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Isoclinic extensions of Lie algebras

Hamid MOHAMMADZADEH,[1,*] **Ali Reza SALEMKAR,**[2] **Zahra RIYAHI**[2]
[1]School of Mathematics, Iran University of Sciences and Technology, Tehran, Iran
[2]Faculty of Mathematical Sciences, Shahid Beheshti University, G.C., Tehran, Iran

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Abstract: In this article we introduce the notion of the equivalence relation, isoclinism, on the central extensions of Lie algebras, and determine all central extensions occurring in an isoclinism class of a given central extension. We also show that under some conditions, the concepts of isoclinism and isomorphism between the central extensions of finite dimensional Lie algebras are identical. Finally, the connection between isoclinic extensions and the Schur multiplier of Lie algebras are discussed.

Key words: Lie algebra, isoclinic extensions, Schur multiplier, stem cover

1. Introduction

In 1940, P. Hall [6] introduced an equivalence relation on the class of all groups called isoclinism, which is weaker than isomorphism and plays an important role in classification of finite p-groups. This notion has since been further studied by a number of authors, including Bioch [4], Hekster [7], Jones and Wiegold [8], and Weichsel [16]. In 1994, K. Moneyhun [10] gave a Lie algebra analogue of isoclinism as follows: Two Lie algebras L_1 and L_2 are isoclinic if there exists an isomophism γ between the central quotients $L_1/Z(L_1)$ and $L_2/Z(L_2)$ and an isomorphism β between the derived subalgebras L_1^2 and L_2^2 such that γ and β are compatible with the commutator maps of L_1 and L_2. Evidently, this produces a partition on the class of all Lie algebras into equivalence classes, the so-called isoclinism families. Note that the class of all abelian Lie algebras, whose classification is completely known, constitutes the isoclinism family of the zero Lie algebra. In [10], it has been proved that each isoclinism family contains a special Lie algebra, called a stem algebra, such that its centre is contained in its derived subalgebra. The result yields that the concepts of isoclinism and isomorphism between Lie algebras of the same finite dimension are identical. The structure of all Lie algebras occurring in an isoclinism family has been extensively studied in [12] and also some applications have been given in [2, 11, 12, 14]. Furthermore, Salemkar et. al [15] generalized the notion of isoclinism to the notion of n-isoclinism, that is the isoclinism with respect to the variety of nilpotent Lie algebras of class at most n.

In the next section, we introduce the concept of isoclinism on the central extensions of Lie algebras, which is a generalization of the above mentioned work of Moneyhun, and give some equivalent conditions under which two central extensions are isoclinic. In Section 3, we show that under some conditions, the concepts of isoclinism and isomorphism between the central extensions of finite dimensional Lie algebras are identical.

*Correspondence: h_mohammadzadeh@iust.ac.ir
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Finally, Section 4 is devoted to the study of the connection between isoclinism and the subalgebras of the Schur multipliers.

Note that some results on the isoclinism and the Schur multiplier of Lie algebras also hold for the group case (see [5, 12, 14]). However, the results in Section 3 are not generally true in the case of groups (see Example 3.8).

Throughout, all Lie algebras are considered over a fixed field Λ, and the square bracket $[\ ,\ ]$ denotes the Lie multiplication. An exact sequence $e : 0 \to M \overset{\subseteq}{\to} K \to L \to 0$ of Lie algebras is a *central extension* of L if $M \subseteq Z(K)$. Furthermore, if $M \subseteq K^2$ then e is called a *stem extension.* Obviously, $e_K : 0 \to Z(K) \to K \to K/Z(K) \to 0$ is always a central extension. The sequence e is said to be finite dimensional when K is of finite dimension. If the following diagram of Lie algebras and Lie homomorphisms is commutative

$$\begin{array}{ccccccccc} e_1 & : 0 & \longrightarrow & M_1 & \longrightarrow & K_1 & \xrightarrow{\pi_1} & L_1 & \longrightarrow & 0 \\ & & & \downarrow \beta|_{M_1} & & \downarrow \beta & & \downarrow \gamma & & \\ e_2 & : 0 & \longrightarrow & M_2 & \longrightarrow & K_2 & \xrightarrow{\pi_2} & L_2 & \longrightarrow & 0, \end{array}$$

where the rows are central extensions and $\beta|_{M_1}$ is the restriction of β to M_1, then the triple $(\beta|_{M_1}, \beta, \gamma) : e_1 \to e_2$ is called a *morphism* from e_1 to e_2. In particular, if β, γ are isomorphisms then e_1 and e_2 are said to be *isomorphic* and are denoted by $e_1 \cong e_2$.

2. Isoclinic extensions

The following definition is vital in our investigation which is similar to [9; Definitions 1.1, 1.2] for the group case.

Definition 2.1 *Let* $e_i : 0 \to M_i \overset{\subseteq}{\to} K_i \overset{\pi_i}{\to} L_i \to 0$, $i = 1, 2$, *be two central extensions.*

(i) The extensions e_1 *and* e_2 *are said to be isoclinic if there exist Lie isomorphisms* $\gamma : L_1 \to L_2$ *and* $\beta' : K_1^2 \to K_2^2$ *such that for all* $k_1, k_2 \in K_1$ *we have* $\beta'([k_1, k_2]) = [k_1', k_2']$, *where* $k_i' \in K_2$ *and* $\gamma\pi_1(k_i) = \pi_2(k_i'), i = 1, 2$. *In this case, the pair* (γ, β') *is called an isoclinism from* e_1 *to* e_2 *and we write* $(\gamma, \beta') : e_1 \sim e_2$. *In particular,* K_1 *and* K_2 *are isoclinic as* [10] *if their corresponding relative central extensions* e_{K_1} *and* e_{K_2} *are isoclinic.*

(ii) A morphism $(\beta|_{M_1}, \beta, \gamma) : e_1 \to e_2$ *is called isoclinic if the pair* $(\gamma, \beta|_{K_1^2})$ *is an isoclinism from* e_1 *to* e_2. *Moreover,* $(\beta|_{M_1}, \beta, \gamma)$ *is said to be an isoclinic epimorphism or monomorphism if* β *is onto or one-to-one, respectively.*

Evidently, isoclinism between the central extensions is an equivalence relation, and then it produces the class of all central extensions of Lie algebras into equivalence classes, called *isoclinism families.*

There are many examples of isoclinic morphisms. We list only a few of them. Suppose that $e : 0 \to M \overset{\subseteq}{\to} K \overset{\pi}{\to} L \to 0$ is a central extension.

(1) Let A be an abelian Lie algebra and for any Lie algebra X, the maps $\pi_X : X \oplus A \to X$ and $i_X : X \to X \oplus A$ denote the projective and canonical homomorphisms, respectively. Then the extension $e \oplus A :\ 0 \to M \oplus A \to K \oplus A \overset{\pi\pi_K}{\to} L \to 0$ is central, and the morphisms $(\pi_M, \pi_K, 1_L) : e \oplus A \to e$ and $(i_M, i_K, 1_L) : e \to e \oplus A$ are isoclinic epimorphism and isoclinic monomorphism, respectively.

(2) Let N be an ideal of M with $N \cap K^2 = 0$, $\bar{\pi} : K/N \to L$ is the epimorphism induced by π and for any ideal Y of K containing N, the map $\rho_Y : Y \to Y/N$ denotes the natural epimorphism. Then the extension $e/N : \ 0 \to M/N \overset{\subseteq}{\to} K/N \overset{\bar{\pi}}{\to} L \to 0$ is central, and $(\rho_M, \rho_K, 1_L) : e \to e/N$ is an isoclinic epimorphism. In particular, if $(\beta|_{M_1}, \beta, \gamma) : e_1 \to e_2$ is an isoclinic epimorphism of the central extensions, then there is an ideal N of M_1 such that $N \cap K_1^2 = 0$ and $e_2 \cong e_1/N$.

(3) Let T be a subalgebra of K such that $K = T + M$ and $\pi(T) = L$. Then the extension $e_T \ : \ 0 \to T \cap M \to T \overset{\pi|_T}{\to} L \to 0$ is central and $(i|_{T\cap M}, i, 1_L) : e_T \to e$ is an isoclinic monomorphism, where $i : T \to L$ is the inclusion map. In particular, if $(\beta|_{M_1}, \beta, \gamma) : e_1 \to e_2$ is an isoclinic monomorphism, then there is a subalgebra T of K_2 such that $K_2 = T + M_2$ and $e_1 \cong e_{2_T}$.

In what follows, we use the above notations. The following lemmas give some fundamental properties of isoclinic extensions whose proofs are simple and left to the reader.

Lemma 2.2 *Let $(\gamma, \beta') : e_1 \sim e_2$ be an isoclinism. Then*
(i) $\gamma\pi_1(k) = \pi_2\beta'(k)$ *for all* $k \in K_1^2$.
(ii) $\beta'(M_1 \cap K_1^2) = M_2 \cap K_2^2$.
(iii) $\beta'([k_1, x]) = [k_2, \beta'(x)]$ *for all* $x \in K_1^2$, $k_i \in K_i$, $i = 1, 2$, *with* $\gamma\pi_1(k_1) = \pi_2(k_2)$.

Lemma 2.3 *(i) If $(\gamma, \beta') : e_1 \sim e_2$ is an isoclinism, then γ induces an isomorphism $\gamma' : K_1/Z(K_1) \to K_2/Z(K_2)$ such that the pair (γ', β') is an isoclinism from K_1 to K_2, and moreover $M_1 = Z(L_1)$ if and only if $M_2 = Z(L_2)$.*

(ii) A morphism $(\beta|_{M_1}, \beta, \gamma) : e_1 \to e_2$ is isoclinic if and only if γ is an isomorphism and $\ker \beta \cap K_1^2 = 0$.

(iii) The composition of isoclinic morphisms is an isoclinic morphism.

The following main result of this section gives some equivalent conditions under which two central extensions are isoclinic. In proving the result, we have used the argument given by Beyl and Tappe [3] in the case of groups (see also [9; Theorems 2.4, 2.5]).

Theorem 2.4 *Let e_1 and e_2 be two central extensions. Then the following statements are equivalent:*

(i) e_1 and e_2 are isoclinic.

(ii) There exists a central extension e' together with isoclinic epimorphisms from e' onto e_1 and e_2.

(iii) There exists a central extension e'' together with isoclinic monomorphisms from e_1 and e_2 into e''.

(iv) There exist an abelian Lie algebra A, a central extension e', an isoclinic monomorphism from e' into $e_1 \oplus A$, and an isoclinic epimorphism from e' onto e_2.

(v) There exist an abelian Lie algebra B, a central extension e'', an isoclinic epimorphism from $e_1 \oplus B$ onto e'', and an isoclinic monomorphism from e_2 into e''.

In the above theorem, if we restrict ourselves to isoclinic Lie algebras, the following consequence is obtained, which was already proved in [12] using another technique.

Corollary 2.5 *Let K_1 and K_2 be two arbitrary Lie algebras. Then the following statements are equivalent:*

(i) K_1 and K_2 are isoclinic.

(ii) There exist an abelian Lie algebra A, a subalgebra L of $K_1 \oplus A$ with $K_1 \oplus A = L + Z(K_1 \oplus A)$ and an ideal N of L with $N \cap L^2 = 0$ such that L/N is isomorphic to K_2.

(iii) There exist an abelian Lie algebra B, an ideal M of $K_1 \oplus B$ with $M \cap (K_1 \oplus B)^2 = 0$, and a subalgebra T of $(K_1 \oplus B)/M$ with $(K_1 \oplus B)/M = T + Z((K_1 \oplus B)/M)$ such that T is isomorphic to K_2.

The rest of this section will provide a proof for Theorem 2.4.

Using the assumptions of Theorem 2.4, let $\gamma : L_1 \to L_2$ be an isomorphism and $\bar{K} = \{(k_1, k_2) | k_i \in K_i, i = 1, 2, \gamma\pi_1(k_1) = \pi_2(k_2)\}$. Then $\bar{K}$ is a subalgebra of $K_1 \oplus K_2$ and $\bar{e} : 0 \to M_1 \oplus M_2 \overset{\subseteq}{\to} \bar{K} \overset{\delta}{\to} L_1 \to 0$ is a central extension, in which $\delta(k_1, k_2) = \pi_1(k_1)$, for all $(k_1, k_2) \in \bar{K}$. Assuming $A = \bar{K}/\bar{K}^2$, we now have the following lemmas.

Lemma 2.6 *The isomorphism $\gamma : L_1 \to L_2$ induces an isoclinism from e_1 to e_2 if and only if there exist isoclinic epimorphisms from $\bar{e}$ onto e_1 and e_2.*

Proof The necessity of the condition holds trivially.

Conversely, let γ induce an isoclinism from e_1 to e_2. Then there is an isomorphism $\beta' : K_1^2 \to K_2^2$ induced by γ such that the pair (γ, β') is an isoclinism from e_1 to e_2. It is readily verified that $\bar{K}^2 = \{(k_1, \beta'(k_1)) | k_1 \in K_1^2\}$. Assume that $\beta_i : \bar{K} \to K_i$, $i = 1, 2$, are the projective homomorphisms, $\gamma_1 = 1_{L_1}$ and $\gamma_2 = \gamma$. As $\ker \beta_i \cap \bar{K}^2 = 0$, it follows that $(\beta_i|_{M_1 \oplus M_2}, \beta_i, \gamma_i) : \bar{e} \to e_i$ is an isoclinic epimorphism for $i = 1, 2$, as required. □

Lemma 2.7 *Let (γ, β') be an isoclinism from e_1 to e_2. Then there exist isoclinic monomorphisms from $\bar{e}$ into $e_1 \oplus A$ and $e_2 \oplus A$.*

Proof Consider the map $\bar{\beta}_i : \bar{K} \to K_i \oplus A$ given by $\bar{\beta}_i(k) = (\beta_i(k), k + \bar{K}^2)$, for $i = 1, 2$. It is obvious that $\bar{\beta}_i$ is a homomorphism of Lie algebras. In the proof of Lemma 2.6, we regarded that $\bar{K}^2 = \{(k_1, \beta'(k_1)) | k_1 \in K_1^2\}$. So, if $k = (k_1, k_2) \in \ker \bar{\beta}_i$ then $k_i = 0$ and $k_2 = \beta'(k_1)$, implying that $\ker \bar{\beta}_i = 0$. Hence the morphisms $(\bar{\beta}_1|_{M_1 \oplus M_2}, \bar{\beta}_1, 1_{L_1}) : \bar{e} \to e_1 \oplus A$ and $(\bar{\beta}_2|_{M_1 \oplus M_2}, \bar{\beta}_2, \gamma) : \bar{e} \to e_2 \oplus A$ are isoclinic monomorphisms, as required. □

Lemma 2.8 *The isomorphism $\gamma : L_1 \to L_2$ induces an isoclinism from e_1 to e_2 if and only if for some ideal T of $M_1 \oplus A$ with $T \cap (K_1 \oplus A)^2 = 0$, there exist isoclinic monomorphisms from e_1 and e_2 into the extension $(e_1 \oplus A)/T$.*

Proof The necessity of the condition holds trivially.

Conversely, suppose that γ induces an isoclinism (γ, β') from e_1 to e_2. Put $T = \{(m_1, (m_1, 0) + \bar{K}^2) | m_1 \in M_1\}$. It is easy to see that T is an ideal of $M_1 \oplus A$ with $T \cap (K_1 \oplus A)^2 = 0$. We define two homomorphisms $\delta_1 : K_1 \to (K_1 \oplus A)/T$ and $\delta_2 : K_2 \to (K_1 \oplus A)/T$ as follows:

$$\delta_1(k_1) = (k_1, \bar{K}^2) + T \quad \text{and} \quad \delta_2(k_2) = (k_1, (k_1, k_2) + \bar{K}^2) + T,$$

where $k_i \in K_i$ and $\gamma\pi_1(k_1) = \pi_2(k_2)$. We claim that δ_1 and δ_2 are monomorphisms. If $k_1 \in \ker \delta_1$, then $k_1 \in M_1$ and $\beta'(k_1) = 0$. But this follows that $k_1 = 0$. Similarly, $\ker \delta_2 = 0$. We therefore conclude that the morphisms $(\delta_1|_{M_1}, \delta_1, 1_{L_1}) : e_1 \to (e_1 \oplus A)/T$ and $(\delta_2|_{M_2}, \delta_2, \gamma^{-1}) : e_2 \to (e_1 \oplus A)/T$ are isoclinic monomorphisms, as required. □

Now, the proof of Theorem 2.4 is easily deduced from the above lemmas.

3. Isoclinic extensions of finite dimensional Lie algebras

It was established by Moneyhun in [10] that the members of an isoclinism family of finite dimensional Lie algebras are exactly the direct sums of a stem algebra T with finite dimensional abelian Lie algebras. She also proved that all isoclinic Lie algebras of the same finite dimension are isomorphic. In this section, we extend these results to finite dimensional central extensions. We first recall a special case of factor sets. Suppose that $e : 0 \to M \overset{\subseteq}{\to} K \overset{\pi}{\to} L \to 0$ is a central extension. Then a bilinear map $f : L \times L \to M$ is called a factor set on e if the following conditions hold:

(*i*) $f(l_1, l_1) = 0$;

(*ii*) $f([l_1, l_2], l_3) + f([l_2, l_3], l_1) + f([l_3, l_1], l_2) = 0$,

for all $l_1, l_2, l_3 \in L$. Now, assume that f is a factor set on the extension e. It is checked that the direct product $M \times L$ of vector spaces M and L can be converted into a Lie algebra by the formula $[(m_1, l_1), (m_2, l_2)] = (f(l_1, l_2), [l_1, l_2])$. If we denote by $(M \oplus L)_f$ the above Lie algebra, then $e_f : 0 \to M_f \overset{\subseteq}{\to} (M \oplus L)_f \overset{\sigma}{\to} L \to 0$ is a central extension, in which $\sigma : (M \oplus L)_f \to L$ is the projective map and $M_f = \ker \sigma$. We henceforth assume that the extension e_f is given as just described.

We need the following lemmas and proposition for the proofs of our main results.

Lemma 3.1 *Let $e_i : 0 \to M_i \overset{\subseteq}{\to} K_i \overset{\pi_i}{\to} L_i \to 0$, $i = 1, 2$, be two central extensions. Then*

(*i*) *e_1 is stem if and only if M_1 contains no non-zero ideal N satisfying $N \cap K_1^2 = 0$.*

(*ii*) *If e_1 and e_2 are two isoclinic stem extensions, then $M_1 \cong M_2$. In particular, if the Lie algebras K_i, $i = 1, 2$, are finite dimensional, then $\dim K_1 = \dim K_2$.*

Proof (*i*) Suppose first that e_1 is stem, and N is an ideal of M_1 with $N \cap K_1^2 = 0$. Then $N = N \cap M_1 \subseteq N \cap Z(K_1) \cap K_1^2 = N \cap K_1^2 = 0$.

Conversely, assume that on the contrary $M_1 \nsubseteq Z(K_1) \cap K_1^2$. Then there exists a non-zero element $x \in M_1 \setminus (Z(K_1) \cap K_1^2)$. Obviously, $\langle x \rangle$ is a central ideal of K_1 and consequently, by hypothesis, we must have $\langle x \rangle \cap K_1^2 \neq 0$. So, for some non-zero element $c \in \Lambda$, $cx \in K_1^2$ and the contradiction follows. Therefore $M_1 \subseteq Z(K_1) \cap K_1^2$.

(*ii*) It is a straightforward consequence of Lemma 2.2(*ii*). □

Using Lemma 3.1(*i*) and arguing as in the proof of [15; Theorem 3.3], we get the following corollary whose proof we omit.

Corollary 3.2 *Any central extension is isoclinic to a stem extension.*

Lemma 3.3 *Let $e_i : 0 \to M_i \overset{\subseteq}{\to} K_i \overset{\pi_i}{\to} L_i \to 0$, $i = 1, 2$, be two central extensions. Then*

(*i*) *there exists a factor set $f : L_1 \times L_1 \to M_1$ such that the extensions e_1 and e_{1f} are isomorphic;*

(*ii*) *if e_1 and e_2 are stem extensions and $(\gamma, \beta') : e_1 \sim e_2$ is an isoclinism, then there exists a factor set $g : L_1 \times L_1 \to M_1$ such that the extensions e_2 and e_{1g} are isomorphic.*

Proof (i) Let T_1 be a vector space complement of M_1 in K_1. Then for any $l_1 \in L_1$, there is a unique element $t_{l_1} \in T_1$ with $\pi_1(t_{l_1}) = l_1$. It is easily seen that the map $f : L_1 \times L_1 \to M_1$ given by $f(l_1, l_2) = [t_{l_1}, t_{l_2}] - t_{[l_1,l_2]}$ is a factor set and the isomorphism $\beta : (M_1 \oplus L_1)_f \to K_1$ given by $\beta(m_1, l_1) = m_1 + t_{l_1}$ induces an isomorphism between the extensions e_1 and e_{1f}.

(ii) Owing to Lemma 2.2(ii), $\beta'(M_1) = M_2$ and by part (i), there exists a factor set $h : L_2 \times L_2 \to M_2$ such that the extensions e_2 and e_{2_h} are isomorphic. Consider the map $g : L_1 \times L_1 \to M_1$ given by $g(l_1, l_2) = \beta'^{-1}(h(\gamma(l_1), \gamma(l_2)))$. Evidently, g is a factor set and the isomorphism $\theta : (M_1 \oplus L_1)_g \to (M_2 \oplus L_2)_h$ defined by $\theta(m_1, l_1) = (\beta'(m_1), \gamma(l_1))$ yields the morphism $(\beta'|_{M_1}, \theta, \gamma)$ from e_{1g} to e_{2_h}, as required. □

Proposition 3.4 *Let $e : 0 \to M \overset{\subseteq}{\to} K \overset{\pi}{\to} L \to 0$ be a finite dimensional central extension, and $f, g : L \times L \to M$ be factor sets on e such that $\dim(M \oplus L)_f = \dim(M \oplus L)_g$, e_f is a stem extension, and the extensions e_f and e_g are isoclinic. Then e_f and e_g are isomorphic.*

Proof Assume that (γ, β') is an isoclinism from e_f to e_g. It follows from the assumption and Lemma 2.2(ii) that $\beta'(M_f) = M_g$. For all $l_1, l_2 \in L$, we have

$$\beta'([(0, l_1), (0, l_2)]) = [(0, \gamma(l_1)), (0, \gamma(l_2))] = (g(\gamma(l_1), \gamma(l_2)), \gamma([l_1, l_2])),$$

and

$$\beta'([(0, l_1), (0, l_2)]) = \beta'(f(l_1, l_2), [l_1, l_2]) = \beta'(f(l_1, l_2), 0) + \beta'(0, [l_1, l_2]).$$

If we take $d([l_1, l_2])$ to be the first component of $\beta'(0, [l_1, l_2])$, then we get a linear map $d : L^2 \to M$ satisfying $\rho\beta'(f(l_1, l_2), 0) + d([l_1, l_2]) = g(\gamma(l_1), \gamma(l_2))$, in which $\rho : M_g \to M$ is the projective map. We extend d to all of L by taking d to be zero on the vector space complement of L^2 in L. It is fairly easy to see that the map $\lambda : (M \oplus L)_f \to (M \oplus L)_g$ given by $\lambda(m, l) = \beta'(m, 0) + (d(l), \gamma(l))$ is an isomorphism. As $\lambda|_{M_f} = \beta'|_{M_f}$ and $(\lambda|_{M_f}, \lambda, \gamma) : e_f \to e_g$ is a morphism, we deduce that the extensions e_f and e_g are isomorphic, which completes the proof. □

Combining this with Lemmas 3.1 and 3.3, we get the following important result.

Corollary 3.5 *Let $e_i : 0 \to M_i \overset{\subseteq}{\to} K_i \overset{\pi_i}{\to} L_i \to 0$, $i = 1, 2$, be finite dimensional stem extensions. Then e_1 and e_2 are isoclinic if and only if they are isomorphic.*

Theorem 3.6 *Let $\mathcal{C}$ be an isoclinism family of finite dimensional central extensions. Then $\mathcal{C}$ contains a stem extension e_1, and every central extension lying in $\mathcal{C}$ is the form of $e_1 \oplus A$, in which A is a finite dimensional abelian Lie algebra.*

Proof In view of Corollary 3.2, $\mathcal{C}$ admits a stem extension e_1. Plainly, for any finite dimensional abelian Lie algebra A, the extensions e_1 and $e_1 \oplus A$ are isoclinic and consequently $e_1 \oplus A \in \mathcal{C}$. Now, suppose $e : 0 \to M \overset{\subseteq}{\to} K \overset{\pi}{\to} L \to 0$ is an arbitrary central extension in $\mathcal{C}$. We can find an ideal N of M such that $M = N \oplus (M \cap K^2)$, and e/N is a stem extension in $\mathcal{C}$. We choose the vector subspace T of K such that $K^2 \subseteq T$ and T is the complement of N in K. Then T is an ideal of K with $\pi(T) = L$, and the stem extensions e_1 and $e/N \cong e_T$ are isoclinic. According to Corollary 3.5, we have $e_T \cong e_1$ and hence $e \cong e_T \oplus N \cong e_1 \oplus N$, in which N is a finite dimensional abelian Lie algebra and the result is proved. □

The following important corollary is an immediate consequence of the above theorem.

Corollary 3.7 *Let* $e_i : 0 \to M_i \overset{\subseteq}{\to} K_i \overset{\pi_i}{\to} L_i \to 0$*,* $i = 1, 2$*, be finite dimensional central extensions with* $\dim K_1 = \dim K_2$*. Then* e_1 *and* e_2 *are isoclinic if and only if they are isomorphic.*

The following example shows that the results of this section are not necessarily true for groups.

Example 3.8 *Consider the following groups:*

$$G = \langle a, b \mid a^2 = b^4 = 1, ab = ba \rangle,$$
$$G_1 = \langle a_1, b_1, c_1 \mid a_1^2 = c_1^2 = 1, b_1^4 = c_1, a_1c_1 = c_1a_1, b_1c_1 = c_1b_1, a_1b_1 = c_1b_1a_1 \rangle,$$
$$G_2 = \langle a_2, b_2, c_2 \mid a_2^2 = c_2^2 = b_2^4 = 1, c_2a_2 = a_2c_2, c_2b_2 = b_2c_2, a_2b_2 = c_2b_2a_2 \rangle.$$

It is straightforward to verify that G_1 *and* G_2 *are non-isomorphic groups of order* 16*, and the extensions* $e_i : 1 \to \langle c_i \rangle \to G_i \to G \to 1$*,* $i = 1, 2$*, are isoclinic stem extensions.*

4. Isoclinism and the Schur multiplier

In this section, using isoclinic extensions we obtain some interesting results on the Schur multiplier of Lie algebras. Recall that the *Schur multiplier* of a Lie algebra L is the abelian Lie algebra $\mathcal{M}(L) = (R \cap F^2)/[R, F]$, in which $F/R \cong L$ is a free presentation of L. We note that the Schur multiplier of L is independent of the choice of the free presentation of L and is isomorphic to $H_2(L, \Lambda)$, the second homology group of L with coefficients in the trivial L-module Λ (see [1, 5, 11 or 14] for more information on the Schur multiplier of Lie algebras). It is easily checked that any Lie homomorphism $\gamma : L_1 \to L_2$ induces a homomorphism $\mathcal{M}(\gamma) : \mathcal{M}(L_1) \to \mathcal{M}(L_2)$. Furthermore, for any central extension $e : 0 \to M \overset{\subseteq}{\to} K \overset{\pi}{\to} L \to 0$, it follows from [11; Proposition 4.1] that there exists a homomorphism $\theta(e) : \mathcal{M}(L) \to K^2$ such that the sequence

$$0 \to \ker\theta(e) \to \mathcal{M}(L) \overset{\theta(e)}{\to} K^2 \overset{\pi|}{\to} L^2 \to 0$$

is exact (see [12; Lemma 1.2] for a special case).

Using the above assumptions and notations, we have the following useful lemmas.

Lemma 4.1 *Let* $(\beta|_{M_1}, \beta, \gamma) : e_1 \to e_2$ *be a morphism of the central extensions, in which* $\gamma : L_1 \to L_2$ *is an isomorphism. Then* $(\beta|_{M_1}, \beta, \gamma)$ *is an isoclinic morphism if and only if* $\mathcal{M}(\gamma)(\ker\theta(e_1)) = \ker\theta(e_2)$*.*

Proof It is readily verified that the morphism $(\beta|_{M_1}, \beta, \gamma) : e_1 \to e_2$ yields the commutativity of the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\theta(e_1) & \overset{\subseteq}{\longrightarrow} & \mathcal{M}(L_1) & \overset{\theta(e_1)}{\longrightarrow} & K_1^2 & \overset{\pi_1|}{\longrightarrow} & L_1^2 & \longrightarrow & 0 \\
 & & \big\downarrow \mathcal{M}(\gamma)|_{\ker\theta(e_1)} & & \big\downarrow \mathcal{M}(\gamma) & & \big\downarrow \beta|_{K_1^2} & & \big\downarrow \gamma|_{L_1^2} & & \\
0 & \longrightarrow & \ker\theta(e_2) & \overset{\subseteq}{\longrightarrow} & \mathcal{M}(L_2) & \overset{\theta(e_2)}{\longrightarrow} & K_2^2 & \overset{\pi_2|}{\longrightarrow} & L_2^2 & \longrightarrow & 0,
\end{array} \qquad (1)$$

where rows are the exact sequences induced by the central extensions e_i, $i = 1, 2$. As γ is an isomorphism, we conclude that the maps $\gamma|_{L_1^2}$ and $\mathcal{M}(\gamma)$ are isomorphisms, the restriction of $\mathcal{M}(\gamma)$ to $\ker\theta(e_1)$ is an monomorphism, and $\beta|_{K_1^2}$ is an epimorphism. By Lemma 2.3(ii), $(\beta|_{M_1}, \beta, \gamma)$ is an isoclinic morphism if and only if $\beta|_{K_1^2}$ is an isomorphism, being equivalent to the equality $\mathcal{M}(\gamma)(\ker\theta(e_1)) = \ker\theta(e_2)$ by the above diagram.
□

Lemma 4.2 *Let* $\gamma : L_1 \to L_2$ *be an isomorphism and* $\bar{e}$ *be the extension defined in Section* 2. *Then* $\ker\theta(\bar{e}) = \mathcal{M}(\gamma)^{-1}(\ker\theta(e_2)) \cap \ker\theta(e_1)$.

Proof Let β_1 and β_2 be homomorphisms defined in the proof of Lemma 2.6. Then we have $\beta_1\theta(\bar{e})(x) = \theta(e_1)(x)$ and $\beta_2\theta(\bar{e})(x) = \theta(e_2)\mathcal{M}(\gamma)(x)$, for all $x \in \mathcal{M}(L_1)$. But this follows that $\theta(\bar{e})(x) = (\theta(e_1)(x), \theta(e_2)\mathcal{M}(\gamma)(x))$, implying the result. □

In the following we additional additional criteria for two central extensions to be isoclinic.

Theorem 4.3 *Let* $e_i : 0 \to M_i \overset{\subseteq}{\to} K_i \overset{\pi_i}{\to} L_i \to 0$, $i = 1, 2$, *be two central extensions, and* $\gamma : L_1 \to L_2$ *be an isomorphism. Then the following statements are equivalent:*

(i) e_1 *and* e_2 *are isoclinic.*

(ii) *There is an isomorphism* $\beta' : K_1^2 \to K_2^2$ *with* $\beta'\theta(e_1) = \theta(e_2)\mathcal{M}(\gamma)$.

(iii) $\mathcal{M}(\gamma)(\ker\theta(e_1)) = \ker\theta(e_2)$.

Proof Let $\bar{e}$ be the central extension introduced in Section 2, $\gamma_1 = 1_{L_1}$ and $\gamma_2 = \gamma$.

$(i) \Longrightarrow (ii)$ In view of Lemma 2.6, we have the following commutative diagrams:

$$\begin{array}{ccccccccc} \bar{e}\ :0 & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & \bar{K} & \longrightarrow & L_1 & \longrightarrow & 0 \\ & & \big\downarrow \beta_i|_{M_1\oplus M_2} & & \big\downarrow \beta_i & & \big\downarrow \gamma_i & & \\ e_i\ :0 & \longrightarrow & M_i & \longrightarrow & K_i & \longrightarrow & L_i & \longrightarrow & 0, \end{array}$$

where $(\beta_i|_{M_1\oplus M_2}, \beta_i, \gamma_i) : \bar{e} \to e_i$ is an isoclinic epimorphism for $i = 1, 2$. From this one can deduce that $\beta_i|_{\bar{K}^2}\theta(\bar{e}) = \theta(e_i)\mathcal{M}(\gamma_i)$. Therefore, it is enough to take $\beta' = (\beta_2|_{\bar{K}^2})(\beta_1|_{\bar{K}^2})^{-1}$.

$(ii) \Longrightarrow (iii)$ It follows from the diagram (1), and the fact that $\mathcal{M}(\gamma)$ is an isomorphism.

$(iii) \Longrightarrow (i)$ From hypothesis and Lemma 4.2, we have $\ker\theta(\bar{e}) = \ker\theta(e_1)$. Consequently, $\mathcal{M}(\gamma_i)(\ker\theta(\bar{e})) = \ker\theta(e_i)$ for $i = 1, 2$, and so there exist isoclinic epimorphisms from $\bar{e}$ onto e_1 and e_2 thanks to Lemma 4.1. We therefore conclude from Lemma 2.6 that the extensions e_1 and e_2 are isoclinic. □

A stem extension $e : 0 \to M \overset{\subseteq}{\to} K \overset{\pi}{\to} L \to 0$ is called a *stem cover* of the Lie algebra L if $M \cong \mathcal{M}(L)$. In [13], it has been established that any Lie algebra admits at least one stem cover. In view of Corollary 3.7, one can readily regard that finite dimensional Lie algebras, up to isomorphic, have unique stem covers. In the following corollary we show that the stem covers of an arbitrary Lie algebra are isoclinic, which is a vast generalization of a result obtained in [11].

Corollary 4.4 *All stem covers of a given Lie algebra* L *are mutually isoclinic.*

Proof Follows from the above theorem and the fact that if e is a stem cover of L then $\ker\theta(e) = 0$. □

Corollary 4.5 *Using the assumptions and notations of Theorem* 4.3, *let* e_1 *be a stem cover of* L_1, K_1 *and* K_2 *be of the same finite dimension, and* $\dim K_1^2 = \dim K_2^2$. *Then* e_1 *and* e_2 *are isomorphic.*

Proof Set $N = \theta(e_1)\mathcal{M}(\gamma^{-1})(\ker\theta(e_2))$. Then N is an ideal of K_1 contained in $K_1^2 \cap M_1$, and we have

$$\begin{aligned} \mathcal{M}(\gamma^{-1})(\ker\theta(e_2)) &= \{x \in \mathcal{M}(L_1) \mid \theta(e_1)(x) \in N\} \\ &= \{x \in \mathcal{M}(L_1) \mid \theta(e_1/N)(x) = N\} = \ker\theta(e_1/N). \end{aligned}$$

So, the extensions e_1/N and e_2 are isoclinic by Theorem 4.3, and consequently, $\dim(K_1^2/N) = \dim(K_2^2)$. But this follows that $N = 0$. Hence Corollary 3.7 implies that $e_1 \cong e_2$, which completes the proof. □

The above corollary deduces that if K_1 and K_2 are two Lie algebras of the same finite dimension with $\dim K_1^2 = \dim K_2^2$, then K_1 is a stem cover of a Lie algebra if and only if so is K_2.

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