eta_n|\mathcal{F}_s], Z_s) - g(s, \mathcal{E}_g[\eta_n|\mathcal{F}_s], Z_s^n)}{Z_s - Z_s^n} 1_{(Z_s - Z_s^n \neq 0)}, \end{array}$$

which imply $|a_t| \leq \alpha(t)$, $|b_t| \leq \beta(t)$.

Relation (22) can be rewritten as follows:

$$\mathcal{E}_g[\xi|\mathcal{F}_t] - \mathcal{E}_g[\eta_n|\mathcal{F}_t] = \xi - \eta_n + \int_t^\infty a_s\left(\mathcal{E}_g[\xi|\mathcal{F}_s] - \mathcal{E}_g[\eta_n|\mathcal{F}_s]\right) \mathrm{d}s - \int_t^\infty \left(Z_s - Z_s^n\right) \mathrm{d}\overline{W}_s, \tag{23}$$

where $\overline{W}_t = W_t - \int_0^t b_s \mathrm{d}s$. By the Girsanov theorem, we know that $\left(\overline{W}_t\right)_{t\geq 0}$ is Q^b-Brownian motion, where $\frac{\mathrm{d}Q^b}{\mathrm{d}P} = \mathrm{e}^{-\frac{1}{2}\int_0^\infty |b_s|^2 \mathrm{d}s + \int_0^\infty b_s \mathrm{d}W_s}$.

Solving (23), we obtain

$$\mathcal{E}_g[\xi|\mathcal{F}_t] - \mathcal{E}_g[\eta_n|\mathcal{F}_t] = (\xi - \eta_n)\mathrm{e}^{\int_t^\infty a_s \mathrm{d}s} - \int_t^\infty \left(Z_s - Z_s^n\right) \mathrm{e}^{\int_t^s a_r \mathrm{d}r} \mathrm{d}\overline{W}_s. \tag{24}$$

By the Burkholder-Davis-Gundy inequality (for example, see Sect. 3 of Chap. VII of Dellacherie and Meyer [12]), Hölder's inequality and noting the fact that

$$E\left[\mathrm{e}^{-\frac{1}{2}\int_0^\infty |b_s|^2 \mathrm{d}s + \int_0^\infty b_s \mathrm{d}W_s}\right] = 1$$

and

$$E\left[\mathrm{e}^{-\frac{1}{2}\int_0^\infty |qb_s|^2 \mathrm{d}s + \int_0^\infty qb_s \mathrm{d}W_s}\right] = 1,$$

we have

$$\begin{array}{rl} & E_{Q^b}\left[\left|\int_0^t \left(Z_s - Z_s^n\right) \mathrm{e}^{\int_0^s a_r \mathrm{d}r} \mathrm{d}\overline{W}_s\right|\right] \\ \leq & \mathrm{e}^{\int_0^\infty \alpha(t)\mathrm{d}t} E_{Q^b}\left[\left(\int_0^\infty |Z_s - Z_s^n|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right] \\ \leq & \mathrm{e}^{\int_0^\infty \alpha(t)\mathrm{d}t} \left(E\left[\left(\int_0^\infty |Z_s - Z_s^n|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}} \left(E\left[\left(\frac{\mathrm{d}Q^b}{\mathrm{d}P}\right)^q\right]\right)^{\frac{1}{q}} \\ \leq & \mathrm{e}^{\left[\frac{1}{2}(q-1)\int_0^\infty \beta^2(t)\mathrm{d}t + \int_0^\infty \alpha(t)\mathrm{d}t\right]} \left(E\left[\left(\int_0^\infty |Z_s - Z_s^n|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}} \\ < & \infty, \end{array}$$

where $\frac{1}{p}+\frac{1}{q}=1$. It then follows that $\left(\int_0^t (Z_s-Z_s^n)\,\mathrm{e}^{\int_0^s a_r \mathrm{d}r}\mathrm{d}\overline{W}_s\right)_{t\geq 0}$ is a martingale with respect to Q^b. Hence $E_{Q^b}\left[\int_0^t (Z_s-Z_s^n)\,\mathrm{e}^{\int_0^s a_r \mathrm{d}r}\mathrm{d}\overline{W}_s\right]=0$. Taking conditional expectation $E_{Q^b}[\cdot|\mathcal{F}_t]$ on both sides of (24), we have

$$\mathcal{E}_g[\xi|\mathcal{F}_t]-\mathcal{E}_g[\eta_n|\mathcal{F}_t]=E_{Q^b}\left[(\xi-\eta_n)\mathrm{e}^{\int_t^\infty a_s \mathrm{d}s}|\mathcal{F}_t\right].$$

Note that $|a_t|\leq \alpha(t)$ and hence

$$|\mathcal{E}_g[\xi|\mathcal{F}_t]-\mathcal{E}_g[\eta_n|\mathcal{F}_t]|\leq \mathrm{e}^{\int_0^\infty \alpha(t)\mathrm{d}t}E_{Q^b}[|\xi-\eta_n||\mathcal{F}_t].$$

By Hölder's inequality, we obtain

$$E_{Q^b}[|\xi-\eta_n||\mathcal{F}_t]=\frac{E\left[|\xi-\eta_n|\frac{\mathrm{d}Q^b}{\mathrm{d}P}|\mathcal{F}_t\right]}{E\left[\frac{\mathrm{d}Q^b}{\mathrm{d}P}|\mathcal{F}_t\right]}\leq \frac{\left(E\left[|\xi-\eta_n|^p|\mathcal{F}_t\right]\right)^{\frac{1}{p}}\left(E\left[(\frac{\mathrm{d}Q^b}{\mathrm{d}P})^q|\mathcal{F}_t\right]\right)^{\frac{1}{q}}}{E\left[\frac{\mathrm{d}Q^b}{\mathrm{d}P}|\mathcal{F}_t\right]}.$$

Since $\left(\mathrm{e}^{-\frac{1}{2}\int_0^t |b_s|^2\mathrm{d}s+\int_0^t b_s\mathrm{d}W_s}\right)_{t\geq 0}$ and $\left(\mathrm{e}^{-\frac{1}{2}\int_0^t |qb_s|^2\mathrm{d}s+\int_0^t qb_s\mathrm{d}W_s}\right)_{t\geq 0}$ are both martingales with respect to $(\mathcal{F}_t)_{t\geq 0}$, hence

$$\frac{\left(E\left[(\frac{\mathrm{d}Q^b}{\mathrm{d}P})^q|\mathcal{F}_t\right]\right)^{\frac{1}{q}}}{E\left[\frac{\mathrm{d}Q^b}{\mathrm{d}P}|\mathcal{F}_t\right]}\leq \mathrm{e}^{\frac{1}{2}(q-1)\int_0^\infty \beta^2(t)\mathrm{d}t}\frac{\left(\mathrm{e}^{-\frac{1}{2}\int_0^t |qb_s|^2\mathrm{d}s+\int_0^t qb_s\mathrm{d}W_s}\right)^{\frac{1}{q}}}{\mathrm{e}^{-\frac{1}{2}\int_0^t |b_s|^2\mathrm{d}s+\int_0^t b_s\mathrm{d}W_s}}\leq \mathrm{e}^{\frac{1}{2}(q-1)\int_0^\infty \beta^2(t)\mathrm{d}t}.$$

Thus for all $t\in[0,\infty)$,

$$|\mathcal{E}_g[\xi|\mathcal{F}_t]-\mathcal{E}_g[\eta_n|\mathcal{F}_t]|\leq \mathrm{e}^{\left[\frac{1}{2}(q-1)\int_0^\infty \beta^2(t)\mathrm{d}t+\int_0^\infty \alpha(t)\mathrm{d}t\right]}\left(E[|\xi-\eta_n|^p|\mathcal{F}_t]\right)^{\frac{1}{p}}. \tag{25}$$

Noting that $E[|\xi-\eta_n|^p|\mathcal{F}_t]\to 0$, as $n\to\infty$, $t\in[0,\infty)$, then

$$|\mathcal{E}_g[\xi|\mathcal{F}_t]-\mathcal{E}_g[\eta_n|\mathcal{F}_t]|\to 0,\quad \text{as}\quad n\to\infty.$$

The proof of Theorem 4.1 is complete. □

Remark 4.1 (i) *In Theorem 4.1, if we replace (A.5) by (A.3), the following result* $\lim_{n\to\infty} Y_t^n = Y_t$, *a.s.,* $t\in[0,\infty)$ *holds.*

(ii) *For any* $\xi\in\mathcal{L}(\Omega,\mathcal{F},P)$, *let* $\xi^n:=(\xi\wedge n)\vee(-n)$, $n=1,2,\cdots$, *then by Theorem 4.1, we have:*

$$\lim_{n\to\infty}\mathcal{E}_g[\xi^n|\mathcal{F}_t]=\mathcal{E}_g[\xi|\mathcal{F}_t],\quad a.s.,\quad \forall t\in[0,\infty).$$

(iii) *By the proof of Theorem 4.1, we have: if* $\xi\in L^p(\Omega,\mathcal{F},P)$, *then there exists a constant* $C>0$ *such that* $\mathcal{E}_g[|\xi||\mathcal{F}_t]\leq C(E[|\xi|^p|\mathcal{F}_t])^{\frac{1}{p}}$, $\forall t\in[0,\infty)$.

At the end of the paper, we introduce the definition of generalized g*-martingale (resp. generalized* g*-supermartingale, generalized* g*-submartingale).*

Definition 4.3 *Suppose g satisfies (A.2) and (A.5). A process $(X_t)_{t\geq 0}$ satisfying that for each t, $X_t \in \mathcal{L}(\Omega, \mathcal{F}_t, P)$ is called a generalized g-martingale (resp. generalized g-supermartingale, generalized g-submartingale), if for any t and s satisfying $t \leq s$,*

$$\mathcal{E}_g[X_s|\mathcal{F}_t] = X_t \quad (\textit{resp.} \leq X_t, \geq X_t), \quad a.s.$$

Example 4.1 *Suppose that $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$ and $(A_t)_{t\geq 0}$ is an RCLL increasing process with $(A_t)_{t\geq 0} \in \mathcal{S}^2(\mathbb{R})$. Consider the BSDE:*

$$Y_t = \xi + \int_t^{\infty} \frac{1}{(1+s)^2}|Z_s|\mathrm{d}s + A_{\infty} - A_t - \int_t^{\infty} Z_s \mathrm{d}W_s. \tag{26}$$

Let $g(t,y,z) := \frac{1}{(1+t)^2}|z|$. Obviously, g satisfies (A.2) and (A.5). By Theorem 3.2, for any t and s satisfying $t \leq s$, $\mathcal{E}_g[Y_s|\mathcal{F}_t] \leq Y_t$, a.s.. Thus $(Y_t)_{t\geq 0}$ is a generalized g-supermartingale.

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