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Products of conjugacy classes and products of irreducible characters in finite groups

Mohammad Reza DARAFSHEH,* **Sajjad MAHMOOD ROBATI**
School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran

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Abstract: Let G be a finite group. If A and B are two conjugacy classes in G, then AB is a union of conjugacy classes in G and $\eta(AB)$ denotes the number of distinct conjugacy classes of G contained in AB. If χ and ψ are two complex irreducible characters of G, then $\chi\psi$ is a character of G and again we let $\eta(\chi\psi)$ be the number of irreducible characters of G appearing as constituents of $\chi\psi$. In this paper our aim is to study the product of conjugacy classes in a finite group and obtain an upper bound for η in general. Then we study similar results related to the product of two irreducible characters.

Key words: Conjugacy classes, irreducible characters, products

1. Introduction

Let G be a finite group, $a \in G$ and $Cl(a) = a^G = \{a^g | g \in G\}$ be the conjugacy class of a in G. If the subset X of G is G-invariant, i.e. $X^g = \{x^g | x \in X\} = X$ for all $g \in G$, then X is the union of m distinct conjugacy classes of G, for some positive integer m. Set $\eta(X) = m$. Given any conjugacy classes $Cl(a)$ and $Cl(b)$, we can check that the product $Cl(a)Cl(b) = \{a^h b^k | k, h \in G\}$ is a G-invariant set. Hence it is a union of conjugacy classes of G. Let $\eta(Cl(a)Cl(b))$ denote the number of conjugacy classes contained in $Cl(a)Cl(b)$. In this note, we study the product $Cl(a)Cl(b)$ when G is a finite group.

Let $[a, G]$ be the set of all commutators $[a, g] = a^{-1}g^{-1}ag$ where $g \in G$. Our first result is the following.

Theorem A. Let G be a finite group and $a \in G$. Then the following conditions are equivalent:

(i) $[a, G]$ is a subgroup of G;

(ii) $Cl(a)Cl(a^{-1}) = [a, G]$;

(iii) $|Cl(a)Cl(a^{-1})| = |Cl(a)|$;

(iv) $|(Cl(a))^n| = |Cl(a)|$ for all $n \in \mathbb{N}$.

An application of Theorem A is the following.

Theorem B. Let G be a finite group and $a, b \in G$ and $[a, G]$ be a subset of $Z(G)$. Then

(i) $\eta(Cl(a)Cl(b)) = |Cl(a)||Cl(b)|/|[a, G] \cap (Cl(b^{-1})Cl(b))||Cl(ab)|$;

(ii) If $Cl(a)Cl(b) \cap Z(G) \neq \emptyset$, then $\eta(Cl(a)Cl(b)) = |Cl(a)|$;

*Correspondence: darafsheh@ut.ac.ir
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(iii) If $|Cl(a)|$ is an odd number, then $\eta(Cl(a)Cl(a)) = 1$;

(iv) If $|Cl(a)|$ is an even number, then $\eta(Cl(a)Cl(a)) = 2^n$ (where n is the number of cyclic direct factors in the decomposition of the Sylow 2-subgroup of $[a, G]$).

In [1], Adan-Bante shows that if G is a finite p-group and $a \in G$ and $|Cl(a)| = p^n$, then $\eta(Cl(a)Cl(a^{-1})) \geq n(p-1)+1$. In this paper, we prove that if G is a finite group and $|Cl(a)|$ is an odd number, then $\eta(Cl(a)Cl(a)) \leq (|Cl(a)|+1)/2$ for all $a \in G$. Also, in Proposition 2.5, we prove that if G is a finite group and $a \in G$ and $|a|$ is an odd number, then $[a, G]$ is a subgroup of G if and only if $\eta(Cl(a)Cl(a)) = 1$, where $|a|$ is the order of a.

The case of products of two irreducible characters of a finite group has been of interest to researchers as well. For a finite group G, let χ and φ be irreducible characters of G, then obviously $\chi\varphi$ is a character of G, hence $\chi\varphi = \sum_{i=1}^{k} m_i\chi_i$, $m_i \in \mathbb{N}$, and $\chi_i \in \mathrm{Irr}(G)$, hence we define $\eta(\chi\varphi) = k$. Analogous results on the products of the irreducible characters of G may be true for the products of conjugacy classes of G. For example we may mention [4], which studies this analogy. But in some cases this analogy may not hold.

In [7] it is proved that, there are two non-identity irreducible characters χ and φ such that $\chi\varphi$ is an irreducible character of A_n if and only if n is a perfect square, while in [3] it is proved that the product of non-trivial conjugacy classes in A_n for $n \geq 5$ is never a conjugacy class. Concerning irreducible characters of a finite p-group G, it is proved in [2] that if $\chi, \varphi \in \mathrm{Irr}(G)$ are faithful, then either $\eta(\chi\varphi) = 1$ or $\eta(\chi\varphi) \geq (p+1)/2$.

Concerning irreducible characters of a finite group G, in this paper the following results are also proved.

Theorem C. Let G be a finite group of odd order and $\chi \in \mathrm{Irr}(G)$. Then χ vanishes on $G - Z(\chi)$ if and only if $\eta(\chi^2) = 1$.

Theorem D. Let G be a finite group and $\chi, \varphi \in \mathrm{Irr}(G)$ such that χ vanishes on $G - Z(\chi)$. Then

(i) If $\chi\varphi \cap \mathrm{Lin}(G) \neq \emptyset$, then $\eta(\chi\psi)) = |\mathrm{Irr}(G/Z(G))|$;

(ii) If $\chi(1)$ is an odd number, then $\eta(\chi^2) = 1$.

We also put forward the following conjecture.

Conjecture. Let G be a finite group of odd order and $\chi \in \mathrm{Irr}(G)$, then $\eta(\chi^2) \leq (\chi(1)+1)/2$.

2. Proof of Theorems A and C

Denote by $C_G(a) = \{g \in G | a^g = a\}$ the centralizer of a in G. Also let $[a, g] = a^{-1}a^g$ and $[a, G] = \{[a, g] | g \in G\}$. Since $a^g = a[a, g]$ for any $g \in G$, we have that $Cl(a) = a^G = \{ax | x \in [a, G]\}$. In particular, we get that $|Cl(a)| = |[a, G]|$. Through this note, we will use the well-known fact that $|Cl(a)| = |G : C_G(a)|$.

Lemma 2.1 *Let G be a finite group and $a \in G$ and $[a, G]$ be a subgroup of G. Then*

(i) $[a, G]$ is a normal subgroup of G;

(ii) $[a, G] = [a^{-1}, G]$.

Proof (i) Observe that

$$[a, g]^h = (a^{-1})^h a^{gh} = [a, h]^{-1}[a, gh]$$

for all $g, h \in G$. Since $[a, G]$ is a subgroup of G, then $[a, h]^{-1}[a, gh] \in [a, G]$. Therefore $[a, G]$ is a normal subgroup of G.

(ii) We have that

$$[a^{-1}, g] = a(a^{-1})^g = (a^{-1})^{ga^{-1}}a = [a, ga^{-1}]^{-1}$$

for all $g \in G$. Also since $[a, G]$ is a subgroup of G, then $[a, ga^{-1}]^{-1} \in [a, G]$. Therefore $[a^{-1}, G] \subseteq [a, G]$. Since $|Cl(a)| = |Cl(a^{-1})|$ we have $[a, G] = [a^{-1}, G]$. □

Proof [**Proof of Theorem A**]
$(i \Rightarrow ii)$

By Lemma 2.1, observe that

$$Cl(a)Cl(a^{-1}) = a[a, G]a^{-1}[a^{-1}, G] = [a, G][a^{-1}, G] = [a, G].$$

$(ii \Rightarrow i)$

Observe that, since $a^G a = aa^G$ we can write

$$Cl(a)Cl(a^{-1}) = a[a, G]a^{-1}[a^{-1}, G] = [a, G][a^{-1}, G].$$

Fix $l, h \in G$. Since $Cl(a)Cl(a^{-1}) = [a, G]$ we have $[a, G][a^{-1}, G] = [a, G]$ and $[a, h][a^{-1}, k] = [a, l]$ for some $k \in G$.

Since $[a^{-1}, k]^{-1} = [a, ka^{-1}]$ we have $[a, h] = [a, l][a, ka^{-1}]$ and $[a, h][a, l]^{-1} = [a, ka^{-1}]$. Thus $1_G \in [a, G]$, and hence $[a, G]$ is a subgroup of G.
$(ii \Leftrightarrow iii)$

Since $a^{-1}a^G = a^G a^{-1}$ we have $[a, G] \subseteq Cl(a)Cl(a^{-1})$. Therefore from $|[a, G]| = |Cl(a)|$, we obtain $Cl(a)Cl(a^{-1}) = [a, G]$ if and only if $|Cl(a)Cl(a^{-1})| = |Cl(a)|$.
$(i \Rightarrow iv)$

Observe that

$$(Cl(a))^n = (a[a, G])^n = a^n[a, G]$$

for all $n \in \mathbb{N}$. Therefore $|(Cl(a))^n| = |Cl(a)|$.
$(iv \Rightarrow i)$

Let $n = 2$. Observe that, since $a^G a = aa^G$, we obtain

$$Cl(a)Cl(a) = a[a, G]a[a, G] = a^2[a, G][a, G].$$

Thus if $|Cl(a)Cl(a)| = |Cl(a)|$, then $|[a, G][a, G]| = |[a, G]|$. Since $1_G \in [a, G]$ we obtain $[a, G] \subseteq [a, G][a, G]$ and $[a, G][a, G] = [a, G]$. Then we conclude that $[a, G]$ is a subgroup of G, since $xy \in [a, G]$ for any x, y in $[a, G]$ and $[a, G]$ is a nonempty finite set. □

Proposition 2.2 *Let G be a finite group and $a, b \in G$ and $Cl(a)Cl(b) \cap Z(G) \neq \emptyset$. Then $[a, G]$ is a subgroup of G if and only if $|Cl(a)Cl(b)| = |Cl(a)|$.*

Proof Suppose that there exists some $z \in Cl(a)Cl(b) \cap Z(G)$. Then there exists some $g \in G$ such that $z = a^g b$ and $b = (a^{-1})^g z$. Thus we have

$$Cl(a)Cl(b) = Cl(a)Cl((a^{-1})^g z) = Cl(a)Cl((a^{-1})^g)z = Cl(a)Cl(a^{-1})z,$$

therefore $|Cl(a)Cl(b)| = |Cl(a)Cl(a^{-1})|$. Thus by Theorem A, $[a,G]$ is a subgroup of G if and only if $|Cl(a)Cl(b)| = |Cl(a)Cl(a^{-1})| = |Cl(a)|$.

□

Proposition 2.3 *Let G be a finite group and $\chi,\psi \in \mathrm{Irr}(G)$ with $\mathrm{Irr}(\chi\psi)\cap\mathrm{Lin}(G)\neq\emptyset$. Then χ vanishes on $G-Z(G)$ if and only if $\widehat{\chi\psi}=\rho_{G/Z(G)}$, where $\rho_{G/Z(G)}$ is the regular character of $G/Z(G)$ and $\widehat{\chi\psi}(gZ(G)) := \chi\psi(g)$ for all $g\in G$.*

Proof Suppose that χ vanishes on $G-Z(G)$ and $\lambda\in\mathrm{Lin}(G)\cap\mathrm{Irr}(\chi\psi)$. Since χ and $\lambda\overline{\psi}$ are irreducible characters then $0\neq[\chi\psi,\lambda]=[\chi,\lambda\overline{\psi}]=1$ and $\chi=\lambda\overline{\psi}$. Therefore $\chi\psi=\lambda\overline{\psi}\psi$ and ψ vanishes on $G-Z(G)$. We can check that $\eta(\chi\psi)=\eta(\overline{\psi}\psi)$.

We can write $\overline{\psi}(g)\psi(g)=|\psi(g)|^2=\psi(1)^2$ for all $g\in Z(G)$ and $\overline{\psi}(g)\psi(g)=0$ for all $g\in G-Z(G)$, thus $Z(G)=\ker(\overline{\psi}\psi)$. Suppose that $\varphi_i\in\mathrm{Irr}(\overline{\psi}\psi)$ for $i=1,...,n$, then $Z(G)=\ker(\overline{\psi}\psi)=\bigcap_{i=1}^{n}\ker(\varphi_i)\subseteq\ker(\varphi_i)$ and

$$\begin{aligned}0\neq[\overline{\psi}\psi,\varphi_i] &= 1/|G|\sum_{g\in G}\overline{\psi}\psi(g)\,\overline{\varphi_i}(g)\\ &= 1/|G|\sum_{g\in Z(G)}\overline{\psi}(g)\psi(g)\,\overline{\varphi_i}(g)\\ &= \frac{|Z(G)|\psi(1)^2\varphi_i(1)}{|G|}=\varphi_i(1)\end{aligned} \tag{2.1}$$

because $\psi(1)^2=|G:Z(G)|$. Thus $\overline{\psi}\psi=\sum_{i=1}^{n}\varphi_i(1)\varphi_i$ and $|G:Z(G)|=\psi(1)^2=\sum_{i=1}^{n}\varphi_i(1)^2$. Since $\mathrm{Irr}(G/Z(G))=\{\theta\in\mathrm{Irr}(G)|Z(G)\subseteq ker\theta\}$ and $\rho_{G/Z(G)}=\sum_{\widehat{\theta}\in\mathrm{Irr}(G/Z(G))}\widehat{\theta}(1)\widehat{\theta}$ then $\widehat{\varphi}_i\in\mathrm{Irr}(G/Z(G))$ and $\widehat{\overline{\psi}\psi}=\rho_{G/Z(G)}$, where $\widehat{\varphi}_i(gZ(G)):=\varphi_i(g)$ for all $g\in G$. It follows that $\widehat{\chi\psi}=\rho_{G/Z(G)}$.

Conversely suppose $\widehat{\chi\psi}=\rho_{G/Z(G)}=\sum_{\widehat{\theta}\in\mathrm{Irr}(G/Z(G))}\widehat{\theta}(1)\widehat{\theta}$, then $\chi\psi$ vanishes on $G-Z(G)$. On the other hand, we have $\mathrm{Irr}(\chi\psi)\cap\mathrm{Lin}(G)\neq\emptyset$, then there exists some $\lambda\in\mathrm{Lin}(G)$ such that $\lambda\in\mathrm{Irr}(\chi\psi)$ and $0\neq[\chi\psi,\lambda]=[\chi,\lambda\overline{\psi}]=1$. Thus we have $\chi=\lambda\overline{\psi}$ and $\chi\psi=\lambda\overline{\psi}\psi$. It follows that $\overline{\psi}\psi$ vanishes on $G-Z(G)$ and χ vanishes on $G-Z(G)$. □

Lemma 2.4 *Let G be a finite group and $a\in G$. If $[a,G]$ is not a subgroup of G, then $\eta(Cl(a)Cl(a))\neq 1$.*

Proof We have that $C_G(a)\subseteq C_G(a^2)$, therefore $|Cl(a^2)|\leq|Cl(a)|$. Since $[a,G]$ is not a subgroup of G, then by proof of Theorem A, $|Cl(a^2)|\leq|Cl(a)|<|Cl(a)Cl(a)|$. Thus $Cl(a)Cl(a)$ is not a single conjugacy class. □

Proposition 2.5 *Let G be a finite group and $a\in G$ and the order of a be an odd number. Then $[a,G]$ is a subgroup of G if and only if $\eta(Cl(a)Cl(a))=1$.*

Proof We have $C_G(a)\subseteq C_G(a^2)$. Since $|a|$ is an odd number, we can check that $C_G(a)=C_G(a^2)$. Thus

$$|Cl(a)|=|G:C_G(a)|=|G:C_G(a^2)|=|Cl(a^2)|.$$

Then by Theorem A, $Cl(a)Cl(a) = Cl(a^2)$. Conversely, by Lemma 2.4, if $\eta(Cl(a)Cl(a)) = 1$ then $[a, G]$ is a subgroup of G and the proof is complete. □

At this point we prove Theorem C, which is an analogue of Proposition 2.5.

Proof [**Proof of Theorem C**] First suppose χ vanishes on $G - Z(\chi)$. By Corollary 2.30 of [5] we have $\chi(1)^2 = |G : Z(\chi)|$. By Theorem 4.5 of [5] the alternating part of the character χ^2 which is denoted by χ_A is a character of G and for $g \in G$ we have $\chi_A(g) = 1/2(\chi(g)^2 - \chi(g^2))$. Let us define $\chi^{(2)}$ by $\chi^{(2)}(g) = \chi(g^2)$, for all $g \in G$, and χ^2 by $\chi^2(g) = \chi(g)^2$, for all $g \in G$. Then $\chi^{(2)} = \chi^2 - 2\chi_A$. Since $|G|$ is odd, by Problem 4.5 of [5], $\chi^{(2)} \in \mathrm{Irr}(G)$, hence

$$\begin{aligned} 0 \neq [\chi^2, \chi^{(2)}] &= 1/|G| \sum_{g \in G} \chi^2(g)\overline{\chi^{(2)}(g)} \\ &= 1/|G| \sum_{g \in Z(\chi)} \chi^2(g)\overline{\chi^{(2)}(g)}. \end{aligned} \tag{2.2}$$

But $\chi|_{Z(\chi)} = \chi(1)\lambda_1$ and $\chi^{(2)}|_{Z(\chi)} = \chi^{(2)}(1)\lambda_2 = \chi(1)\lambda_2$ where $\lambda_1, \lambda_2 \in \mathrm{Lin}(Z(\chi))$. Therefore continuing, (2.2), we can write:

$$\begin{aligned} 0 \neq [\chi^2, \chi^{(2)}] &= \chi(1)^3/|G| \sum_{g \in Z(\chi)} \lambda_1^2(g)\overline{\lambda_2(g)} \\ &= (\chi(1)^3|Z(\chi)|/|G|)[\lambda_1^2, \lambda_2]. \end{aligned} \tag{2.3}$$

But $[\lambda_1^2, \lambda_2] = 0$ or 1, which implies $\lambda_1^2 = \lambda_2$ and $0 \neq [\chi^2, \chi^{(2)}] = \chi(1)$. Therefore $\chi^2 = \chi(1)\chi^{(2)}$, proving $\eta(\chi^2) = 1$.

Conversely suppose G has odd order and $\eta(\chi^2) = 1$. Since $\chi^{(2)} \in \mathrm{Irr}(\chi^2)$ and $\eta(\chi^2) = 1$, then $\chi^2 = \chi(1)\chi^{(2)}$. If $\chi(g) \neq 0$, then

$$|\chi(g)| = \frac{\chi(1)}{|\chi(g)|}|\chi^{(2)}(g)| \geq |\chi^{(2)}(g)|. \tag{2.4}$$

Thus $1 = [\chi^{(2)}, \chi^{(2)}] \leq [\chi, \chi] = 1$ and it follows that $|\chi(g)| = |\chi^{(2)}(g)|$. Therefore by (2.4), we have that $\chi(1) = |\chi(g)|$ and $g \in Z(\chi)$. Thus if $\chi(g) \neq 0$ then $g \in Z(\chi)$, and therefore χ vanishes on $G - Z(\chi)$. □

Example 2.6 *Let D_8 be the dihedral group of order 2^3 and $a \in D_8 \backslash Z(D_8)$. We can check that $Cl(a)Cl(a) = Cl(a^{-1})Cl(a) = Z(D_8)$, therefore $\eta(Cl(a)Cl(a)) = \eta(Cl(a^{-1})Cl(a)) = 2$. Thus Proposition 2.5 may not remain true if $a \in G$ has even order. On the other hand, let χ be the faithful irreducible character of D_8. We can check that χ vanishes on $G - Z(G)$, but $\eta(\chi^2) = 4$. Thus Theorem C may not remain true if G has even order.*

3. Upper bound

Lemma 3.1 *Let G be a group and $a, b \in G$. Then $\eta(Cl(a)Cl(b)) \leq |Cl(a)|$.*

Proof For $c \in Cl(a)Cl(b)$ we have $c = g^{-1}agh^{-1}bh$ where $g, h \in G$, we deduce that c is conjugate to $hg^{-1}agh^{-1}b = a^{gh^{-1}}b$, and hence there is $d \in G$ such that $c^d \in a^G b$. Therefore $c \in \bigcup_{g \in G} Cl(a^g b)$, implying that $Cl(a)Cl(b) \subseteq \bigcup_{g\in G} Cl(a^g b)$. Clearly $\bigcup_{g\in G} Cl(a^g b) \subseteq Cl(a)Cl(b)$, hence $Cl(a)Cl(b) = \bigcup_{g\in G} Cl(a^g b)$, thus $\eta(Cl(a)Cl(b)) = \eta(\bigcup_{g\in G} Cl(a^g b)) \leq |a^G b| = |Cl(a)|$. □

Proposition 3.2 *Let G be a finite group and $a \in G$. If $|Cl(a)|$ is an odd number, then $Cl(a)Cl(a)$ is the union of at most $(|Cl(a)|+1)/2$ distinct conjugacy classes, i.e. $\eta(Cl(a)Cl(a)) \leq (|Cl(a)|+1)/2$.*

Proof By Lemma 3.1 , we have that $Cl(a)Cl(a) = \cup_{g\in G} Cl(a^g a)$. It suffices to show that for any $x \in Cl(a)$ with $x \neq a$, there exists $y \in Cl(a)$, $y \neq x, a$, such that $Cl(xa) = Cl(ya)$. Observe that $a^g a$ and $a^{g^{-1}} a$ are always conjugate; thus, if $x = a^g$ and g^2 does not centralize a, one can take $y = a^{g^{-1}}$.

Suppose on the other hand that $g^2 \in C_G(a)$ whenever $x = a^g$. We claim that this happens only when $x = a$. Indeed, fix a, g with $x = a^g$. Then $(zg)^2 \in C_G(a)$ for all $z \in C_G(a)$ and taking into account that $(zg)^2 = zz^{g^{-1}} g^2$, it follows that g normalizes $C_G(a)$. Now let P be a Sylow 2-subgroup of $C_G(a)$ (which is also a Sylow 2-subgroup of $C_G(a)$ since $|Cl(a)| = n$ is odd). Thus $P^g \subseteq C_G(a)$, therefore $P^{zg} = P$ for some $z \in C_G(a)$. By replacing g with zg, we have that g normalizes P. Thus the 2-part of g is in $P \subseteq C_G(a)$ and since $g^2 \in C_G(a)$ then $g \in C_G(a)$. Therefore we can suppose that g has odd order. But then $g \in C_G(a)$ and $x = a$. Thus since $Cl(a)Cl(a) = \cup_{i=1}^{n-1} Cl(a^{g_i} a) \cup Cl(a^2)$ for some $\{g_i\}_{i=1}^{n-1} \subseteq G \backslash C_G(a)$, then

$$\eta(Cl(a)Cl(a)) \leq \frac{|Cl(a)|-1}{2} + 1 = \frac{|Cl(a)|+1}{2}.$$

□

Proposition 3.3 *Let G be a finite group and $a \in G$ and $|Cl(a)| = 2$. Then $\eta(Cl(a)Cl(a)) = \eta(Cl(a)Cl(a^{-1})) = |Cl(a)| = 2$.*

Proof Let $g \in G \backslash C_G(a)$. Observe that $Cl(a) = \{a, a^g\}$ and $a^2, aa^g \in Cl(a)Cl(a)$. By Lemma 3.1, we have $Cl(a)Cl(a) = Cl(a^2) \cup Cl(aa^g)$. Since $a^2 \neq aa^g \neq (a^2)^g$ then $aa^g \notin Cl(a^2)$. Therefore $Cl(a^2) \neq Cl(aa^g)$ and $\eta(Cl(a)Cl(a)) = |Cl(a)| = 2$.

Also, by Lemma 3.1, we have $\eta(Cl(a)Cl(a^{-1})) \leq 2$. Since $1_G \in Cl(a)Cl(a^{-1})$ and $a \notin Z(G)$, then $\eta(Cl(a)Cl(a^{-1})) = 2$

□

Remark 3.4 *Let p and q be two prime numbers such that $p \mid q-1$. Let G be a finite nonabelian group of order pq and $a \in G$ and $|a| = q$.*

We can check that $|Cl(a)| = p$ and $|Cl(a)Cl(a)| = p(p+1)/2$. Also if $p \neq 2$, then all of the conjugacy classes contained in $Cl(a)Cl(a)$ are of size p and therefore $\eta(Cl(a)Cl(a)) = (p+1)/2$. Thus the bound in Proposition 3.2 is optimal.

Otherwise, if $p = 2$, then $|Cl(a)Cl(a)| = 3$ and $Cl(a)Cl(a) = 1_G \cup Cl(a^2)$. Therefore $\eta(Cl(a)Cl(a)) = 2$.

4. Simple groups and symmetric groups

Proposition 4.1 *Let G be a finite nonabelian group and $a \in G$. Then* *(i) If* $a \in G'$*, then* $Cl(a) \subsetneqq G'$*;*

(ii) If $a \notin G'$*, then* $G' \cap Cl(a) = \emptyset$*.*

Proof (i) Let $a \in G'$. If $a = 1_G$ then the statement is true. Otherwise, since $a^{-1}a^G = [a, G] \subseteq G'$, therefore $Cl(a) = a^G \subseteq G'$. Since $1_G \notin Cl(a)$, thus $Cl(a) \subsetneqq G'$.

(ii) Let $a \notin G'$. Since $a^{-1}a^G = [a, G]$, therefore $Cl(a) \cap G' = \emptyset$. □

Corollary 4.2 *Let* G *be a finite nonabelian simple group and* $a \in G$*. Then* $[a, G]$ *is a subgroup of* G *if and only if* $a = 1_G$*.*

Proof Assume that $a \neq 1_G$ and $[a, G]$ be a subgroup of G. Therefore by Lemma 2.1, $[a, G]$ is a normal subgroup of G. Since G is a finite nonabelian simple group and $[a, G] \neq 1_G$, then $[a, G] = G' = G$. By Proposition 4.1, $|Cl(a)| \neq G'$, we have that $[a, G] \neq G'$ and the proof is complete. □

Corollary 4.3 *Let* G *be a finite nonabelian simple group and* $1_G \neq a \in G$*. If* $|Cl(a)|$ *is an odd number, then* $2 \leq \eta(Cl(a)Cl(a)) \leq (|Cl(a)| + 1)/2$*.*

Proof It follows from Corollary 4.2 and Lemma 2.4 and Proposition 3.2. □

Proposition 4.4 *Let* S_n *be the symmetric group of degree* n *and* $\alpha \in S_n$*. Then* $[\alpha, S_n]$ *is a subgroup of* S_n *for* $n > 4$ *if and only if* $\alpha = id$*.*

Proof Assume that $\alpha \neq id$ and $[\alpha, S_n]$ be a subgroup of S_n. Since $[\alpha, S_n] \subseteq S_n' = A_n$ for $n > 4$, thus $[\alpha, S_n] = A_n$. Therefore, by Theorem A, $|Cl(\alpha)| = n!/2$, thus

$$|Cl(\alpha)| = \frac{n!}{1^{e_1}e_1!2^{e_2}e_2!\dots n^{e_n}e_n!} = \frac{n!}{2}$$

where $1^{e_1}, \dots, n^{e_n}$ is the cycle structure of $\alpha \in S_n$. It is easy to see that $|Cl(\alpha)| \neq n!/2$ for $n > 4$. Therefore $[\alpha, S_n]$ is not a subgroup of S_n for $n > 4$. □

Remark 4.5 *We can check that* $[\alpha, S_4]$ *is not a subgroup of* S_4 *for all* $id \neq \alpha \in S_4$*.*

Remark 4.6 *Observe that if* $Cl(a)Cl(a^{-1})$ *is a subgroup of* G*, then* $Cl(a)Cl(a^{-1})$ *is a normal subgroup of* G*, since* $Cl(a)Cl(a^{-1})$ *is* G*-invariant.*

The authors in [3] show that if $G \simeq J_1$ *(where* J_1 *is the Janko group of order* 175560*) then* $Cl(a)Cl(a^{-1}) = J_1$ *for all* $a \in J_1 \backslash \{1_G\}$*. But in Corollary 4.2, we showed that if* G *is a finite nonabelian simple group, then* $[a, G] \neq G$ *for all* $a \in G$*.*

Thus if G *is a finite nonabelian simple group and* $Cl(a)Cl(a^{-1})$ *is a subgroup of* G*, then it is not necessary that* $[a, G]$ *is a subgroup of* G*.*

5. Proof of Theorems B and D

Lemma 5.1 *Let G be a finite group and $a \in G$ and $[a, G]$ is a subset of $Z(G)$. Then $[a, G]$ is a subgroup of G.*

Proof Since $[a, G] \subseteq Z(G)$, thus $[a, g_1] = z_1$ and $[a, g_2] = z_2$ for all $g_1, g_2 \in G$ and for some $z_1, z_2 \in Z(G)$. It is easy to see that $[a, g_1][a, g_2] = z_1 z_2 = [a, g_1 g_2]$. Since $1_G \in [a, G]$ and G is a finite group, it follows that $[a, G]$ is a subgroup of G. □

Proposition 5.2 *Let G be a finite group and $a, b \in G$ and $[a, G] \subseteq Z(G)$. Then all of the conjugacy classes contained in $Cl(a)Cl(b)$ are of size $|Cl(ab)|$. Therefore $\eta(Cl(a)Cl(b)) = |Cl(a)Cl(b)|/|Cl(ab)|$.*

Proof Since $Cl(a)Cl(b)$ is a G-invariant set, then

$$Cl(a)Cl(b) = \{Cl(a^g b) | g \in G\} = \{Cl(abz) | z \in [a, G]\} = \{Cl(ab)z | z \in [a, G]\}.$$

Therefore all of the conjugacy classes contained in $Cl(a)Cl(b)$ are of size $|Cl(ab)|$ and the proof is complete. □

Corollary 5.3 *Let G be a finite group and $a, b \in G$ and $[a, G] \subseteq Z(G)$. If $Cl(a)Cl(b) \cap Z(G) \neq \emptyset$, then $\eta(Cl(a)Cl(b)) = |Cl(a)|$.*

Proof By proposition 5.2, we have that

$$Cl(a)Cl(b) = \{Cl(ab)z | z \in [a, G]\}.$$

Observe that since $Cl(a)Cl(b) \cap Z(G) \neq \emptyset$, then $(ab)^g \in Z(G)$ for some $g \in G$, therefore $ab \in Z(G)$. Thus $Cl(a)Cl(b) \subseteq Z(G)$ and $\eta(Cl(a)Cl(b)) = |Cl(a)Cl(b)| = |Cl(a)|$. □

Proof [**Proof of Theorem B**]

(i) Observe that

$$Cl(a)Cl(b) = a[a, G]Cl(b) = aCl(b)[a, G] \tag{5.1}$$

and $[a, G]$ is a subgroup of $Z(G)$.

Assume that $X = \{Cl(g) | g \in G\}$. $[a, G]$ acts on X by right multiplication in G. Thus the stabilizer of $Cl(b)$ in $[a, G]$ is

$$\begin{aligned} St_{[a,G]}(Cl(b)) &= \{z \in [a, G] | Cl(b)z = Cl(b)\} \\ &= \{z \in [a, G] | z \in Cl(b)Cl(b^{-1})\} \\ &= [a, G] \cap (C(b)Cl(b^{-1})) \end{aligned} \tag{5.2}$$

and the orbit of $Cl(b)$ is

$$\text{Orb}_{[a,G]}(Cl(b)) = \{Cl(b)z | z \in [a, G]\}. \tag{5.3}$$

Since $|Cl(b)| = |Cl(b)z|$ for all $z \in [a, G]$, then $|Cl(b)[a, G]| = |Cl(b)||\text{Orb}_{[a,G]}(Cl(b))|$. Thus

$$|Cl(b)[a, G]| = |Cl(b)||[a, G] : St_{[a,G]}(Cl(b))| = \frac{|Cl(b)||[a, G]|}{|[a, G] \cap (Cl(b)Cl(b^{-1}))|}. \tag{5.4}$$

Therefore by 5.1 and 5.4, we have that

$$|Cl(a)Cl(b)| = |Cl(b)[a,G]| = \frac{|Cl(a)||Cl(b)|}{|[a,G] \cap (Cl(b)Cl(b^{-1}))|}. \tag{5.5}$$

Thus Proposition 5.2 and 5.5 imply (i).

(ii) Follows from Corollary 5.3.

(iii) By Lemma 5.1 we have that $[a,G]$ is a subgroup $Z(G)$, thus

$$Cl(a)Cl(a) = a[a,G]a[a,G] = a^2[a,G][a,G] = a^2[a,G].$$

Therefore $|Cl(a)Cl(a)| = |a^2[a,G]| = |Cl(a)|$. Also there exists $n \in \mathbb{N}$ such that $[a,G] = \langle z_1 \rangle \times \cdots \times \langle z_n \rangle$ for some $z_i \in Z(G)$, $1 \leq i \leq n$ (where $|z_i| = p_i^{\alpha_i}$ for $1 \leq i \leq n$ and $\alpha_i \in \mathbb{N}$). Thus

$$Cl(a) = a[a,G] = a\langle z_1 \rangle \times \cdots \times \langle z_n \rangle.$$

Also we have

$$\begin{aligned} Cl(a^2) &= \{az_1^{i_1} \ldots z_n^{i_n} a z_1^{i_1} \ldots z_n^{i_n} | i_j = 0, \ldots, |z_j| - 1, j = 0, \ldots, n\} \\ &= \{a^2 z_1^{2i_1} \ldots z_n^{2i_n} | i_j = 0, \ldots, |z_j| - 1, j = 0, \ldots, n\} \\ &= a^2(\langle z_1^2 \rangle \times \cdots \times \langle z_n^2 \rangle). \end{aligned}$$

Next if $|Cl(a)|$ is an odd number, then $\langle z_j^2 \rangle = \langle z_j \rangle$ for $j = 1, \ldots, n$. Thus $Cl(a)Cl(a) = a^2[a,G] = Cl(a^2)$ and ii) follows.

(iv) Otherwise, let $|Cl(a)|$ be an even number and P be a Sylow 2-subgroup of $[a,G]$. Assume that $P = \langle z_1 \rangle \times \cdots \times \langle z_m \rangle$ (where $|z_i| = 2^{\alpha_i}$ for $1 \leq i \leq m$ and $\alpha_i \in \mathbb{N}$). Thus we have that

$$\begin{aligned} Cl(a^2 z_1^{l_1} \ldots z_m^{l_m}) &= Cl(a^2) z_1^{l_1} \ldots z_m^{l_m} \\ &= a^2 \left(\langle z_1^2 \rangle \times \cdots \times \langle z_m^2 \rangle \times \cdots \times \langle z_n^2 \rangle\right) z_1^{l_1} \ldots z_m^{l_m} \end{aligned}$$

for $l_1, \ldots, l_m = 0, 1$. Therefore

$$Cl(a)Cl(a) = \bigcup_{l_1,\ldots,l_m=0}^{1} Cl(a^2 z_1^{l_1} \ldots z_m^{l_m}).$$

Thus $Cl(a)Cl(a)$ is the union of exactly 2^m distinct conjugacy classes of G of size $|Cl(a^2)|$ and (iv) follows. □

Proof [**Proof of Theorem D**]

(i) Follows from Proposition 2.3.

(ii) We can assume that χ is a faithful irreducible character and then $Z(\chi) = Z(G)$. Since χ vanishes on $G - Z(G)$, by Corollary 2.30 of [5] we can write $\chi(1)^2 = [G : Z(G)]$, therefore $|G : Z(G)|$ is an odd number. If $|G|$ is an odd number then by Theorem C $\eta(\chi^2) = 1$. Otherwise, let $|G|$ be an even number. If $P \in Syl_2(G)$

then $P \subseteq Z(G)$ and $P \trianglelefteq G$, because $|G : Z(G)|$ is an odd number. By Theorem 7.41 of [6], there is a subgroup H of G such that $|H| = |G : P|$. Since $P \trianglelefteq G$ then we can check that $G = HP = H \times P$. By problem 4.5 of [5], $\chi^{(2)} \in \mathrm{Irr}(G)$ because $G = H \times P$ and P is an abelian subgroup of G and $(|H|, 2) = 1$. But by Theorem 4.5 of [5], $\chi^2 = \chi^{(2)} + 2\chi_A$ where $\chi_A \in Char(G)$. By the Proof of Theorem C we can write $0 \neq [\chi^2, \chi^{(2)}] = \chi(1)$ and $\chi^2 = \chi(1)\chi^{(2)}$. It follows that $\eta(\chi^2) = 1$. □

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