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On the centroid of prime semirings

Hasret YAZARLI,[1,*] **Mehmet Ali ÖZTÜRK**[2]
Cumhuriyet University, Faculty of Arts and Sciences, Department of Mathematics, 58140 Sivas, Turkey
Adıyaman University, Faculty of Arts and Sciences, Department of Mathematics, Adıyaman, Turkey

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Abstract: We define and study the extended centroid of a prime semiring. We show that the extended centroid is a semifield and give some properties of the centroid of a right multiplicatively cancellable prime semiring.

Key words: Semiring, prime ideal, prime semiring, quotient semiring

1. Introduction

Semirings abound in the mathematical world around us. Indeed, the first mathematical structure we encounter—the set of natural numbers—is a semiring. Historically, semirings first appear implicitly in [3] and later in [8], [6], [10] and [7], in connection with the study of ideals of a ring. They also appear in [4] and [5] in connection with the axiomatization of the natural numbers and nonnegative rational numbers. Over the years, semirings have been studied by various researchers either in their own right, in an attempt to broaden techniques coming from semigroup theory or ring theory, or in connection with applications. In [9] Martindale first constructed for any prime ring R a "ring of quotients" Q. After, Öztürk and Jun introduced the extended centroid of a prime Γ-ring and obtained some results in Γ-ring M with derivation which was related to Q, and the quotient Γ-ring of M [11, 12]. In this paper, we define and study the extended centroid of a prime semiring.

2. Preliminaries

A semiring is a nonempty set R on which operations of addition and multiplication have been defined such that the following conditions are satisfied:

(1) $(R,+)$ is a commutative monoid with identity element 0_R;

(2) $(R,.)$ is a monoid with identity element 1_R;

(3) Multiplication distributes over addition from either side;

(4) $0_R r = 0_R = r0_R$ for all $r \in R$;

(5) $1_R \neq 0_R$.

An element a of semiring R is right multiplicatively cancellable if and only if $ba = ca$ only when $b = c$. Left multiplicatively cancellable elements are similarly defined. An element of R is multiplicatively cancellable if and only if it is both left and right multiplicatively cancellable.

*Correspondence: hyazarli@cumhuriyet.edu.tr
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An element of semiring R is a unit if and only if there exists element r_1 of R satisfying $rr_1 = 1_R = r_1r$. The set of all units of R is denoted by $U(R)$. $U(R)$ is submonoid of $(R, .)$ which is in fact a group. If $U(R) = R\backslash\{0_R\}$ then R is division semiring. A commutative division semiring is semifield.

A nonzero element a of semiring R is a left zero divisor if and only if there exists nonzero element b of R satisfying $ab = 0_R$. It is a right zero divisor if and only if there exists a nonzero element b of R satisfying $ba = 0_R$. It is a zero divisor if and only if it is both a left and right zero divisor. A semiring R having no zero divisors is entire.

Let a be an element of semiring R. An element b of R is additive inverse of a if and only if $a + b = 0_R$.

A left ideal U of a semiring R is a nonempty subset of R satisfying the following conditions:

(1) If $a, b \in U$ then $a + b \in U$;

(2) If $a \in U$ and $r \in R$ then $ra \in U$;

(3) $U \neq R$ $(1_R \notin U)$.

Note that ideals are proper, namely R is not an ideal of itself.

As in the case of rings, an ideal U of a semiring R is prime if and only if whenever $HK \subseteq U$, for ideals H and K of R, we must have either $H \subseteq U$ or $K \subseteq U$. R is a prime semiring if and only if ideal $\{0_R\}$ of semiring R is prime.

Proposition 1 *[2, Proposition 7.4] The following conditions on ideal U of a semiring R are equivalent:*

(1) U is prime;

(2) $\{arb | r \in R\} \subseteq U$ if and only if $a \in U$ or $b \in U$;

(3) If a and b are elements of R satisfying $(a)(b) \subseteq U$ then either $a \in U$ or $b \in U$.

If R and S are semirings then function $\gamma : R \to S$ is a morphism of semirings if and only if

(1) $\gamma(0_R) = 0_S$,

(2) $\gamma(1_R) = 1_S$,

(3) $\gamma(r_1 + r_2) = \gamma(r_1) + \gamma(r_2)$ and $\gamma(r_1r_2) = \gamma(r_1)\gamma(r_2)$ for all $r_1, r_2 \in R$.

A morphism of semirings which is both injective and surjective is called isomorphism. If there exists isomorphism between semirings R and S we write $R \cong S$. If $\gamma : R \to S$ is a morphism of semirings then $\mathrm{Im}(\gamma) = \{\gamma(r) | r \in R\}$ is a subsemiring of S.

Let R be a semiring. A left R semimodule is a commutative monoid $(M, +)$ with additive identity 0_M for which we have function $R \times M \to M$, denoted by $(r, m) \mapsto rm$ and called scalar multiplication, which satisfy the following conditions for all elements r_1 and r_2 of R and all elements m_1 and m_2 of M:

(1) $(r_1r_2)m = r_1(r_2m)$,

(2) $r(m_1 + m_2) = rm_1 + rm_2$,

(3) $1_Rm = m$,

(4) $r0_M = 0_M = 0_Rm$.

If R is a semiring and M and N are left R semimodules then a function α from M to N is an R homomorphism if and only if the following conditions are satisfied:

(1) $(m_1 + m_2)\alpha = m_1\alpha + m_2\alpha$ for all $m_1, m_2 \in M$;

(2) $(rm)\alpha = r(m\alpha)$ for all $m \in M$ and $r \in R$.

3. Extended centroid

Let R be a prime semiring. Let us denote set of all nonzero ideals of R and R by $N = N(R)$. That is,

$$N = N(R) = \{U \mid \{0_R\} \neq U \text{ is ideal of } R\} \cup \{R\}$$

U and R are regarded as right R semimodules.

Define relation $\equiv_N$ on $M = \{f : U \to R \mid U \in N, f \text{ is right } R \text{ homomorphism}\}$ as follows: Let $f, g \in M$.

$$f \equiv_N g \Leftrightarrow \text{there exists } K \in N \text{ and } K \subseteq U \cap V \text{ such that } f = g \text{ on } K$$

$$\text{where } U \in N \text{ and } V \in N \text{ are domains of } f \text{ and } g \text{ respectively... } (*)$$

For any $U \in N$ and $V \in N$ it is possible to find a nonzero $K \in N$, since R is a prime semiring. For all $U \in N$ which is domain of f, $f \equiv_N f$, since $U \subseteq U \cap U$ and $f = f$ on U. Thus $\equiv_N$ is reflexive. Let f, $g \in M$ where $U \in N$ and $V \in N$ are domains of f and g respectively. Suppose $f \equiv_N g$. Then there exists $K \in N$ and $K \subseteq U \cap V$ such that $f = g$ on K. Thus, $K \in N$ and $K \subseteq V \cap U$ such that $g = f$ on K, i.e., $g \equiv_N f$. Hence $\equiv_N$ is symmetric. Let f, $g, h \in M$ where $U \in N$, $V \in N$ and $H \in N$ are domains of f, g and h respectively. Suppose $f \equiv_N g$ and $g \equiv_N h$. Then there exist $K_1, K_2 \in N$ and $K_1 \subseteq U \cap V$ and $K_2 \subseteq V \cap H$ such that $f = g$ on K_1 and $g = h$ on K_2. Thus, $\{0_R\} \neq K = K_1 \cap K_2 \subseteq (U \cap V) \cap (V \cap H) \subseteq U \cap H$ and $f = h$ on K. This implies then $f \equiv_N h$. Hence $\equiv_N$ is transitive. Consequently, $\equiv_N$ is an equivalence relation on M. This gives a chance for us to get partition of M. We then denote the equivalence class by $\widehat{f} = [U, f]$, where $\widehat{f} := \{g : V \to R \mid f \equiv_N g\}$ and denote by Q_r set of all equivalence classes. That is,

$$Q_r = \left\{ \widehat{f} \,\middle|\, f : U \to R \text{ right } R \text{ homomorphism and } U \in N \right\}.$$

Now we define an addition $''+''$ on Q_r as

$$\widehat{f} + \widehat{g} = \widehat{f + g},$$

for all $\widehat{f}, \widehat{g} \in Q_r$. Let $\widehat{f}, \widehat{g} \in Q_r$, where $U \in N$ and $V \in N$ are domains of f and g, respectively. Therefore $f + g : U \cap V \to R$ is a right R homomorphism. Assume that $f_1 \equiv_N f_2$ and $g_1 \equiv_N g_2$, where U_1, U_2, V_1 and V_2 are domains of f_1, f_2, g_1 and g_2, respectively. Then $\exists K_1 \subseteq U_1 \cap U_2$, such that $f_1 = f_2$ on K_1 and $\exists K_2 \subseteq V_1 \cap V_2$ such that $g_1 = g_2$ on K_2. Taking $K = K_1 \cap K_2$. Then $K \neq \{0_R\}$ and

$$\begin{aligned} K = K_1 \cap K_2 &\subseteq (U_1 \cap U_2) \cap (V_1 \cap V_2) \\ &= (U_1 \cap V_1) \cap (U_2 \cap V_2). \end{aligned}$$

For any $x \in K$, we have $(f_1 + g_1)(x) = f_1(x) + g_1(x) = f_2(x) + g_2(x) = (f_2 + g_2)(x)$, and so $f_1 + g_1 = f_2 + g_2$ on K. Therefore $f_1 + g_1 \equiv_N f_2 + g_2$ where $f_1 + g_1 : U_1 \cap V_1 \to R$ and $f_2 + g_2 : U_2 \cap V_2 \to R$ are right R homomorphisms. That is, addition "+" is well-defined. Now we prove that Q_r is a commutative monoid. Let $\widehat{f}, \widehat{g}, \widehat{h} \in Q_r$ where $U \in N$, $V \in N$ and $H \in N$ are domains of f, g and h respectively. Since $U \cap (V \cap H) = (U \cap V) \cap H$, we get for all $x \in U \cap (V \cap H)$,

$$\begin{aligned} [f + (g + h)](x) &= f(x) + (g + h)(x) = f(x) + [g(x) + h(x)] \\ &= [f(x) + g(x)] + h(x) = (f + g)(x) + h(x) \\ &= [(f + g) + h](x). \end{aligned}$$

Hence $f+(g+h)=\ (f+g)+h$ on $U\cap(V\cap H)$. That is, $\widehat{f}+\left(\widehat{g}+\widehat{h}\right)=\left(\widehat{f}+\widehat{g}\right)+\widehat{h}$.

Taking $\widehat{\theta}\in Q_r$, where $\theta:R\rightarrow R,\ x\mapsto 0_R$ for all $x\in R$.Let $\widehat{f}\in Q_r$, where $U\in N$ is domain of f. Since $U\subseteq U\cap R$, we get for all $x\in U$,

$$(f+\theta)(x)=f(x)+\theta(x)=f(x)+0_R=f(x)$$

and

$$(\theta+f)(x)=\theta(x)+f(x)=0_R+f(x)=f(x).$$

Thus, $\widehat{f}+\widehat{\theta}=\widehat{\theta}+\widehat{f}=\widehat{f}$. Hence $\widehat{\theta}$ is the additive identity in Q_r.

Finally, for any elements $\widehat{f},\widehat{g}\in Q_r$ where $U\in N$ and $V\in N$ are domains of f and g respectively, we have for all $x\in U\cap V=V\cap U$,

$$(f+g)(x)=f(x)+g(x)=g(x)+f(x)=(g+f)(x).$$

That is, $\widehat{f}+\widehat{g}=\widehat{g}+\widehat{f}$. Therefore $(Q_r,+)$ commutative monoid.

Now we define a multiplication "." on Q_r as

$$\widehat{f}\widehat{g}=\widehat{fg},$$

for all $\widehat{f},\widehat{g}\in Q_r$. Let $\widehat{f},\widehat{g}\in Q_r$ where $U\in N$ and $V\in N$ are domains of f and g, respectively. Therefore $fg:VU\rightarrow R$ is a right R homomorphism. Assume that $f_1\equiv_N f_2$ and $g_1\equiv_N g_2$ where U_1,U_2, V_1 and V_2 are domains of f_1,f_2,g_1 and g_2 respectively. Then $\exists K_1\subseteq U_1\cap U_2$ such that $f_1=f_2$ on K_1 and $\exists K_2\subseteq V_1\cap V_2$ such that $g_1=g_2$ on K_2. Also $V_1U_1\cap V_2U_2\subseteq(U_1\cap U_2)\cap(V_1\cap V_2)=(U_1\cap V_1)\cap(U_2\cap V_2)$ and there exists $\{0_R\}\neq K\in N$ such that $K\subseteq V_1U_1\cap V_2U_2$. For any $x\in K$, $x\in V_1U_1\cap V_2U_2$. So that $x\in V_1U_1$ and $x\in V_2U_2$. Then, $x=\sum\limits_{\text{finite}}a_ib_i$, $a_i\in V_1\cap V_2$ and $b_i\in U_1\cap U_2$. Therefore

$$\begin{aligned}(f_1g_1)(x)=f_1(g_1(x))&=f_1\left(g_1\left(\sum_{\text{finite}}a_ib_i\right)\right)\\&=f_1\left(\sum_{\text{finite}}g_1(a_i)\,b_i\right)=f_1\left(\sum_{\text{finite}}g_2(a_i)\,b_i\right)\\&=f_2\left(\sum_{\text{finite}}g_2(a_i)\,b_i\right)=f_2\left(g_2\left(\sum_{\text{finite}}a_ib_i\right)\right)\\&=f_2(g_2(x))=(f_2g_2)(x),\end{aligned}$$

and so $f_1g_1=f_2g_2$ on K. Hence, $f_1g_1\equiv_N f_2g_2$. That is, "." is well-defined. Now we will prove that $(Q_r,.)$ is a monoid. Let $\widehat{f},\widehat{g},\widehat{h}\in Q_r$ where $U\in N$, $V\in N$ and $H\in N$ are domains of f, g and h, respectively. Since $H(VU)=(HV)U$, we get for all $x\in H(VU)$,

$$\begin{aligned}\left[(fg)\,h\right](x)&=(fg)(h(x))=f(g(h(x)))\\&=f((gh)(x))=(f(gh))(x)\end{aligned}$$

Hence $(fg)h = f(gh)$ on $H(VU)$. That is, $\left(\widehat{f}\widehat{g}\right)\widehat{h} = \widehat{f}\left(\widehat{g}\widehat{h}\right)$.

Taking $\widehat{1} \in Q_r$ where $1 : R \to R$, $x \mapsto x$ for all $x \in R$. Let $\widehat{f} \in Q_r$, where $U \in N$ is domain of f. Since $RU \subseteq U$, we get for all $x \in RU$, $(f1)(x) = f(1(x)) = f(x)$ and $(1f)(x) = 1(f(x)) = f(x)$. Thus, $\widehat{f}\widehat{1} = \widehat{1}\widehat{f} = \widehat{f}$. Hence $\widehat{1}$ is the multiplicative identity in Q_r. Therefore $(Q_r, .)$ is a monoid.

Let $\widehat{f}, \widehat{g}, \widehat{h} \in Q_r$ where $U \in N$, $V \in N$ and $H \in N$ are domains of f, g and h respectively. Since $(V \cap H)U \subseteq VU \cap HU$, we get for all $x \in (V \cap H)U$,

$$\begin{aligned}[f(g+h)](x) &= f((g+h)(x)) = f(g(x) + h(x)) \\ &= f(g(x)) + f(h(x)) = (fg + fh)(x).\end{aligned}$$

Hence $f(g+h) = fg + fh$ on $(V \cap H)U$. That is, $\widehat{f}\left(\widehat{g} + \widehat{h}\right) = \widehat{f}\widehat{g} + \widehat{f}\widehat{h}$. Similarly, $\left(\widehat{f} + \widehat{g}\right)\widehat{h} = \widehat{f}\widehat{h} + \widehat{g}\widehat{h}$.

Suppose that $\widehat{\theta} = \widehat{1}$. Then, $\theta \equiv_N 1$. That is, there exist $K \in N$ and $K \subseteq R \cap R$ such that $\theta = 1$ on K. Therefore, $\theta(x) = 1(x)$ for all $x \in K$. This is a contradiction with $\{0_R\} \neq K$. Thus, $\widehat{\theta} \neq \widehat{1}$.

Now we prove that $\widehat{f}\ \widehat{\theta} = \widehat{\theta} = \widehat{\theta}\widehat{f}$ for all $\widehat{f} \in Q_r$. Since $RU \subseteq RU \cap R$ and $f\theta = \theta$ on RU where $U \in N$ is domain of f, we get $\widehat{f}\ \widehat{\theta} = \widehat{\theta}$. Similarly $\widehat{\theta}\widehat{f} = \widehat{\theta}$. Thus, $(Q_r, +, .)$ is a semiring.

Theorem 1 *Let R be prime semiring and the relation $\equiv_N$ on M defined by $(*)$. If R is a right multiplicatively cancellable semiring, then R may be embedded in Q_r as a subsemiring.*

Proof Let $a \in R$. Define $\lambda_a : R \to R$ by $\lambda_a(r) = ar$ for all $r \in R$. It is clear that λ_a is a right R homomorphism. Since $R \in N$, then $\lambda_a \in M$. So λ_a defines element $\widehat{\lambda}_a$ of Q_r. Hence we may define $\lambda : R \to Q_r$ by $\lambda(a) = \widehat{\lambda}_a$ for $a \in R$. Clearly λ is well-defined. Let $\lambda(a) = \lambda(b)$ for any $a, b \in R$. Then, $\widehat{\lambda}_a = \widehat{\lambda}_b$, i.e., $\lambda_a \equiv_N \lambda_b$. Hence there exists $K \in N$ and $K \subseteq R \cap R$ such that $\lambda_a = \lambda_b$ on K. Therefore, $\lambda_a(x) = \lambda_b(x)$ for all $x \in K$. So, $ax = bx$ for all $x \in K$. Since R is right multiplicatively cancellable semiring, we get $a = b$. That is, λ is injective. In order to prove λ is a homomorphism, let a, $b \in R$. Then

$$\begin{aligned}\lambda_{a+b}(x) &= (a+b)x = ax + bx = \lambda_a(x) + \lambda_b(x) \\ &= (\lambda_a + \lambda_b)(x)\end{aligned}$$

and

$$\begin{aligned}\lambda_{ab}(x) &= (ab)x = a(bx) = \lambda_a(bx) \\ &= \lambda_a(\lambda_b(x)) = (\lambda_a\lambda_b)(x),\end{aligned}$$

for all $x \in R$. It follows that $\lambda_{a+b} = \lambda_a + \lambda_b$ and $\lambda_{ab} = \lambda_a\lambda_b$. Hence

$$\lambda(a+b) = \widehat{\lambda}_{a+b} = \widehat{\lambda_a + \lambda_b} = \widehat{\lambda}_a + \widehat{\lambda}_b = \lambda(a) + \lambda(b)$$

and

$$\lambda(ab) = \widehat{\lambda}_{ab} = \widehat{\lambda_a\lambda_b} = \widehat{\lambda}_a\widehat{\lambda}_b = \lambda(a)\lambda(b).$$

Also, $\lambda(0_R) = \widehat{\theta}$ and $\lambda(1_R) = \widehat{1}$. Therefore R is a subsemiring of Q_r. □

Therefore R is a subsemiring of Q_r, we call Q_r the right quotient semiring of R. For purposes of convenience, we use q instead of $\hat{q} \in Q_r$.

Lemma 1 *Let R be a right multiplicatively cancellable prime semiring. For each nonzero $q \in Q_r$, there is nonzero ideal U of R such that $q(U) \subseteq R$.*

Proof Clear. □

Lemma 2 *Let R be a right multiplicatively cancellable prime semiring. Then the quotient semiring Q_r of R is a prime semiring.*

Proof Let p, $q \in Q_r$ be such that $pQ_rq = \theta$. If $p \neq \theta \neq q$, then there exist nonzero ideals U and V of R such that $p(U) \subseteq R$ and $q(V) \subseteq R$. Since $p \neq \theta \neq q$, there exist nonzero elements $x \in U$ and $y \in V$ such that $p(x) \neq 0_R \neq q(y)$. Noticing that R is a subsemiring of Q_r, we have

$$p(x)\, Rq(y) \subseteq p(x)\, Q_r q(y) = \{0_R\}$$

and so $p(x)\, Rq(y) = \{0_R\}$. This is a contradiction. Hence $p = \theta$ or $q = \theta$, ending the proof. □

Definition 1 *The set*

$$C := \{g \in Q_r \mid gf = fg \text{ for all } f \in Q_r\}$$

is called the extended centroid of a semiring R.

It is clear that C is a subsemiring of Q_r. Let $\theta \neq c \in C$. Assume that $cf = \theta$ for any $f \in C$. Then, $gcf = \theta$ for any $g \in Q_r$. Hence, $cgf = \theta$, i.e., $cQ_rf = \theta$. Since $\theta \neq c$, we get $f = \theta$. Thus, C is an entire semiring.

Theorem 2 *The center C of Q_r is a semifield.*

Proof We prove that C is a semifield such that every $\theta \neq c \in C$ is inverse. Let $\theta \neq c \in C$. Since $c \in Q_r$, there exists $\{0_R\} \neq U \in N$ such that $cU \subseteq R$. Since R is a prime semiring, $cU \neq \{0_R\}$. Taking $\{0_R\} \neq V = cU$ ideal of R. Moreover, $d : V \to R$, $ca \mapsto a$ is a R homomorphism. Then, $d = [d, V] \in Q_r$. Also, $dc = 1$ since $dc(a) = a = 1(a)$ for all $a \in U$. Thus, $\theta \neq c \in C$ is inverse. That is, C is a semifield. □

We now let $S = RC$, a subsemiring of Q_r containing R. We shall call S the central closure of R. The same proof used in showing that Q_r was prime may be employed to show that S is prime.

Proposition 2 *Let R be right multiplicatively cancellable prime semiring and S be the central closure of R. Then S is a right multiplicatively cancellable prime semiring.*

Proof Since $S = \left\{ \sum_{\text{finite}} r_i c_i | \text{ for all } r_i \in R, c_i \in C \right\}$, the proof of proposition is sufficient for the finite sums with $i = 1$ to 2. Now, let $ad = bd$, $a = r_1c_1 + r_2c_2$, $b = r_1'c_1' + r_2'c_2'$, $0 \neq d = r_1''c_1'' + r_2''c_2'' \in S$, for all r_1, r_2, $r_1', r_2', r_1'', r_2'' \in R$, c_1, c_2, $c_1', c_2', c_1'', c_2'' \in C$ and $c_1 = [f_1, U_1]$, $c_2 = [f_2, U_2]$, $c_1' = [g_1, V_1]$, $c_2' = [g_2, V_2]$, $c_1'' = [h_1, W_1]$, $c_2'' = [h_2, W_2]$. We have that

$$([\lambda_{r_1}, R]\,[f_1, U_1] + [\lambda_{r_2}, R]\,[f_2, U_2])([\lambda_{r_1''}, R]\,[h_1, W_1] + [\lambda_{r_2''}, R]\,[h_2, W_2]\ )$$

$$= ([\lambda_{r_1'}, R][g_1, V_1] + [\lambda_{r_2'}, R]\,[g_2, V_2])([\lambda_{r_1''}, R]\,[h_1, W_1] + [\lambda_{r_2''}, R]\,[h_2, W_2]\ ).$$

Then,

$$([\lambda_{r_1}f_1, U_1] + [\lambda_{r_2}f_2, U_2])([\lambda_{r_1''}h_1, W_1] + [\lambda_{r_2''}h_2, W_2]\ )$$

$$= ([\lambda_{r_1'}g_1, V_1] + [\lambda_{r_2'}g_2, V_2])([\lambda_{r_1''}h_1, W_1] + [\lambda_{r_2''}h_2, W_2])$$

$$([\lambda_{r_1}f_1 + \lambda_{r_2}f_2, U_1 \cap U_2])([\lambda_{r_1''}h_1 + \lambda_{r_2''}h_2, W_1 \cap W_2]\ )$$

$$= ([\lambda_{r_1'}g_1 + \lambda_{r_2'}g_2, V_1 \cap V_2])([\lambda_{r_1''}h_1 + \lambda_{r_2''}h_2, W_1 \cap W_2])$$

$$[(\lambda_{r_1}f_1 + \lambda_{r_2}f_2)(\lambda_{r_1''}h_1 + \lambda_{r_2''}h_2), (W_1 \cap W_2)(U_1 \cap U_2)]$$

$$= [(\lambda_{r_1'}g_1 + \lambda_{r_2'}g_2)(\lambda_{r_1''}h_1 + \lambda_{r_2''}h_2), (W_1 \cap W_2)(V_1 \cap V_2)].$$

Thus, there exist $K \in N$ and $K \subseteq (W_1 \cap W_2)(U_1 \cap U_2) \cap (W_1 \cap W_2)(V_1 \cap V_2)$ such that $(\lambda_{r_1}f_1 + \lambda_{r_2}f_2)(\lambda_{r_1''}h_1 + \lambda_{r_2''}h_2) = (\lambda_{r_1'}g_1 + \lambda_{r_2'}g_2)(\lambda_{r_1''}h_1 + \lambda_{r_2''}h_2)$ on K. $\forall x \in K$,

$$((\lambda_{r_1}f_1 + \lambda_{r_2}f_2)(\lambda_{r_1''}h_1 + \lambda_{r_2''}h_2))(x) = ((\lambda_{r_1'}g_1 + \lambda_{r_2'}g_2)(\lambda_{r_1''}h_1 + \lambda_{r_2''}h_2))(x)$$

$$(\lambda_{r_1}f_1 + \lambda_{r_2}f_2)((\lambda_{r_1''}h_1 + \lambda_{r_2''}h_2)(x)) = (\lambda_{r_1'}g_1 + \lambda_{r_2'}g_2)((\lambda_{r_1''}h_1 + \lambda_{r_2''}h_2)(x))$$

and so $\lambda_{r_1}f_1 + \lambda_{r_2}f_2 = \lambda_{r_1'}g_1 + \lambda_{r_2'}g_2$ on $(\lambda_{r_1''}h_1 + \lambda_{r_2''}h_2)(K)$. Since h_1 and h_2 are right R homomorphisms, and $K \subseteq (W_1 \cap W_2)(U_1 \cap U_2) \cap (W_1 \cap W_2)(V_1 \cap V_2)$, we get $h_1(K) \subseteq h_1(W_1 \cap W_2)(U_1 \cap U_2) \cap h_1(W_1 \cap W_2)(V_1 \cap V_2) \subseteq (U_1 \cap U_2) \cap (V_1 \cap V_2)$ and $h_2(K) \subseteq h_2(W_1 \cap W_2)(U_1 \cap U_2) \cap h_2(W_1 \cap W_2)(V_1 \cap V_2) \subseteq (U_1 \cap U_2) \cap (V_1 \cap V_2)$. So that $Rh_1(K) + Rh_2(K) \subseteq (U_1 \cap U_2) \cap (V_1 \cap V_2)$. If we use that $h_1(K)$ and $h_2(K)$ are right ideals, then $Rh_1(K) + Rh_2(K)$ is an ideal of R. We assume that $Rh_1(K) + Rh_2(K) = \{0_R\}$. Therefore we get $Rh_1(K) \cap Rh_2(K) \subseteq Rh_1(K) + Rh_2(K) = \{0_R\}$. Since R is a prime semiring, $Rh_1(K) = \{0_R\}$ or $Rh_2(K) = \{0_R\}$. Hence we obtain that $\left(\lambda_{r_1''}h_1 + \lambda_{r_2''}h_2\right)(K) = \{0_R\}$ for all $r_1'', r_2'' \in R$ from $\left(\lambda_{r_1''}h_1\right)(K) = \{0_R\}$ and $\left(\lambda_{r_2''}h_2\right)(K) = \{0_R\}$, this contradicts with $d \neq 0$. So $Rh_1(K) + Rh_2(K)$ must be nonzero. Therefore $\lambda_{r_1}f_1 + \lambda_{r_2}f_2 = \lambda_{r_1'}g_1 + \lambda_{r_2'}g_2$ on $Rh_1(K) + Rh_2(K)$. Hence $[\lambda_{r_1}, R][f_1, U_1] + [\lambda_{r_2}, R][f_2, U_2] = [\lambda_{r_1'}, R][g_1, V_1] + [\lambda_{r_2'}, R][g_2, V_2]$. That is, $r_1c_1 + r_2c_2 = r_1'c_1' + r_2'c_2'$ and so $a = b$, ending the proof. □

Theorem 3 *Let R be a right multiplicatively cancellable prime semiring. If a and b are nonzero elements in S such that $axb = bxa$ for all $x \in R$ and S is left cancellable then there exists $q \in C$ such that $qa = b$.*

Proof We may assume that $a \neq \theta$ and $b \neq \theta$. Let U be a nonzero ideal of R such that $aU \subseteq R$ and $bU \subseteq R$, and $V = UaU$. We define a mapping $f : V \to R$, $f\left(\sum_i x_i a y_i\right) = \sum_i x_i b y_i$, x_i, $y_i \in U$. Let $\sum_i x_i a y_i = \sum_i x_i' a y_i'$, x_i, y_i, x_i', $y_i' \in U$. Then,

$$\left(\sum_i x_i a y_i\right) b = \left(\sum_i x_i' a y_i'\right) b$$

$$\sum_i x_i (a y_i b) = \sum_i x_i' (a y_i' b)$$

$$\sum_i x_i\,(by_ia) = \sum_i x_i'\,(by_i'a)$$

$$\left(\sum_i x_iby_i\right)a = \left(\sum_i x_i'by_i'\right)a.$$

Since S is a right multiplicatively cancellable semiring by Proposition 2, we get $\sum_i x_iby_i = \sum_i x_i'by_i'$. That is, f is well-defined. f is a right R homomorphism. Because $f\left((xay)\,r\right) = f\left(xa\,(yr)\right) = xb\,(yr) = (xby)\,r = f\,(xay)\,r$ for all $x, y \in U$ and $r \in R$. Let q denote the element of Q_r determined by f and let p be any element of Q_r, with $pK \subseteq R$ for some nonzero ideal K of R. For $x, y \in U$ and $z \in K$ we have $(qp)\,(zxay) = q\left((pz)\,xay\right) = (pz)\,xby = p\,(zxby) = pq\,(zxay)$. Since $KU \subseteq K \cap U$ and $pq = qp$ on KU, we get $qp = pq$ for all $p \in Q_r$, and so $q \in C$. Since S is multiplicatively cancellable, $xqay = xby$, for all $x, y \in U$ implies $qa = b$. □

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