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On theta pair for a proper subalgebra

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Abstract: For a proper subalgebra K of a finite dimensional Lie algebra L, a pair (A,B) of subalgebras of L is called a θ-pair if $L=\langle A,K\rangle$, B is the largest ideal of L contained in $A\cap K$ and for each proper subalgebra C/B of A/B which is an ideal of L/B, we have $L\neq C+K$. In this article, using this concept, we give some characterizations of solvability and supersolvability of a finite dimensional Lie algebra.

Key words: θ-pair, Lie algebra, solvable, supersolvable

1. Introduction

All Lie algebras referred to in this article are of finite dimension at least two over a fixed field Λ. Let L be a Lie algebra with a subalgebra K. Then the Frattini ideal, the core (with respect to L) of K, which is the largest ideal of L in K, and the centralizer of K in L are denoted by $\varphi(L)$, K_L and $C_L(K)$, respectively. Also, K is called a 2-maximal subalgebra of L if K is a maximal subalgebra of some maximal subalgebra M of L.

The relation between the properties of subalgebras of a Lie algebra L and the structure of L has been studied extensively. In [8] and [10], Towers introduced, respectively, the concepts of c-ideality of subalgebras and the ideal index of maximal subalgebras of a Lie algebra and showed that these concepts play important roles in the study of Lie algebra theory (see also [6]). The first two authors in [5] presented the notion of θ-pair for maximal subalgebras and investigated its influence on the structure of some certain Lie algebras. In this article, we define the θ-pair for proper subalgebras and give some equivalent conditions for solvability and supersolvability of a Lie algebra.

Definition. *Let K be a proper subalgebra of a Lie algebra L. A pair (A,B) of subalgebras of L is said to be a θ-pair for K in L if it satisfies the following conditions:*

(i) $L=\langle A,K\rangle$ and $B=(A\cap K)_L$;

(ii) if C/B is a proper subalgebra of A/B which is an ideal of L/B, then $L\neq C+K$.

In addition, if A is an ideal of L, then the pair (A,B) is called an ideal θ-pair for K.

This is analogous to the concept of θ-pair for any proper subgroup of a finite group given by Li and Li in [4]; and it has since been studied by a number of authors (see [2,3]).

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Given a proper subalgebra K of a Lie algebra L, we denote by $\theta_L(K)$ the set of all θ-pairs for K in L. It is easily seen that $\theta_L(K)$ is the same as the set of θ-pairs for maximal subalgebras as defined in [5] when K is a maximal subalgebra of L. We define a partial order on $\theta_L(K)$ by means of $(A,B) \preceq (C,D)$ if and only if $A \leq C$, and the maximal elements with respect to ordering are called *maximal θ-pairs*.

The following two results give some information on θ-pairs which will be needed in our investigation.

Lemma 1.1 *Let L be a Lie algebra with a proper subalgebra K and an ideal N. Then*

(i) if (A,B) is a maximal θ-pair in $\theta_L(K)$ and $N \subseteq B \cap K$, then $(A/N, B/N)$ is a maximal θ-pair in $\theta_{L/N}(K/N)$. Conversely, if $(A/N, B/N)$ is a maximal θ-pair in $\theta_{L/N}(K/N)$, then (A,B) is a maximal θ-pair in $\theta_L(K)$;

(ii) if $(A,B),(C,D) \in \theta_L(K)$ and $A \leq C$, then $B \leq D$;

(iii) if (A,B) is an ideal θ-pair in $\theta_L(K)$, then $A \cap K_L = B$;

(iv) if (A,B) is a maximal ideal θ-pair in $\theta_L(K)$ and N is contained in K, then $N \subseteq B$. In particular, $B = K_L$.

Proof (i) It is straightforward from the definition of θ-pair.

(ii) It is obvious that $B + D$ is an ideal of L contained in $C \cap K$ and then $B \leq D$.

(iii) It is readily seen that $(A \cap K)_L = A \cap K_L$.

(iv) Suppose that on the contrary $N \nsubseteq B$. Then $A + N$ is an ideal of L properly containing N and $(A+N) \cap K_L = B + N$. If C is an ideal of L with $B + N \subseteq C \subseteq A + N$ and $L = C + K$, then $(A \cap C)/B$ is an ideal of L/B contained in A/B and

$$(A \cap C) + K = ((A+N) \cap C) + K = C + K.$$

Hence, using the assumption, we have $A \cap C = B$ or $A \cap C = A$. In the former case $C = B + N$ and in the second case $C = A + N$, We therefore conclude that $(A+N, B+N)$ is an ideal θ-pair in $\theta_L(K)$, contradicting the maximality of (A,B) in $\theta_L(K)$. □

Lemma 1.2 *Let L be a Lie algebra with a proper subalgebra K, and $(A,B),(C,D) \in \theta_L(K)$ in which A is an ideal of L contained in C. Then C is an ideal of L and $A/B \cong C/D$. In addition, if $A \cap K = B$, then $C \cap K = D$.*

Proof As $A + D$ is an ideal of L contained in C and $L = (A+D) + K$, the condition (ii) in the definition of θ-pair yields that $A + D = C$ and so C is an ideal of L. We also have $A \cap D \leq A \cap K_L = B$, whence $A \cap D = B$ and $A/B \cong C/D$. Now, if $A \cap K = B$, then $C \cap K = (A+D) \cap K = (A \cap K) + D = D$. This completes the proof of the lemma. □

Using the above lemma, we regard that maximal ideal θ-pairs for a proper subalgebra K of L are maximal elements in $\theta_L(K)$.

Recall that the abelian socle of a Lie algebra L, $\mathrm{Asoc}(L)$, is the union of all minimal abelian ideals. It is easily checked that $\mathrm{Asoc}(L)$ is the direct sum of minimal abelian ideals. Also, we denote the nil radical of L by $N(L)$. The following proposition will be used in the third section.

Proposition 1.3 ([7]) *Let L be a Lie algebra. Then* *(i)* $N(L/\varphi(L)) = N(L)/\varphi(L)$.

(ii) *If* $\varphi(L) = 0$, *then* $Asoc(L) = N(L)$.

2. Some characterizations of solvable Lie algebras

In this section, we first characterize the solvable Lie algebras in terms of ideal θ-pairs.

Theorem 2.1 *A Lie algebra* L *is solvable if for each* 2*-maximal subalgebra* K *of* L, *there is a maximal ideal* θ*-pair* (A,B) *in* $\theta_L(K)$ *such that* $C_{L/B}(A/B) \neq 0$.

Proof It is obvious that L cannot be simple. Let N be a minimal ideal of L and K/N be a 2-maximal subalgebra of L/N. By hypothesis, there is a maximal ideal θ-pair (A,B) in $\theta_L(K)$ with $C_{L/B}(A/B) \neq 0$. Note that $N \leq B$ by Lemma 1.1(iv). Invoking Lemma 1.1(i), $(A/N, B/N)$ is a maximal ideal θ-pair in $\theta_{L/N}(K/N)$ and moreover, $C_{(L/N)/(B/N)}((A/N)/(B/N)) \cong C_{L/B}(A/B) \neq 0$. By induction, L/N is solvable. If N' is another minimal ideal of L, then L/N' is solvable and $N \cap N' = 0$. Hence L is solvable. We thus assume that L has a unique minimal ideal N and L/N is solvable. If $\varphi(L) \neq 0$, then the result holds thanks to [7; Theorem 5]. If $\varphi(L) = 0$, then there is a maximal subalgebra M of L with $M_L = 0$. Suppose now that K is a maximal subalgebra of M and $(E, 0)$ is a maximal ideal θ-pair in $\theta_L(K)$ such that $C_L(E) \neq 0$. It follows that $N \subseteq E \cap C_L(E) = Z(E)$, and therefore L is solvable, which gives the required result. □

Combining this with Lemma 1.2, we have the following corollary.

Corollary 2.2 *A Lie algebra* L *is solvable if for each* 2*-maximal subalgebra* K *of* L, $\theta_L(K)$ *contains an ideal* θ*-pair* (A,B) *such that* A/B *is abelian.*

By arguments similar to those used in the proof of Theorem 2.1 and applying Lemma 1.2, we can further improve the above corollary as follows.

Theorem 2.3 *A Lie algebra* L *is solvable if and only if for each* 2*-maximal subalgebra* K *of* L, *there is an ideal* θ*-pair* (A,B) *in* $\theta_L(K)$ *such that the factor Lie algebra* A/B *is solvable.*

In proving the following result, we have used the argument given by Towers ([8; Theorem 3.4]).

Theorem 2.4 *A Lie algebra* L *is solvable if and only if for each maximal solvable subalgebra* K *of* L, *there is an ideal* θ*-pair* (A,B) *in* $\theta_L(K)$ *such that* $A \cap K = B$.

Proof The necessity holds by [5; Theorem A(v)]. Conversely, let $S(L)$ denote the largest solvable ideal of L. Suppose to the contrary that L is non-solvable and $x_1 \in L - S(L)$. Then $x_1 \in K$ for some maximal solvable subalgebra K of L. By the assumption, there is an ideal θ-pair (A_1, B_1) in $\theta_L(K)$ such that $A_1 \cap K = B_1$. Since $B_1 \subseteq S(L)$, it follows that $x_1 \notin A_1$ and the factor Lie algebra L/A_1 is solvable. $A_1 \subseteq S(L)$ implies that L is solvable, a contradiction. So, suppose that $A_1 \nsubseteq S(L)$ and $x_2 \in A_1 - S(L)$. The same argument shows that there is an ideal θ-pair (A_2, B_2) for K such that $x_2 \notin A_2$ and L/A_2 is solvable.
Clearly, $A_1 \cap A_2$ is a proper subalgebra of A_1, and $A_1 \cap A_2 \subseteq S(L)$ implies that L is solvable. So, we assume that $A_1 \cap A_2 \nsubseteq S(L)$ and $x_3 \in (A_1 \cap A_2) \backslash S(L)$. Continuing in this way, we find ideals $A_1, ..., A_n$ of L such that

$A_1 \cap A_2 \cap ... \cap A_n \subseteq S(L)$ and L/A_i is solvable for each $1 \le i \le n$. Since $L/\cap_{i=1}^{n} A_i$ is isomorphic to a subalgebra of $\bigoplus_{i=1}^{n}(L/A_i)$, the factor Lie algebra $L/\cap_{i=1}^{n} A_i$ and then L are solvable, which is again a contradiction. □

Note that Theorem 2.4 remains valid if the statement "maximal solvable subalgebra" is replaced by "maximal nilpotent subalgebra".

We now give additional criteria for the solvability of Lie algebras by using maximal θ-pairs.

Theorem 2.5 *A Lie algebra L is solvable if and only if for every 2-maximal subalgebra K of L and each maximal θ-pair (A, B) in $\theta_L(K)$, the factor Lie algebra A/B is solvable.*

Proof The necessity holds trivially. Conversely, let L be a non-solvable Lie algebra of the smallest dimension satisfying the hypothesis and N be a minimal ideal of L. Owing to Lemma 1.1(i), the factor Lie algebra L/N is solvable and then, without loss of generality, we may assume that N is the unique minimal ideal of L and $\varphi(L) = 0$. So, there is a maximal subalgebra M of L with $M_L = 0$. Now, suppose that K is a maximal subalgebra of M and $(A, 0)$ is a maximal pair in $\theta_L(K)$. If N is contained in A then N is solvable and so is L, a contradiction. Otherwise, the maximality of $(A, 0)$ yields that $(A + N, 0)$ does not belong to $\theta_L(K)$. Hence, we find an ideal B of L such that $B < A+N$ and $L = B+K$. If we choose the ideal C to be minimal with respect to $L = C + K$, then $(C, 0)$ is a θ-pair for K. Alternatively, $C \le D$ for some maximal θ-pair $(D, 0)$ in $\theta_L(K)$. Since D is solvable by hypothesis, it follows that N and then L are solvable, which is again a contradiction. This completes the proof. □

If Lie algebras are considered over a field of characteristic zero, we can improve the above theorem by showing the following.

Theorem 2.6 *Let L be a Lie algebra over a field of characteristic zero. If for every 2-maximal subalgebra K of L, there is a maximal θ-pair (A, B) in $\theta_L(K)$ such that the factor Lie algebra A/B is solvable, then L is solvable.*

Proof It is proved using similar arguments as in the proof of [5; Theorem A(iv)]. □

3. Some characterizations of supersolvable Lie algebras

This section is devoted to presenting some sufficient conditions implying a Lie algebra L to be supersolvable.

Theorem 3.1 *Let L be a Lie algebra in which for every maximal subalgebra K of each maximal nilpotent subalgebra of L, there is an ideal θ-pair (A, B) in $\theta_L(K)$ such that* $\dim(A/B) = 1$. *Then L is supersolvable.*

Proof It is readily seen from the hypothesis that $\mathrm{Asoc}(L) \neq 0$. Let L be a minimal counterexample and N a minimal abelian ideal of L with $\dim(L/N) > 1$. We first prove that L/N is supersolvable. Suppose that U/N is a maximal nilpotent subalgebra of L/N and K/N is a maximal subalgebra of U/N. We consider the following two cases.

Case 1: N is contained in $\varphi(U)$. Then $U/\varphi(U)$ is nilpotent and consequently U is a maximal nilpotent subalgebra of L by a theorem of Barnes [1]. By hypothesis, there is an ideal θ-pair (A, B) in $\theta_L(K)$ such that $\dim(A/B) = 1$. If $N \le A$, it follows that $(A/N, B/N)$ is an ideal θ-pair in $\theta_{L/N}(K/N)$. So, we assume that $N \not\le A$. Then $A \cap N = B \cap N = 0$ and $A/B \cong (A + N)/(B + N)$. We can thus deduce that $((A + N)/N, (B + N)/N) \in \theta_{L/N}(K/N)$ and $(A + N)/(B + N)$ is one-dimensional.

Case 2: N is not contained in $\varphi(U)$. Then $U = N + M$ for some maximal subalgebra M of U. If we choose the subalgebra D to be minimal with respect to $U = N + D$, then Lemma 7.1 of [9] follows that $N \cap D < \varphi(D)$. However, $U/N \cong D/(N \cap D)$ is nilpotent, implying that D is nilpotent. Consider the subalgebra V of U to be a maximal element in the (non-empty) collection of nilpotent subalgebras, D say, of U such that $U = N + D$. One may readily verify that V is a maximal subalgebra of L and moreover, $K = K \cap (N + V) = N + (K \cap V) = N + H$ in which H is a maximal nilpotent subalgebra of V with $K \cap V \subseteq H$. By the assumption, there is an ideal θ-pair (A, B) in $\theta_L(K)$ such that $\dim(A/B) = 1$. If $N \leq A$, then $(A/N, B/N)$ is an ideal θ-pair in $\theta_{L/N}(K/N)$. So, suppose that $N \cap A = 0$ and $(A + N)/N$ is a minimal ideal of L/A. As L/A is a nilpotent Lie algebra, we conclude that $(A + N)/A \subseteq Z(L/A)$ and then $[N, L] \subseteq N \cap A = 0$. Thus, N is central and $N + V$ is a nilpotent subalgebra of L containing V. But this follows that $N \leq V$ and $K = H$. Now, proceeding as in case 1, it may be inferred that the pair $((A + N)/N, (B + N)/N)$ is an ideal θ-pair in $\theta_{L/N}(K/N)$ such that $(A + N)/(B + N)$ is one-dimensional. Therefore L/N satisfies the hypothesis of the theorem, and hence L/N is supersolvable.

Since the class of all supersolvable Lie algebras is a saturated formation, we may assume that N is the unique minimal abelian ideal of L and furthermore, $\varphi(L) = 0$. We show that $\dim(N) = 1$. We have that $L = N + M$ for some maximal subalgebra M of L with $M_L = 0$. Let V be a maximal nilpotent subalgebra of L containing N. Then $V = N + (V \cap M)$. We choose H to be a maximal subalgebra of V containing $V \cap M$, and assume that (A, B) is an ideal θ-pair in $\theta_L(H)$ such that $\dim(A/B) = 1$. If $B \neq 0$ then $N \subseteq H$, implying that $V = H$, a contradiction. Hence we must have $B = 0$ and then $\dim N = \dim A = 1$, which is again a contradiction. Therefore L is supersolvable, as desired. □

An argument similar to that employed in proving Theorem 3.1 allows us to establish the following result (the proof of which is omitted).

Theorem 3.2 *Let L be a Lie algebra with $Asoc(L) \neq 0$. If for every maximal subalgebra K of each maximal nilpotent subalgebra of L, there is an ideal θ-pair (A, B) in $\theta_L(K)$ with $A \cap K = B$, then L is supersolvable.*

In the next result, we establish, under some conditions, that Theorem 3.2 holds with "maximal nilpotent subalgebras" replaced by "the largest nilpotent ideal", which is much easier to find.

Theorem 3.3 *Let L be a solvable Lie algebra over a field of characteristic zero. If for every maximal subalgebra K of $N(L)$, there is an ideal θ-pair (A, B) in $\theta_L(K)$ with $A \cap K = B$, then L is supersolvable.*

Proof We use induction on the dimension of L. We distinguish two cases:

Case 1: $\varphi(L) \neq 0$. Because of Proposition 1.3(i), any maximal subalgebra of $N(L/\varphi(L))$ may be regarded as $K/\varphi(L)$, in which K is a maximal subalgebra of $N(L)$. By hypothesis, there is an ideal θ-pair (A, B) in $\theta_L(K)$ with $A \cap K = B$. If $\varphi(L)$ is contained in B, then $(A/\varphi(L), B/\varphi(L))$ is an ideal θ-pair in $\theta_{L/\varphi(L)}(K/\varphi(L))$ with $A/\varphi(L) \cap K/\varphi(L) = B/\varphi(L)$. We therefore assume that $\varphi(L) \nleq B$. Then $A \cap \varphi(L) = B \cap \varphi(L)$, implying that $A/B \cong (A + \varphi(L))/(B + \varphi(L))$. Hence $(A + \varphi(L)/\varphi(L), B + \varphi(L)/\varphi(L))$ is an ideal θ-pair in $\theta_{L/\varphi(L)}(K/\varphi(L))$ such that $(A + \varphi(L)) \cap K = B + \varphi(L)$. So, by applying induction, we deduce that the factor Lie algebra $L/\varphi(L)$ and then, by [1; Theorem 7], L are supersolvable.

Case 2. $\varphi(L) = 0$. Invoking Proposition 1.3(ii), $N(L) = C_1 \oplus C_2 \oplus \ldots \oplus C_n$, in which the C_i, $i = 1, \ldots, n$, are minimal abelian ideals of L. Note that the derived subalgebra L^2 of L is nilpotent, and then L^2 is a subalgebra of $N(L)$. We first assume that $L^2 = N(L)$. If K is a maximal subalgebra of $N(L)$, it follows from the hypothesis of the theorem that there is an ideal θ-pair (A, B) in $\theta_L(K)$ with $A \cap K = B$. As L/A is isomorphic to a quotient of K, L^2 is contained in A. Consequently, we must have $A = L$ and $B = K$. This shows that every maximal subalgebra of $N(L)$ is ideal in L. We claim that any ideal C_i is one-dimensional. To see this, suppose that C_i^* is any maximal subalgebra of C_i. It is obvious that $K := C_i^* + (C_1 \oplus \ldots \oplus \widehat{C}_i \oplus \ldots \oplus C_n)$ is a maximal subalgebra of $N(L)$ (here $\widehat{C}_i$ means that C_i is omitted from the direct sum). Therefore K and $C_i^* = K \cap C_i$ are ideals in L. The minimality of C_i leads to the conclusion that $C_i^* = 0$ and that $\dim C_i = 1$, as claimed. Now, we put $N_i = C_1 \oplus \ldots \oplus C_i$ for $i = 1, \ldots, n$. Then $N(L)$ has a chain $0 = N_0 \leq N_1 \leq N_2 \leq \ldots \leq N_n = N(L)$, where N_i is an ideal in L and $\dim(N_i/N_{i-1}) = 1$, for each i in the internal $1 \leq i \leq n$. Noting that $L/N_n = L/L^2$ is abelian, an easy induction shows that the factor Lie algebra L/N_i is supersolvable for each $0 \leq i \leq n$. In particular, $L \cong L/N_0$ is supersolvable.

We now assume that L^2 is a proper subalgebra of $N(L)$. Choose N to be a minimal abelian ideal of L with $N \cap L^2 = 0$. We show that $N(L/N) = N(L)/N$. Put $N(L/N) = H/N$ for some ideal H of L. Certainly, L^2 lies in H and then H is isomorphic to a subalgebra of the nilpotent Lie algebra $N(L/N) \oplus H/L^2$. But this follows that $H = N(L)$. An argument similar to that used in case 1 may be used to show that the factor Lie algebra L/N is supersolvable. Therefore $L/N \oplus L/L^2$ and $L \cong L/(N \cap L^2)$ are supersolvable. The proof of the theorem is complete. □

We end the paper by giving some other sufficient conditions for supersolvability of Lie algebras. We omit the proof which is quite similar to the proof of [5; Theorem B].

Theorem 3.4 *A Lie algebra L is supersolvable if one of the following conditions holds:*

(i) For any 2-maximal subalgebra K, there is a maximal θ-pair (A, B) in $\theta_L(K)$ with $Z^*(L/B) \neq 0$, where $Z^*(X)$ denotes the terminal member in the upper central series of a Lie algebra X.

(ii) For any 2-maximal subalgebra K, there is a maximal θ-pair (A, B) in $\theta_L(K)$ with $\varphi(L) \neq 0$.

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