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On some results on IP-graphs

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Abstract: The IP-graph of a naturally valenced association scheme and some of its properties have been studied recently. In this paper we introduce the bipartite version of this graph for a naturally valenced association scheme (X, S), denoted by $\mathrm{BIP}(S)$. We also investigate some of its properties.

Key words: Association scheme, common divisor graph, prime graph, bipartite divisor graph

1. Introduction

Let G be a group acting transitively on a set X such that all subdegrees are finite. Isaacs and Praeger introduced the concept of the common divisor graph of (G, X) in order to study the relations among all subdegrees of (G, X). They investigated the connectivity of this graph. The main result in [5] deals with the number of connected components of the graph, and the diameter of each connected component. They proved that the common divisor graph of (G, X) has at most two nontrivial components. If (G, X) has only one nontrivial component, then the diameter of that component is at most four, otherwise one of these components is a complete graph and the other has diameter at most two. The common divisor graph of (G, X) is called an IP-*graph of* (G, X) due to Neumann [7]. The common divisor graph of (G, X) is also studied by Kaplan [6]. Other related research can be found in [1].

Let G be a group acting transitively on a set X such that all subdegrees are finite. Actually this action of G on X induces a naturally valenced association scheme S on X. By the motivation of the common divisor graph of (G, X), Camina [2] introduced the IP-graph of a naturally valenced association scheme. The common divisor graph of (G, X) is the IP-graph of the naturally valenced association scheme (X, S) arising from the action of G on X. Under a very strong assumption that all paired valencies are equal R. Camina [2] proved that the main results in [5] are also true for the IP-graph of a naturally valenced association scheme whose paired valencies are equal. However, the common divisor graph of (G, X) defined in [5] does not satisfy this assumption.

Later, Xu [8] studied the IP-graph of a general naturally valenced association scheme. He proved similar results for the IP-graph of any naturally valenced association scheme, without the assumption that all paired valencies are equal, generalized results in [2, 5, 6]. Since an arbitrary naturally valenced association scheme may not arise from a transitive action of a group on a set (for example direct products and wreath products can

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be used to construct new association schemes that do not arise from groups), then we should note that most results in [5] are not true for naturally valenced association schemes.

Let $\mathcal{G} = (X, E)$ be a graph, where X is a non-empty set of positive integers. *The bipartite version* of $\mathcal{G}$, B$\mathcal{G}$, is a bipartite graph with vertex set the disjoint union of $\rho(X)$ and X, where $\rho(X)$ is the set of all primes dividing the elements of X and two distinct vertices $p \in \rho(X)$ and $x \in X$ are joined by an edge whenever p divides x.

In this paper we introduce the bipartite version of IP(S) for a naturally valenced association scheme (X, S). We also prove similar results to the results in [2, 5, 6] for the BIP-graph of a naturally valenced association scheme. Theorem 5 and Theorem 6 are the main theorems of this paper so that we talk about invariants such as the number of connected components, the diameter of each component, and also the girth of the IP-graph of a naturally valenced association scheme.

2. Preliminaries

In this section we give some required notations and definitions. For this section the fundamental reference is [10]. First let us begin with the definition of an association scheme.

Definition 1.*[Association Scheme] Let X be a set and S be a partition of $X \times X$. Then S is called an* association scheme *on X if the following properties hold.*

(i) $1_X \in S$, *where* $1_X = \{(x, x)|\ x \in X\}$. *We simply denote it by* 1.

(ii) For any $s \in S$, $s^* = \{(y, z)|(z, y) \in s\}$ *is also in* S.

(iii) For any $p, q, r \in S$, *there exists a cardinal number* a_{pqr} *such that for any* $(y, z) \in r$, *we have* $|\{x \in X|\ (y, x) \in p$ *and* $(x, z) \in q\}| = a_{pqr}$, *where the numbers* a_{pqr} *are called* structure constants *of* S.

From [2] we have the following definition.

Definition 2. *Let (X, S) be an association scheme. For any $x \in X$ and $s \in S$, define xs as the set $xs = \{y \in X|\ (x, y) \in s\}$.*

Then we note that the equation in the last property in Definition 1 can be written as $|yp \cap zq^*| = a_{pqr}$.

Definition 3. *For any $s \in S$,* valency *of s is defined as $n_s = a_{ss^*1}$.*

Recall that for each $x \in X$, $n_s = |xs|$, the cardinality of the set xs. If for any $s \in S$, n_s is finite, then (X, S) is called a *naturally valenced association scheme.*

Note that there may exist $s \in S$ such that $n_s \neq n_{s^*}$.

If for all $s \in S$, $n_s = n_{s^*}$, then we say that paired valencies are *equal.* Also an element $s \in S$ is called a *thin element* of S if $n_s = 1$.

Since $n_1 = 1$, then 1_X is a thin element of S. There may also exist $s \in S$ such that s is a thin element but s^* is not thin. Examples of these types of elements can be found in [5]. However, if X is a finite set, then for any $s \in S$, $n_s = n_{s^*}$; hence s is thin if and only if s^* is also thin.

Let G be a group acting transitively on X. Then G acts on $X \times X$ by $g(x, y) := (gx, gy)$, for any $g \in G$ and any $x, y \in X$. Let S be the set of all orbits of G on $X \times X$. Then (X, S) is an association scheme arising from the action of G on X. Let G_x denote the stabilizer of $x \in G$. Then $\{xs : s \in S\}$ is the set of all orbits of G_x on X. From [5] we can give the definition of subdegree.

Definition 4. *Let G be a group acting transitively on X, and (X,S) be the related association scheme. For any $s \in S$, valency $n_s = |xs|$ is called a* subdegree *of (G,X).*

So the set of valencies of elements in S and the set of subdegrees of (G,X) are exactly the same. Note that 1 is always a subdegree of (G,X).

Now we introduce the definition of the IP-graph of a naturally valenced association scheme from [2].

Definition 5. *Let (X,S) be a naturally valenced association scheme. The* IP-graph *of S, denoted by* IP(S)*, is an undirected graph with vertex set $\{n_s : s \in S\}$ such that two distinct vertices n_r and n_s are joined by an edge if they are not coprime.*

The concept of the IP-graph of a naturally valenced association scheme is a generalization of the notion of common divisor graph of a group acting transitively on a set with all subdegrees finite. In the case where all subdegrees of (G,X) are finite we have the following definition. The common divisor graph related to subdegrees of a group action is defined in [5] as follows.

Definition 6. *Let G be a group acting transitively on a set X such that all subdegrees of (G,X) are finite. Let D be the set of all subdegrees of (G,X).* Common divisor graph *of (G,X) is an undirected graph with vertex set D such that two distinct subdegrees m and n are joined by an edge if and only if m and n are not coprime.*

Let (X,S) be the naturally valenced association scheme arising from the action of G on X. Then the common divisor graph of (G,X) is the IP-graph of S. For any $p \in S$, the component of IP(S) that has the vertex n_p is denoted by $C(n_p)$. If the component $C(n_p)$ has infinitely many vertices, then we say that $C(n_p)$ is *infinite*. Since $n_1 = 1$, the component $C(n_1)$ is a trivial component.

For any $p \in S$ such that $n_p = 1$, the vertex n_p of IP(S) is called a *trivial* vertex.

Isaacs and Praeger [5] (Theorem A) proved that the common divisor graph of (G,X) has at most two nontrivial connected components. R. Camina [2] generalized this result to a class of naturally valenced association schemes and proved that, for a naturally valenced association scheme (X,S) with paired valencies equal, IP(S) has at most two connected components (not including the trivial component $C(n_1)$). Then Xu [8] generalized results of both Isaacs and Praeger [5] and Camina [2].

Definition 7. *[10] Let (X,S) be an association scheme. For any subset $P \subseteq S$, define $P^* = \{s^* | s \in P\}$. For any $P,Q \subseteq S$, define*

$$PQ = \{r \in S |\, \textit{there exists}\ p \in P\ \textit{and there exists}\ q \in Q\ \textit{such that}\ a_{pqr} \neq 0\}.$$

A nonempty subset $T \subseteq S$ is called closed *if $T^*T \subseteq T$. The* thin radical *of S is defined as the set*

$$O_v(S) = \{s \in S |\, n_s = 1\}.$$

If the thin radical is a closed subset of S, then we say that (X,S) is an association scheme with closed thin radical.

We should note that there are some examples of thin radicals which are not closed [5] (Section 5). If the thin radical is not a closed subset of S, then there exists $p \in S$ such that $n_p > 1$ but $n_{p^*} = 1$.

Let G be a group acting transitively on a set X such that all subdegrees are finite, and let (X, S) be the naturally valenced association scheme arising from the action of G on X. Then we have the following fact. The thin radical $O_\upsilon(S)$ is not a closed subset of S if and only if the subdegree 1 is paired with some subdegree $m > 1$. Isaacs and Praeger [5] proved that if the subdegree 1 is paired with some subdegree $m > 1$, then the common divisor graph of (G, X) has just one nontrivial component, and the diameter of that component is at most three. Xu in [8] (Theorem 1.2) generalized this result and proved the following.

Theorem 1. *If (X, S) is a naturally valenced association scheme whose thin radical $O_\upsilon(S)$ is not a closed subset of S, then the graph $\mathrm{IP}(S)$ has just one nontrivial component $C(n_p)$ such that $n_p > 1$ but $n_{p^*} = 1$. Furthermore, the nontrivial component $C(n_p)$ is infinite, and the diameter of $C(n_p)$ is at most three.*

Also Isaacs and Praeger [5] proved the following results.

(i) If the common divisor graph of (G, X) has just one nontrivial component, the component has diameter at most 4.

(ii) If the common divisor graph of (G, X) has two nontrivial components, one of these is a complete graph and the other has diameter at most 2.

The statement of the first part of the above theorem, however, would allow the nontrivial component of the common divisor graph of (G, X) to have diameter 4, but the authors have been unable to determine whether an example with diameter 4 actually exists or not.

Let G be a group acting transitively on a set X such that all subdegrees are finite. The common divisor graph of (G, X) is called *stable* if any two paired subdegrees m and m^* lie in the same component [6].

Definition 8. *The graph $\mathrm{IP}(S)$ is called stable if for any $p \in S$, n_p and n_{p^*} lie in the same component.*

Note that if the graph $\mathrm{IP}(S)$ is stable, then the thin radical $O_\upsilon(S)$ is a closed subset of S.

Theorem 2. *[8] (Theorem 5.2) Let (X, S) be a naturally valenced association scheme such that the graph $\mathrm{IP}(S)$ has two nontrivial components. Assume that there exists $p \in S$, such that $n_p > 1$ and n_{p^*} is not a vertex of the component $C(n_p)$. Then the following hold.*

(i) Both components of $C(n_p)$ and $C(n_{p^*})$ of $\mathrm{IP}(S)$ are infinite.

(ii) For any $s \in S$ such that $n_s \neq n_{s^*}$, one of the vertices n_s and n_{s^*} lies in the component $C(n_p)$, and the other vertex lies in the component $C(n_{p^*})$.

(iii) If there exists $s \in S$ such that $n_s = n_{s^*} > 1$, then only one of the components $C(n_p)$ and $C(n_{p^*})$ contains such vertices.

(iv) If $n_s \neq n_{s^*}$ for any $s \in S$ with $n_s > 1$, then both components $C(n_p)$ and $C(n_{p^*})$ are complete graphs. If there exists $s \in S$ such that $n_s = n_{s^*} > 1$, then one of the components $C(n_p)$ and $C(n_{p^*})$ that contains such vertices has diameter at most two, and the other component is a complete graph.

Definition 9. *Let (X, S) be a naturally valenced association scheme, and $p \in S$. If $n_p > 1$ and $n_p \leq n_s$ for any $s \in S$ such that $n_s \neq 1$, then n_p is called a* minimal vertex *of the graph* $\mathrm{IP}(S)$.

Xu proved the following theorem in [8] (Theorem 5.3).

Theorem 3. *Let (X, S) be a naturally valenced association scheme such that the graph $\mathrm{IP}(S)$ is stable and has two nontrivial components. Let $p \in S$ such that n_p is the minimal vertex of the graph $\mathrm{IP}(S)$, and let $q \in S$ such that $n_q > 1$ is not a vertex of the component $C(n_p)$. Then the following hold.*

(i) The component $C(n_q)$ is a complete graph.

(ii) For any $r \in S$ such that n_r is a vertex of the component $C(n_p)$, we have that $n_r = n_{r^*}$, and n_r is less than the greatest common divisor of the vertices (integers) in the component $C(n_q)$.

(iii) The component $C(n_p)$ has a maximal vertex n_t, i.e. $n_t > n_s$ for any other vertex n_s of $C(n_p)$. Furthermore, the maximal vertex n_t is adjacent to any other vertex of $C(n_p)$. In particular, the component $C(n_p)$ has diameter at most two.

By using Theorem 2 and Theorem 3, Xu [8] (Theorem 1.3) proved the second part of the following theorem.

Theorem 4. *Let (X, S) be a naturally valenced association scheme such that the thin radical $O_v(S)$ is a closed subset of S. Then we have*

(i) If $\mathrm{IP}(S)$ has only one nontrivial component, then the diameter of that component is at most 5;

(ii) If $\mathrm{IP}(S)$ has two nontrivial components. One of them has diameter at most two and the other one is a complete graph.

For this case, some examples can be found in [5, 6]. Throughout this paper, when the graph $\mathrm{IP}(S)$ has the properties of the second part of Theorem 4, we denote the complete component by $C(n_q)$ and the component which has diameter at most two with $C(n_p)$.

By $d(n_p, n_q)$ we denote the distance between vertices n_p and n_q. Thus,

$$d(n_p, n_q) = \begin{cases} 0 & \text{if}\, n_p = n_q, \\ \infty & \text{if}\, n_p \,\text{and}\, n_q \,\text{do not lie in the same component.} \end{cases}$$

Furthermore, the greatest common divisor of the two integers n_p and n_q is denoted by (n_p, n_q). So for any two distinct vertices n_p and n_q, $d(n_p, n_q) \geq 2$ if and only if $(n_p, n_q) = 1$.

Remark 1. *Suppose that X is a set of positive integers. By X^* we mean $X \setminus \{1\}$. Let G be a finite group acting transitively on a set X. By G_x we mean the stabilizer of the element $x \in X$ in G. The concept of the prime vertex graph $\Delta(Z)$, the common divisor graph $\Gamma(Z)$ and the bipartite divisor graph $B(Z)$ for a set Z, which is a set of integers, has been defined in [4]. So throughout this paper by Δ, Γ and B we mean graphs related to a set. Also by [8] we know that $\mathrm{IP}(S)$ has a trivial component, but throughout this paper we do not consider this trivial component. By P_n, C_n and K_n we mean a path of length n, a cycle of length n and a complete graph on n vertices, respectively. Suppose that T is a graph. By $n(T)$ we mean the number of connected components of T. By* girth *of the graph T, we mean the length of the shortest cycle in T and we denote it by $\mathbf{g}(T)$. If the graph does not have any cycle, then we write $\mathbf{g}(T) = \infty$.*

3. Some results about the girth of IP-graph

In this section we discuss the girth of an IP-graph.

Theorem 5. *Let (X,S) be a naturally valenced association scheme. If* $\mathrm{IP}(S)$ *has a cycle and if* $O_v(S)$ *is not closed or* $\mathrm{IP}(S)$ *has at least two nontrivial components, then we have* $\mathbf{g}(\mathrm{IP}(S)) \leq 5$.

Proof First suppose that $O_v(S)$ is not closed. By Theorem 1, we know that $\mathrm{IP}(S)$ has only one nontrivial component $C(n_p)$ which is infinite. If $C(n_p)$ is complete then $\mathbf{g}(\mathrm{IP}(S)) = 3$. Since $\mathrm{IP}(S)$ has a cycle, it can not be a star graph. So we may find a path of length three in the graph $\mathrm{IP}(S)$ in the following form

$$n_s - n_t - n_r - n_q \qquad (I).$$

Here we have two distinct cases. First suppose that n_p is not a vertex of the path (I). By [8] (Lemma 4.2), we know that $d(n_p, n_s) \leq 2$ and $d(n_p, n_t) \leq 2$. So there exist two elements $n_k, n_l \in S$ such that $n_p - n_k - n_s$ and $n_p - n_l - n_t$ are two paths of length two. Now $n_p - n_k - n_s - n_t - n_l - n_p$ is a cycle of length five. Now suppose that n_p is a vertex of the path (I). We have the following cases.

(i) $n_p = n_s$. By [8] (Lemma 4.2) we have $d(n_p, n_q) \leq 2$, so there exists a vertex n_j which is adjacent to both n_p and n_q. It is easy to see that $n_p - n_j - n_q - n_r - n_t - n_p$ is a cycle of length five.

(ii) $n_p = n_t$. First suppose that the degree of n_q is at least two. Then there exists $l \in S$, such that $d(n_q, n_l) = 1$. By [8] (Lemma 4.2) we have $d(n_p, n_l) \leq 2$, so there exists $k \in S$ such that $n_p - n_k - n_l$ is a path in $C(n_p)$. Now we can see that there is a cycle of length five in $C(n_p)$.

Now suppose that n_q is a vertex of degree one. We have two cases here:

First suppose that n_s is a vertex of degree at least two. Suppose that n_s is adjacent to a vertex say $n_u \neq n_p$ and $u \in S$. Since by Theorem 1 $\mathrm{diam}(C(n_p)) \leq 3$, there must be a path of length three between n_q and n_u. As n_q is a vertex of degree one, so there must be a path of length two say

$$n_u - n_h - n_r$$

such that $h \in S$. Now we can find a cycle of length five.

So suppose that n_s is also a vertex of degree one. Remember that since $C(n_p)$ is infinite and connected, there are always infinitely many vertices which have paths to n_p or n_r. Now if n_p and n_r are only adjacent to vertices of degree one, then $C(n_p)$ is acyclic, which contradicts our assumption, so at least one of them is adjacent to a vertex of degree more than one. In each case, as previously, we can find a cycle of length five in $C(n_p)$.

Now suppose that $O_v(S)$ is closed and $\mathrm{IP}(S)$ has two nontrivial components. If $\mathrm{IP}(S)$ is not stable, then by Theorem 2, both components of $\mathrm{IP}(S)$ are infinite. By Theorem 4 one of these components is a complete graph, so we conclude that $\mathbf{g}(\mathrm{IP}(S)) = 3$. Now suppose that $\mathrm{IP}(S)$ is stable. By Theorem 3 we know that one of these components is a complete graph K_m and the other one has diameter at most two. If $m \geq 3$, then $\mathbf{g}(\mathrm{IP}(S)) \leq 3$, otherwise by the assumption we have a cycle of the following form in the other component

$$n_{t_1} - n_{t_2} - n_{t_3} - n_{t_4} - \ldots - n_{t_m} - n_{t_1}.$$

We claim that $m \leq 5$. Since the diameter of this component is at most two, so $d(n_{t_1}, n_{t_4}) \leq 2$. Hence there exists an element $i \in S$ such that $n_{t_1} - n_i - n_{t_4}$ is a path of length two. Then we can see that we have a cycle of length five, so $\mathbf{g}(\mathrm{IP}(S)) \leq 5$. □

Suppose that $O_v(S)$ is a closed subset of S and $\mathrm{IP}(S)$ has just one nontrivial component. By Theorem 4, the diameter of that component is at most five. If the graph $\mathrm{IP}(S)$ has a cycle, we can be sure that the length of this cycle is at most ten, so here we may ask the following question.

Question 1. *Suppose that $O_v(S)$ is a closed subset of S and $\mathrm{IP}(S)$ has just one nontrivial component. If $\mathrm{IP}(S)$ has a cycle, then what can we say about the girth of this graph?*

4. BIP-graph and some of its properties

Throughout this section we suppose that (X, S) is a naturally valenced association scheme. First we introduce the bipartite version of the IP-graph and investigate connectivity of this graph. The main result deals with some invariants such as the number of connected components, the diameter and also the girth of each one.

Definition 10. *Let (X, S) be a naturally valenced association scheme.* The bipartite version *of $\mathrm{IP}(S)$, denoted by $\mathrm{BIP}(S)$, is an undirected bipartite graph with vertex set $\rho(Y)\dot{\cup}Y$, where $Y = \{n_s | s \in S, n_s \neq 1\}$ and $\rho(Y)$ is the set of all primes dividing the elements of Y such that two distinct vertices $p \in \rho(Y)$, $n_s \in Y$ are joined by an edge whenever p divides n_s.*

Example 1. *Let $X = \{x_1, x_2, x_3, x_4\}$, and $G =< (x_1x_2x_3x_4) >$ be a cyclic group of order four, in which G acts on X, transitively. Let δ, γ, ϕ be the orbitals of G on $X \times X$ containing (x_1, x_2), (x_1, x_3), (x_1, x_4), respectively. Then we have*

$$\begin{aligned}
\delta &= \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_1)\};\\
\gamma &= \{(x_1, x_3), (x_2, x_4), (x_3, x_1), (x_4, x_2)\};\\
\phi &= \{(x_1, x_4), (x_2, x_1), (x_3, x_2), (x_4, x_3)\};
\end{aligned}$$

and I is the diagonal orbital of G. Now we have

$$\begin{aligned}
n_\delta = \quad |x_1\delta| = \quad |\{x_2\}| = 1;\\
n_\gamma = \quad |x_1\gamma| = \quad |\{x_3\}| = 1;\\
n_\phi = \quad |x_1\phi| = \quad |\{x_4\}| = 1.
\end{aligned}$$

So in this case $\mathrm{BIP}(S)$ has only a trivial component.

Example 2. *Let X be the set of all ordered pairs of m elements, $m \geq 2$, and let G be the wreath product $S_m \wr S_2$ of the symmetric groups of degree m and two. Then G acts on X as a permutation group with rank three. We may easily see the set of the valencies of the association scheme arising from this action and it is $\{k = 2(m-1), l = (m-1)^2\}$. So in this case $\mathrm{BIP}(S)$ is a connected graph with diameter at most three. Now let $m = 16$, then $\mathrm{BIP}(S)$ has the following form*

$$2 - 30 - 3 - 225 - 5 - 30.$$

Clearly $\mathrm{diam}(\mathrm{BIP}(S)) = 3$, *and* $\mathbf{g}(\mathrm{BIP}(S)) = 4$.

Theorem 6. *We have the following cases for* $\mathrm{BIP}(S)$*.*

(i) $\mathrm{BIP}(S)$ *is a connected graph with diameter at most* 12*;*

(ii) $\mathrm{BIP}(S)$ *is a disconnected graph with two components. One of them has diameter at most four and the other one has diameter at most six.*

Proof First suppose that $O_v(S)$ is not a closed subset of S. By Theorem 1 $\mathrm{IP}(S)$ has only one non-trivial component with diameter at most three. By [4] we have $n(\mathrm{BIP}(S))$ and $n(\mathrm{IP}(S))$ have equal size one, so $\mathrm{BIP}(S)$ is connected. Let $Y = \{n_s | s \in S, n_s \neq 1\}$. By [4] we have the following two cases.

1. $\mathrm{diam}(\mathrm{BIP}(S)) = 2\max\{\mathrm{diam}(\mathrm{IP}(S)), \mathrm{diam}(\Delta(Y))\}$;

2. $\mathrm{diam}(\mathrm{BIP}(S)) = 2\,\mathrm{diam}(\mathrm{IP}(S)) + 1$.

Now let p, q be two distinct elements of $\rho(Y)$, so there exist elements n_s, n_t of Y such that p divides n_s and q divides n_t. Since $\mathrm{diam}(\mathrm{IP}(S)) \leq 3$, so $d_{\mathrm{IP}(S)}(n_s, n_t) \leq 3$. Therefore there exists a path of length at most three between these elements in $\mathrm{IP}(S)$. By [4] (Lemma 1(d)) we can see that $\mathrm{diam}(\Delta(Y)) \leq 4$ and we have the following cases

1. $\mathrm{diam}(\mathrm{BIP}(S)) \leq 2\ \max\{3, 4\}$, so $\mathrm{diam}(\mathrm{BIP}(S)) \leq 8$;

2. $\mathrm{diam}(\mathrm{BIP}(S)) \leq 7$.

Suppose that $O_v(S)$ is a closed subset of S. By [8] (Theorem 1.1), we have two different cases. First suppose that $\mathrm{IP}(S)$ has just one nontrivial component. By Theorem 4 the diameter of this component is at most five. By [4] we have $n(\mathrm{BIP}(S)) = n(\mathrm{IP}(S)) = 1$, so $\mathrm{BIP}(S)$ is connected. Similar to the previous argument we conclude that $\mathrm{diam}(\mathrm{BIP}(S)) \leq 12$.

Now suppose that $\mathrm{IP}(S)$ has two nontrivial components. By Theorem 4 one of them is a complete graph and the other has diameter at most two. We denote them by $C(n_q)$, $C(n_p)$, such that $C(n_q)$ is a complete graph and $C(n_p)$ has diameter at most two. As before, by [4] $n(\mathrm{BIP}(S)) = n(\mathrm{IP}(S)) = 2$, so $\mathrm{BIP}(S)$ is disconnected with two components say T_q, T_p. T_q is a bipartite graph with vertex set $\rho(V(C(n_q)))\dot{\cup}V(C(n_q))$ and T_p is a bipartite graph with vertex set $\rho(V(C(n_p)))\dot{\cup}V(C(n_p))$. Now we check the diameter of T_q, T_p. Let $V_1 = V(C(n_q))$, and $V_2 = V(C(n_p))$. Let $p \in \rho(V_1)$, $n_l \in V_1$. Then there exists $n_k \in V_1$ such that p divides n_k. Since $C(n_q)$ is complete, then n_l, n_k is an edge in $\mathrm{IP}(S)$. So there exists $q \in \rho(V_1)$ that q divides (n_l, n_k). Now $p - n_k - q - n_l$ is a path in T_q, and $d(p, n_l) \leq 3$. Let $p, q \in \rho(V_1)$. Then there are $n_l, n_k \in V_1$ that p divides n_l, q divides n_k. Since $C(n_q)$ is complete, by the above argument, we conclude that $d(p, q) \leq 4$. Also for each two elements of V_1 there is a path of length two between them, so $\mathrm{diam}(T_q) \leq 4$. Let $p \in \rho(V_2)$, $n_l \in V_2$, so there is $n_k \in V_2$ that p divides n_k. Since $d_{\mathrm{IP}(S)}(n_l, n_k) \leq 2$, so $d(p, n_l) \leq 5$. For each two elements of V_2 say n_s, n_t, we may find a path of length four between them. Also for $p, q \in \rho(V_2)$ we have $d(p, q) \leq 6$. Hence $\mathrm{diam}(T_p) \leq 6$. □

Let (X, S) be an association scheme, and T a closed subset of S. By [10] (Lemma 2.1.4(b)), we conclude that $\{xT | x \in X\}$ is a partition of X. Let $X/T = \{xT | x \in X\}$. For any $s \in S$, let us define

$$s^T = \{(yT, zT) | z \in yTsT\}, \quad \text{and} \quad S//T = \{s^T | s \in S\}.$$

Thus, for any $s \in S$ and any $y, z \in X$, $(yT, zT) \in s^T$ if and only if $(y, z) \in TsT$. If S is a naturally valenced association scheme and T is a finite closed subset of S, then by [10] (Theorem 4.1.3), $S//T$ is an association scheme on X/T, and for any $p, r \in S$, we have $n_{p^T} = \frac{n_{TpT}}{n_T}$. The association scheme $(X/T, S//T)$ is called *quotient scheme of* S *over* T.

By [9] (Lemma 3.6), we have the following hypothesis.

Hypothesis 1. *Assume that the graph* $\mathrm{IP}(S)$ *is stable and has two non-trivial components. If* $T = \{s \in S | n_s = 1\} \bigcup \{s \in S | n_s \in V(C(n_p))\}$*, where* $p \in S$ *such that* n_p *is the minimal vertex of the graph* $\mathrm{IP}(S)$*, then* T *is a closed subset of* S*.*

Theorem 7. *Suppose that* $\mathrm{IP}(S)$ *and* T *are given as in Hypothesis 1,* $n_T < \infty$ *and for each* $s \in S \setminus T$*,* n_T *divides* $|TsT|$*. Then* $\mathrm{BIP}(S//T)$ *has only one non-trivial component which has diameter at most four.*

Proof It is clear that for any $s \in T$, $n_{s^T} = 1$. Now suppose that $r, s \in S \setminus T$ such that $n_{s^T} \neq 1$ and $n_{r^T} \neq 1$. Since $s \in S \setminus T$, by [9] (Corollary 3.9), for any $r \in TsT$, $n_r = n_s$. Now we have

$$n_{TsT} = \sum_{p \in TsT} n_p = \sum_{p \in TsT} n_s = n_s |TsT|.$$

In a similar way we conclude that $n_{TrT} = n_r |TrT|$. If n_r, n_s are not coprime, by the hypothesis we conclude that n_{s^T} and n_{r^T} are not coprime. So these two vertices are adjacent in the graph $\mathrm{IP}(S//T)$. By the definition of T we conclude that $\mathrm{IP}(S//T)$ has only one non-trivial component which is a complete graph, so $\mathrm{BIP}(S//T)$ has only one non-trivial component which has diameter at most four. □

Remark 2. *From [4] we know that*

(i) *If* B *contains a cycle* C *which has length more than four, then* Γ *has a cycle and we have* $\mathbf{g}(\Gamma) = 3$ *or* $\mathbf{g}(\Gamma) = \frac{1}{2}\mathbf{g}^*(B)$*, where* $\mathbf{g}^*(B)$ *denotes the length of a cycle with minimum length and more than four vertices;*

(ii) *At least one of* Δ *or* Γ *has a triangle if and only if* B *has* C_6 *or* $K_{1,3}$*;*

(iii) *Both* Γ, Δ *are acyclic if and only if each component of* B *is a path or a cycle of length four;*

(iv) *Both* Γ, Δ *are trees if and only if* B *is a path.*

So the following hold;

(i) *If* $\mathrm{BIP}(S)$ *has a cycle of length more than* 4*, then* $\mathrm{IP}(S)$ *has a cycle and* $\mathbf{g}(\mathrm{IP}(S)) = 3$ *or* $\mathbf{g}(\mathrm{IP}(S)) = \frac{1}{2}\mathbf{g}^*(\mathrm{BIP}(S))$*;*

(ii) *If* $\mathrm{IP}(S)$ *has a triangle then* $\mathrm{BIP}(S)$ *has* C_6 *or* $K_{1,3}$ *(the inverse does not necessarily hold);*

(iii) *If each component of* $\mathrm{BIP}(S)$ *is a path or* C_4*, then* $\mathrm{IP}(S)$ *is acyclic;*

(iv) *If* $\mathrm{BIP}(S)$ *is a path, then* $\mathrm{IP}(S)$ *is a tree.*

Example 3._(Johnson Scheme)_
Let X be the set of all two-element subsets of $\{1,2,3,\ldots,m\}$, $m \geq 4$. The symmetric group $G = S_m$ acts on X as a transitive permutation group. If $x = (a,b) \in X$, then G_x has precisely two orbits on $X \setminus \{(a,b)\}$, namely

$$\lambda(x) = \{(a,c) : c \neq a,b\} \cup \{(b,c) : c \neq a,b\} \text{ and } \quad \eta(x) = \{(c,d) : c,d \neq a,b\}.$$

Let $m = 5$, and $x = (1,2)$. In this case $\mathrm{BIP}(S)$ is isomorphic to P_3, so by the last part of Remark 2, we conclude that $\mathrm{IP}(S)$ is a tree.

Corollary 1. _Let (X,S) be the naturally valenced association scheme. Then $\mathrm{BIP}(S)$ contains C_6, or $K_{1,3}$ if one of the following cases holds:_

_(i) $O_v(S)$ is a closed subset of S, $\mathrm{IP}(S)$ is not stable, and contains two nontrivial components;_

_(ii) $O_v(S)$ is a closed subset of S, $\mathrm{IP}(S)$ is stable, and contains two nontrivial components, such that the complete component is K_m, $m \geq 3$;_

(iii) $O_v(S)$ is not a closed subset of S, $\mathrm{IP}(S)$ contains just one nontrivial component say $C(n_p)$, such that n_s is the only neighbor of n_p, and there exist two vertices other than n_p, n_s in $C(n_p)$, such that they are adjacent.

Proof By Remark 2, we know that if $\mathrm{IP}(S)$ has a triangle, then $\mathrm{BIP}(S)$ contains C_6 or $K_{1,3}$. So it is enough to find a triangle in $\mathrm{IP}(S)$ in each case. In the first case, since $\mathrm{IP}(S)$ is not stable, so there exists $p \in S$, such that n_{p^*} is not a vertex of the component $C(n_p)$. Since $\mathrm{IP}(S)$ has two nontrivial components, by Theorem 2, both components of $\mathrm{IP}(S)$ are infinite. Since one of them is a complete graph, we have a triangle. In the second case since $m \geq 3$, it is trivial. Finally in the last case, by using the assumption and also [8] (Lemma 4.2), we can see there exists a triangle in $\mathrm{IP}(S)$. □

Let G_1 and G_2 be two groups that act transitively on sets X_1, and X_2, respectively. Suppose that D_1, D_2 and D are the sets of subdegrees related to G_1, G_2 and $G_1 \times G_2$, respectively. Let S_1, S_2 and S be the sets of valencies of G_1, G_2 and $G_1 \times G_2$, respectively. Since $D = D_1 D_2 = \{uv | u \in D_1, v \in D_2\}$, so D is the product of two sets of positive integers, now by [3] we have the following properties:

(i) $\mathrm{BIP}(S)$ is always connected and $\mathrm{diam}(\mathrm{BIP}(S)) \leq 6$;

(ii) $\mathrm{IP}(S)$ is connected and $\mathrm{diam}(\mathrm{IP}(S)) \leq 3$;

(iii) $\mathbf{g}(\mathrm{BIP}(S)) = 4$ if one of the following conditions holds:

- (a) $\mathrm{BIP}(S_1)$ has a cycle and $|D_2^* = D_2 \setminus \{1\}| \geq 1$;
- (b) $\mathrm{BIP}(S)$ is connected , $|D_1^* = D_1 \setminus \{1\}| \geq 2$, and there exists $q \in \rho(D_2) \setminus \rho(D_1)$;
- (c) Both $\mathrm{BIP}(S_1), \mathrm{BIP}(S_2)$ are acyclic and disconnected, such that there is a component of $\mathrm{BIP}(S_1)$, which has at least three vertices (or on the other hand contains P_2 which is a path of length two).

Example 4. _Let H be a nonabelian group of order pq, where p and q are primes, and $p \neq q$. This group has conjugacy class sizes $1, p, q$. By the discussion in [5] (Section 1), it follows that $D = \{1, p, q\}$ occurs as a subdegree set (for the semidirect product $H \ltimes H$ with conjugation action). Now let r be a different prime from p, q. The set $\{1, r\}$ occurs as a subdegree set in any group having a non-normal subgroup of order r. It follows_ *that the product set* $\{1, p, q\}.\{1, r\} = \{1, p, q, r, pr, qr\}$ *occurs as a subdegree set. This provides an example of* BIP *-graph, which is acyclic and has diameter* 6 *.*

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