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Complex symplectic geometry with applications to vector differential operators

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Abstract: Let $l(y)$ be a formally self-adjoint vector-valued differential expression of order n on an interval $(a,\infty)(-\infty \leq a < \infty)$ with complex matrix-valued function coefficients and finite equal deficiency indices. In this paper, applying complex symplectic algebra, we give a reformulation for self-adjoint domains of the minimal operator associated with $l(y)$ and classify them.

Key words: Symplectic algebra, Lagrangian subspace, vector-valued differential operator, self-adjoint domains

1. Introduction

Let $l(y)$ be a formally self-adjoint vector-valued differential expression of order n on an interval $I=(a,\infty)(-\infty \leq a < \infty)$ with complex matrix-valued coefficients and finite equal deficiency indices. It is well known from the general operator theory that the minimal operator associated with $l(y)$ can be extended to a self-adjoint operator in a Hilbert space. The study of boundary value problems involving linear differential equations is becoming a well-established area of analysis. Applying the extension theory of symmetric operators to concrete differential operators, a general characterization of self-adjoint extensions of symmetric differential operators is established. For details of some of this work we refer to [1]–[17], etc.

Recently, in [13] Wang, Sun and Zettl give a representation of self-adjoint conditions in terms of certain solutions for real parameter, which leads to a classification of solutions as limit-point or limit-circle in analogy with the celebrated Weyl classification in the second-order case. In [7] Hao, Sun, Wang and Zettl, applying results from [13], characterize self-adjoint domains of general even order linear ordinary differential operators in terms of real-parameter solutions of the differential equation, which is a follow up of [13].

In [3, 4, 5], the complex symplecto-algebraic complete characterizations of self-adjoint extensions of symmetric operators are given. This paper presents a generalization to the case of vector-valued functions of the approach presented in [4]. This approach is based on the following idea. Let $l(y)$ be some ordinary formally self-adjoint differential operator considered in Hilbert space $L^2(I)$ on some interval I. We can define in the standard way the minimal and maximal operators T_0 and T_1 associated to $l(y)$, with domains denoted by $D(T_0), D(T_1)$. On the domain $D(T_1)$ we introduce skew-Hermitian form $[y:z]=(T_1y,z)-(y,T_1z)$, where $(\cdot,\cdot)$ denotes the scalar product in $L^2(I)$. This form generates sympletic structure on the space $S=D(T_1)/D(T_0)$ and there is one-to-one correspondence between complete (maximal) Lagrangian space L in S and self-adjoint

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extensions T of T_0. In this way study of self-adjoint extensions is reduced to study of Lagrangian subspace in the space S and one can try to find the relations between the geometric and algebra properties of L as subspace in S, and the structure of boundary conditions that define the self-adjoint extension T corresponding to L. Applying complex symplectic algebra, we present complete characterizations and classifications for self-adjoint domains associated with $l(y)$.

The layout of this paper is as follows. In Section 1 we summarize the results of symplectic algebra and vector-valued differential operators. In Section 2 complex symplecto-algebraic characterizations of self-adjoint boundary conditions of vector-valued differential operator are given at the case with a finite regular endpoint. Section 3 presents some results at the case with two singular endpoints.

2. Preliminaries

Definition 1 (Definition 1.1 of [4]) *A complex symplectic space S is a complex linear space with a prescribed symplectic form $[\cdot:\cdot]$, namely a sesquilinear form*

(i) $u, v \to [u:v], S\times S \to \mathbb{C}$, so for all vectors $u, v, \omega \in S$ and complex scalars $c_1, c_2 \in \mathbb{C}$, $[c_1u + c_2v : \omega] = c_1[u:\omega] + c_2[v:\omega]$, which is skew-Hermitian;

(ii) $[u:v] = -\overline{[v:u]}$, so for all vectors $u, v, \omega \in S$ and complex scalars $c_1, c_2 \in \mathbb{C}$, $[u : c_1v + c_2\omega] = \overline{c_1}[u:v] + \overline{c_2}[u:\omega]$, and which is also non-degenerate;

(iii) $[u:S] = 0$ implies $u = 0$, for all $u \in S$.

Definition 2 (Definition 1.2 of [4]) *A linear subspace L in the complex symplectic space S is called Lagrangian in case $[L:L] = 0$, that is, for all $u, v \in L$, $[u:v] = 0$.*

Definition 3 (Definition 1.2 of [4]) *A Lagrangian space $L \subset S$ is complete in case $u \in S$ and $[u:L] = 0$ imply $u \in L$.*

Definition 4 (Definition 2.2 of [4]) *Let S be a complex symplectic space with symplectic form $[\cdot:\cdot]$. Then linear subspace S_+ and S_- are symplectic ortho-complements in S, written as*

$$S = S_+ \oplus S_-,$$

in case
(i) $S = span\{S_+, S_-\}$;
(ii) $[S_- : S_+] = 0$.

Consider the formally self-adjoint vector-valued differential expression introduced by J. Weidmann [16]:

$$\begin{aligned} l[y](x) = \quad & r(x)^{-1}\Big\{\sum_{k=0}^{[\frac{n}{2}]}(-1)^k(p_k(x)y^{(k)}(x))^{(k)} \\ & + \sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k[(q_k(x)y^{(k)}(x))^{(k+1)} - (q_k(x)^*y^{(k+1)}(x))^{(k)}]\Big\}, \end{aligned} \tag{1.1}$$

where $y(x) = (y_1(x), \dots, y_m(x))^t$ is defined in the interval $I = (a, \infty)$, $-\infty \le a < \infty$; $[\alpha]$ denotes the greatest integer not greater than α. The $m \times m$ matrix-valued functions $r(x), p_j(x) (j = 0, 1, \dots, [\frac{n}{2}])$ and

$q_j(x)(j=0,1,\dots,[\frac{n-1}{2}])$ satisfy
(i) $r(x), p_j(x)$ and $q_j(x)$ are measurable over I;
(ii) $r(x)$ is a positive, definite matrix and $p_j(x)$ are Hermitian; and $p_k^{-1} \in AC_{\rm loc}(I)$ if $n=2k$; $q_k, (q_k-q_k^*)^{-1} \in AC_{\rm loc}(I)$ if $n=2k+1$;
(iii) $p_k^{-1}, p_k^{-1}q_{k-1}, p_{k-1}-q_{k-1}^*p_k^{-1}q_{k-1}, p_j$ $(j=0,1,\dots,k-2), q_j$ $(j=0,1,\dots,k-2)$ and r are absolutely Lebesgue integrable on all compact subset of I if $n=2k$; $(q_k-q_k^*)^{-1}, (q_k-q_k^*)^{-1}(p_k+q_k'), (q_k-q_k^*)^{-1}q_{k-1}, p_j$ $(j=0,1,\dots,k-1), q_j$ $(j=0,1,\dots,k-1)$ and r are absolutely Lebesgue integrable on all compact subsets of I if $n=2k+1$.

Thus n is the order of $l(y)$. Define the quasi-derivatives $y^{[r]}(r=0,1,\dots,n)$ as in pages 26-30 of [16], then the differential expression (1.1) can be rewritten as

$$l[y](x)=r(x)^{-1}y^{[n]}(x).$$

In the complex vector space $\mathbb{C}^m=\{\alpha:\alpha=(c_1,\dots,c_m)^t, c_i(i=1,2,\dots,m)\in\mathbb{C}\}$, define inner product

$$(\xi,\eta)=\sum_{i=1}^{m}\xi_i\overline{\eta_i},\ \xi=(\xi_1,\dots,\xi_m)^t,\eta=(\eta_1,\dots,\eta_m)^t.$$

A Hilbert space

$$H=\{f:I\to\mathbb{C}^m, f \text{ measurable} \mid \int_I (r(x)f(x),f(x))dx<\infty\}$$

with inner product

$$\langle y,z\rangle=\int_I (r(x)y(x),z(x))dx,\ \text{for all } y,z\in H,$$

denoting Hilbert space H as $L_r^2(I)$.

For the differential expression $l(y)$ defined as above, its maximal operator T_1: $T_1y=l(y)$ on

$$\mathcal{D}(T_1)=\{y:I\to\mathbb{C}^m,\ y^{[k]}\in AC_{\rm loc}(I)(k=0,1,\dots,n-1), y \text{ and } l(y)\in L_r^2(I)\},\tag{1.2}$$

where $AC_{\rm loc}(I)$ denotes a set of complex-vector valued functions which are absolutely continuous on all compact subintervals of I and its minimal operator T_0:

$$T_0y=l(y) \text{ on } \mathcal{D}(T_0)=\{y\in D(T_1)\mid [y:\mathcal{D}(T_1)]=0\}.\tag{1.3}$$

Here, the skew-Hermitian form $[\cdot:\cdot]$ on $\mathcal{D}(T_1)$ is given by

$$[y:z]=\langle T_1y,z\rangle-\langle y,T_1z\rangle,\ \text{for } y,z\in\mathcal{D}(T_1),\tag{1.4}$$

where $[y:z]=\langle T_1y,z\rangle-\langle y,T_1z\rangle$ is the Lagrange bilinear form associated with $l(y)$.

It is known from Theorem 3.1 of [16] that $T_0\subset T_1$ on $\mathcal{D}(T_0)\subset\mathcal{D}(T_1)\subset L_r^2(I)$ satisfy
(i) $\mathcal{D}(T_0)$ is dense in $L_r^2(I)$, so also $\mathcal{D}(T_1)$ is dense in $L_r^2(I)$;
(ii) adjoints $T_0^*=T_1$ and $T_1^*=T_0$,
so both T_0 and T_1 are closed operators, T_0 is symmetric.

For any $[c,d] \subset I, y, z \in \mathcal{D}(T_1)$, we have Green's formula (see pages 35–40 in [16] or see equations (1.3) and (1.4) in [17]),

$$\int_c^d \{(r(x)ly(x), z(x)) - (r(x)y(x), lz(x))\}dx = [y,z]_n(d) - [y,z]_n(c), \tag{1.5}$$

where

$$[y,z]_n(x) = R_n(y)(x) A R_n^*(z)(x), \quad [y,z]_n(\infty) = \lim_{x\to\infty} [y,z]_n(x) \text{ exists}, \tag{1.6}$$

$$R_n(y)(x) = (y^{[0]}(x)^t, y^{[1]}(x)^t, \ldots, y^{[n-1]}(x)^t), \quad x \in I. \tag{1.7}$$

Here, t denotes the transpose of matrix and

$$A(x) = \begin{cases} \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & E_m \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & E_m & \cdots & 0 \\ 0 & \cdots & -E_m & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -E_m & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} & \text{if } n \text{ is even ;} \\ \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & E_m \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & E_m & \cdots & 0 \\ 0 & \cdots & 0 & q_k^*(x) - q_k(x) & 0 & \cdots & 0 \\ 0 & \cdots & -E_m & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -E_m & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} & \text{if } n \text{ is odd,} \end{cases} \tag{1.8}$$

where E_m denotes the identity matrix of size m.

Hereafter, $[\cdot,\ \cdot]_n(x)$ is called the Lagrange bi-linear form corresponding to $l(y)$ on I.

Since $l(y)$ is a formally self-adjoint differential expression on I, we easily get from (1.1) and (1.8) that

$$A^* = -A, \qquad \text{rank } A = nm. \tag{1.9}$$

In order to describe the self-adjoint boundary conditions of differential operator we introduce the deficiency indices of T_0: these are the integers

$$d_\pm = \dim\ (\ker(T_1 \mp i)).$$

In this paper, we assume $l(y)$ is a regular, formally self-adjoint differential expression with finite, equal deficiency indices (d,d). Obviously,

$$0 \le d \le nm.$$

Define an endpoint space $\mathbf{S}$, for $l(y)$ on I, as the quotient or identification vector space

$$\mathbf{S} = \mathcal{D}(T_1)/\mathcal{D}(T_0), \tag{1.10}$$

so there is a natural projection map

$$\psi : \mathcal{D}(T_1) \to \mathbf{S}, f \mapsto \mathbf{f} = \{f + \mathcal{D}(T_0)\}, \text{ for } \mathbf{f} \in S, f \in \mathcal{D}(T_1).$$

Define the symplectic form $[\cdot : \cdot]$ in $\mathbf{S}$, for $\mathbf{f} = \{f + \mathcal{D}(T_0)\}$ and $\mathbf{g} = \{g + \mathcal{D}(T_0)\}$,

$$[\mathbf{f} : \mathbf{g}] = [f : g]. \tag{1.11}$$

3. The case with a finite regular endpoint

The endpoint a is called regular if, for some $a < c < \infty$, $p_k^{-1}, p_k^{-1}q_{k-1}, p_{k-1} - q_{k-1}^* p_k^{-1} q_{k-1}, p_j$ $(j = 0, 1, \ldots, k-2)$, q_j $(j = 0, 1, \ldots, k-2)$ and r are Lebesgue integrable on (a, c) when $n = 2k$ or $(q_k - q_k^*)^{-1}$, $(q_k - q_k^*)^{-1}(p_k + q_k')$, $(q_k - q_k^*)^{-1} q_{k-1}, p_j$ $(j = 0, 1, \ldots, k-1)$, q_j $(j = 0, 1, \ldots, k-1)$ and r are Lebesgue integrable on (a, c) when $n = 2k+1$; otherwise a is said to be singular. Without loss of generality we consider the case with a finite regular endpoint a. In this section, we apply the theory of complex symplectic spaces to the boundary value problems of linear vector-valued differential operators with order $n(\geq 1)$ and complex matrix valued coefficients defined on $[a, \infty)(-\infty < a < \infty)$ and equal deficiency indices (d, d) (obviously, $[\frac{nm+1}{2}] \leq d \leq nm$). This section treats the positioning of a Lagrangian subspace within S, and gives necessary and sufficient conditions for k-grade $(0 \leq k \leq d - [\frac{nm+1}{2}])$ complete Lagrangian subspaces.

Let

$$\widetilde{\mathcal{D}}(l) = \mathcal{D}(T_0) \oplus \mathrm{span}\{\chi_{11}, \chi_{12}, \ldots, \chi_{mn}\}, \tag{2.1}$$

where the symbol $\oplus$ denotes a direct sum and span$\{\chi_{11}, \ldots, \chi_{mn}\}$ denotes the linear span of $\chi_{11}, \ldots, \chi_{mn}$ and $\chi_{ij}(i = 1, \cdots, m; j = 1, \ldots, n)$ be a set of functions in $\mathcal{D}(T_1)$ which satisfy the following conditions:

$$\chi_{ij}^{[k-1]}(a) = \begin{cases} 0_{m\times 1} & \text{for } j \neq k; \\ e_i & \text{for } j = k, \end{cases} \quad \chi_{ij}(t) = 0 \text{ for all } t \geq a+1, \tag{2.2}$$

where e_i is the ith canonical unit vector in $\mathbb{C}^m$ and $0_{m\times 1} = (\overbrace{0, \ldots, 0}^{m})^t$. Clearly, $\chi_{ij} \overline{\in} D_0(l)$.

For any $\widetilde{z} \in \widetilde{D}(l)$, it is not difficult to see that

$$[y, \widetilde{z}]_n(\infty) = 0, \text{for all } y \in D(T_1). \tag{2.3}$$

Let $N = 2d - nm$, then $l(y) = \lambda y$ $(Im\lambda \neq 0)$ has N linearly independent square integrable solutions on $[a, \infty)$, denoted by $\theta_1, \ldots, \theta_N$, which satisfy

$$\mathrm{rank}\ K = N, \quad \mathcal{D}(T_1) = \widetilde{\mathcal{D}}(l) \oplus \mathrm{span}\{\theta_1, \ldots, \theta_N\},$$

where $K = ([\theta_i, \theta_j]_n(\infty))_{1\leq i,j\leq N}$, $K^* = -K$ (cf. Lemma 3 of [15]).

Since $K^* = -K$, $(iK)^* = iK$ (where $i^2 = -1$), iK is symmetric Hermitian, there exists some complex non-singular matrix T such that

$$T(iK)T^* = \mathrm{diag}\{E_q, -E_p\},$$

where $p + q = N, p \geq 0, q \geq 0$. So $TKT^* = \mathrm{diag}\{-iE_q, iE_p\}$.

Define $\widetilde{\theta_1}, \ldots, \widetilde{\theta_N}$, such that

$$\begin{pmatrix} \widetilde{\theta_1} \\ \vdots \\ \widetilde{\theta_N} \end{pmatrix} = T \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_N \end{pmatrix},$$

obviously, $\widetilde{\theta_1}, \ldots, \widetilde{\theta_N}$ are N linearly independent square integrable solutions of $l(y) = \lambda y$ on $[a, \infty)$, which satisfy

$$([\widetilde{\theta_i}, \widetilde{\theta_j}]_n(\infty))_{1\leq i,j\leq N} = \mathrm{diag}\{-iE_q, iE_p\}, \ \mathcal{D}(T_1) = \widetilde{\mathcal{D}}(l) \oplus \mathrm{span}\{\widetilde{\theta_1}, \ldots, \widetilde{\theta_N}\}.$$

Thus we have

Lemma 1 *Let $N = 2d - nm$. Then $l(y) = \lambda y$ $(Im\,\lambda \neq 0)$ has N linearly independent square integrable solutions on $[a, \infty)$, denoted by $\theta_1, \ldots, \theta_N$, which satisfy*

$$J \triangleq ([\theta_i, \theta_j]_n(\infty))_{1\leq i,j\leq N} = diag\{-iE_q, iE_p\},\ \mathcal{D}(T_1) = \widetilde{\mathcal{D}}(l) \oplus span\{\theta_1, \ldots, \theta_N\}. \tag{2.4}$$

Obviously, $J^* = -J$ and $J = -J^{-1}$.

Lemma 2 *The complex vector space*

$$\mathbf{S} = \mathcal{D}(T_1)/\mathcal{D}(T_0)$$

with the Skew-Hermitian from $[\cdot : \cdot]$, as in (1.11), is a complex symplectic space and dim $\mathbf{S} = 2d$.

Proof Since the operators T_1 and T_0 arise from the differential expression of order $n \geq 1$ on $[a, \infty)$, it follows from the theory of linear ordinary differential equation and von Neumann's formula (Theorem 4.3 of [16]) that $\mathbf{S}$ has a finite dimension, and $\dim \mathbf{S} = 2d$. It is easy to verify by Definition 1 that $\mathbf{S}$ is a complex symplectic space. Thus we have the following lemma.

Lemma 3 *Let linear subspace of* $\mathbf{S}$

$$\mathbf{S}_- = \{y \in \mathbf{S} \mid [y, \theta_i]_n(\infty) = 0, i = 1, 2, \ldots, 2d - nm\},$$

$$\mathbf{S}_+ = \{y \in \mathbf{S} \mid y^{[k]}(a) = 0, k = 0, 1, \ldots, n-1\},$$

then $\mathbf{S} = \mathbf{S}_- \oplus \mathbf{S}_+$ and dim $\mathbf{S}_- = nm$, dim $\mathbf{S}_+ = 2d - nm$.

Proof In fact, the definition of minimal operator (Theorem 3.12 of [16]) implies

$$\mathcal{D}(T_0) = \{f \in \mathcal{D}(T_1) | f^{[k]}(a) = 0, k = 0, 1, \ldots, n-1; [f, \theta_i]_n(\infty) = 0, 1 \leq i \leq 2d - nm\},$$

together with the decomposition of maximal operator domains $\mathcal{D}(T_1)$ and Definition 4, it is easy to see that the results hold. □

Applying GKN-Theorem (Corollary 1 of Appendix in [4]) and the balanced intersection principle (Theorem 2.4 in [4]) to the quotient vector space $\mathbf{S}$, we have this next lemma.

Lemma 4 *(GKN-Theorem) (i) There exists a self-adjoint extension T of T_0 if and only if there exists a complete Lagrangian subspace $\mathbf{L} \subseteq \mathbf{S}$;*

(ii) A Lagrangian subspace $\mathbf{L} \subset \mathbf{S}$ is complete if and only if dim $\mathbf{L} = d$, where (d, d) is the deficiency indices of $l(y)$;

(iii) For each self-adjoint operator T on domains $\mathcal{D}(T) \subset L_r^2(I)$, which is an extension of T_0 on $\mathcal{D}(T_0)$, the corresponding complete Lagrangian subspace $\mathbf{L}$ is defined by

$$\mathbf{L} = \mathcal{D}(T)/\mathcal{D}(T_0),$$

so $\mathcal{D}(T) = c_1 f_1 + \cdots + c_d f_d + \mathcal{D}(T_0)$. Here $\{f_1, \ldots, f_d\}$ is any basis of $\mathbf{L}$, with any corresponding representative functions $f_1, \ldots, f_d \in \mathcal{D}(T_1)$, and $c_1, \ldots, c_d$ are arbitrary complex numbers.

Lemma 5 (Balanced intersection principle) *For each complete Lagrangian space* $\mathbf{L}$ *in* $\mathbf{S}$*, then*

$$0 \le d - [\frac{nm+1}{2}] - dim\, \mathbf{L} \cap \mathbf{S}_+ = [\frac{nm}{2}] - dim\, \mathbf{L} \cap \mathbf{S}_- \le d - [\frac{nm+1}{2}].$$

Definition 5 *For each complete Lagrangian space* $\mathbf{L}$ *in* $\mathbf{S}$*, let*

$$k = d - [\frac{nm+1}{2}] - dim\, \mathbf{L} \cap \mathbf{S}_+ = [\frac{nm}{2}] - dim\, \mathbf{L} \cap \mathbf{S}_-.$$

Then $\mathbf{L}$ *is called* k*-grade, or* $\mathcal{D}(T_L)$ *is called* k*-grade.*

From Lemma 2, we see that dim $\mathbf{S} = 2d$, so the complex symplectic space $\mathbf{S}$ is linearly isomorphic to $\mathbb{C}^{2d} = \{\alpha | \alpha = (c_1, c_2, \ldots, c_{2d})^t, c_i \in \mathbb{C}, i = 1, 2, \ldots, 2d\}$. We can use the customary unit basis vectors in $\mathbb{C}^{2d}$,

$$e^1 = (1, 0, \ldots, 0)^t, e^2 = (0, 1, 0, \ldots, 0)^t, \ldots, e^{nm} = (\overbrace{0, \ldots, 0, 1}^{nm}, 0, \ldots, 0)^t,$$
$$f^1 = (\overbrace{0, \ldots, 0, 1}^{nm+1}, 0, \ldots, 0)^t, f^2 = (\overbrace{0, \ldots, 0, 1}^{nm+2}, 0, \ldots, 0)^t, f^{2d-nm} = (0, \ldots, 0, 1)^t,$$

so

$$\mathbf{S} = \text{span}\{e^1, e^2, \ldots, e^{nm}, f^1, f^2, \ldots, f^{2d-nm}\}.$$

Lemma 6 *Let the deficiency index of* $l(y)$ *on* $[a, \infty)$*,* $def(l) = (d, d)$ $([\frac{nm+1}{2}] \le d \le nm)$*,* $N = 2d - nm$*. For all* $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}^m$*,* $\beta_1, \beta_2, \ldots, \beta_N \in \mathbb{C}$*, there exists* $y \in \mathcal{D}(T_1)$*, such that*

$$y^{[i-1]}(a) = \alpha_i \ (i = 1, 2, \ldots, n); \ [y, \theta_k]_n(\infty) = \beta_k \ (k = 1, 2, \ldots, N), \tag{2.5}$$

where $\theta_k (k = 1, 2, \ldots, N)$ *defined in Lemma 1.*

Proof By Lemma 1 and (2.1), for all $y \in \mathcal{D}(T_1)$, there exist $d_{ij} \in \mathbb{C}(1 \le i \le m; 1 \le j \le n)$ and $c_k (1 \le k \le N) \in \mathbb{C}$, such that

$$y = y_0 + \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} \chi_{ij} + \sum_{k=1}^{N} c_k \theta_k,$$

where $y_0 \in \mathcal{D}(T_0)$, χ_{ij} defined in (2.2). Choose

$$\begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} = -J^t \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix}, \begin{pmatrix} d_{11} \\ \vdots \\ d_{m1} \\ \vdots \\ d_{1n} \\ \vdots \\ d_{mn} \end{pmatrix} = \begin{pmatrix} \alpha_1^t \\ \vdots \\ \alpha_n^t \end{pmatrix} + \Phi(a) J^t \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix},$$

where J is defined in Lemma 1 and

$$\Phi(a) = \begin{pmatrix} \theta_1^{[0]}(a)^t & \theta_2^{[0]}(a)^t & \ldots & \theta_N^{[0]}(a)^t \\ \theta_1^{[1]}(a)^t & \theta_2^{[1]}(a)^t & \ldots & \theta_N^{[1]}(a)^t \\ \vdots & \vdots & \ddots & \vdots \\ \theta_1^{[n-1]}(a)^t & \theta_2^{[n-1]}(a)^t & \ldots & \theta_N^{[n-1]}(a)^t \end{pmatrix}.$$

Therefore $y \in \mathcal{D}(T_1)$, the fact that χ_{ij} satisfy (2.2) and a direct computation imply that y satisfies (2.5). This completes the proof of this Lemma. □

From Lemma 6 we can introduce corresponding coordinates in $\mathbf{S}$ by the convenient choice

$$\begin{aligned} \mathbf{f} &= (f(a)^t, f^{[1]}(a)^t, \ldots, f^{[n-1]}(a)^t, [f,\theta_1]_n(\infty), \ldots, [f,\theta_{2d-nm}]_n(\infty)) \\ &= \textstyle\sum_{i=1}^m \sum_{j=1}^n f_i^{[j-1]}(a) e^{i+(j-1)m} + [f,\theta_1]_n(\infty) f^1 + \ldots + [f,\theta_{2d-nm}]_n(\infty) f^{2d-nm}, \end{aligned} \tag{2.6}$$

where $\mathbf{f} = \{f + \mathcal{D}(T_0)\}$, for $f = (f_1, \ldots, f_m)^t \in \mathcal{D}(T_1)$.

In terms of these coordinates, the symplectic form $[\cdot : \cdot]$ in $\mathbf{S}$ can be expressed as the following form as Theorem 1.

Theorem 1 *For* $\mathbf{f}, \mathbf{g} \in \mathbf{S}$, *we have* $[\mathbf{f} : \mathbf{g}] = \mathbf{f} H \mathbf{g}^*$, *where* $\mathbf{f}, \mathbf{g}$ *appearing in the right side of the equation take each corresponding coordinate in* $\mathbf{S}$ *defined in (2.6),*

$$H = \begin{pmatrix} -A(a) & 0_{nm\times(2d-nm)} \\ 0_{(2d-nm)\times nm} & J \end{pmatrix}_{2d\times 2d}$$

and $A(a), J$ *defined in (1.8) and (2.4), respectively.*

Proof By (1.4), (1.5) and (1.11), we get for $\mathbf{f}, \mathbf{g} \in \mathbf{S}$,

$$\begin{aligned} [\mathbf{f} : \mathbf{g}] &= [f : g] = \langle T_1 f, g\rangle - \langle f, T_1 g\rangle \\ &= \langle lf, g\rangle - \langle f, lg\rangle = [f,g]_n(\infty) - [f,g]_n(a). \end{aligned} \tag{2.7}$$

From (1.6), we have

$$[f,g]_n(a) = R_n(f)(a) A(a) R_n^*(g)(a). \tag{2.8}$$

Denotes

$$r_n(f)(\infty) = ([f,\theta_1]_n(\infty), \ldots, [f,\theta_N]_n(\infty).$$

Now we prove

$$[f,g]_n(\infty) = r_n(f)(\infty) J r_n^*(g)(\infty).$$

By Lemma 1, for $f, g \in \mathcal{D}(T_1)$, there exist $\widetilde{f}, \widetilde{g} \in \widetilde{\mathcal{D}}(l)$ and c_i, d_i $(i = 1, 2, \ldots, N) \in \mathbb{C}$, such that

$$f = \widetilde{f} + \sum_{i=1}^N c_i\theta_i, g = \widetilde{g} + \sum_{i=1}^N d_i\theta_i, \tag{2.9}$$

together with (2.3), we get

$$[f,\theta_i]_n(\infty) = (c_1, \ldots, c_N) \begin{pmatrix} [\theta_1,\theta_i]_n(\infty) \\ \vdots \\ [\theta_N,\theta_i]_n(\infty) \end{pmatrix} (i = 1, 2, \ldots, N),$$

which can be written as

$$r_n(f)(\infty) = (c_1, \ldots, c_N) J,$$

that is,

$$(c_1, \ldots, c_N) = r_n(f)(\infty) J^{-1}. \tag{2.10}$$

Similarly,

$$(d_1, \ldots, d_N)^* = (J^*)^{-1} r_n^*(g)(\infty) = J r_n^*(g)(\infty). \tag{2.11}$$

By (2.9), (2.10) and (2.11), we obtain

$$\begin{aligned} [f,g]_n(\infty) &= (c_1, \ldots, c_N) J (d_1, \ldots, d_N)^* \\ &= r_n(f)(\infty) J r_n^*(g)(\infty). \end{aligned} \tag{2.12}$$

Equations (2.7), (2.8) and (2.12) imply

$$\begin{aligned} [\mathbf{f} : \mathbf{g}] &= r_n(f)(\infty) J r_n^*(g)(\infty) - R_n(f)(a) A(a) R_n^*(g)(a) \\ &= (R_n(f)(a), r_n(f)(\infty)) H (R_n(g)(a), r_n(g)(\infty))^* \\ &= f H g^*, \end{aligned}$$

and so the result follows. □

By Theorem 1, we can introduce the corresponding symplectic form $[\cdot : \cdot]$ in $\mathbb{C}^{2d}$ using the skew-Hermitian $2d \times 2d$ matrix H (it is easy to verify that H is a skew-Hermitian matrix from (1.9) and Lemma 1), thus the boundary value problem for the differential expression $l(y)$ on $[a, \infty)$ is reduced, via the GKN-Theorem, to the purely algebraic problem of determining all the complete Lagrangian subspaces L in the complex symplectic space $\mathbb{C}^{2d}$, and a complete Lagrangian subspaces of $\mathbb{C}^{2d}$ is of S by virtue of the symplectic isomorphism of S with $\mathbb{C}^{2d}$.

Theorem 2 *A complete Lagrangian subspace in S is 0-grade, or 1-grade, …, or $(d - [\frac{nm+1}{2}])$-grade.*

Proof Lemma 5 and Definition 5 imply Theorem 2. □

Theorem 3 *For $\mathbf{S}_-$ and $\mathbf{S}_+$ defined in Lemma 3, we have*

$$\mathbf{S}_- = span\{e^1, e^2, \ldots, e^{nm}\}, \quad \mathbf{S}_+ = span\{f^1, f^2, \ldots, f^{2d-nm}\}.$$

Proof First we prove $\mathbf{S}_- = \text{span}\{e^1, e^2, \ldots, e^{nm}\}$. For $\mathbf{f} = \{f + \mathcal{D}(T_0)\} \in \mathbf{S}_-$, then $f \in \mathcal{D}(T_1)$ and $[f, \theta_i]_n(\infty) = 0$ $(i = 1, 2, \ldots, 2d - nm)$. By (2.6), we have

$$\mathbf{f} = \sum_{i=1}^{m} \sum_{j=1}^{n} f_i^{[j-1]}(a) e^{i+(j-1)m} \in \text{span}\{e^1, e^2, \ldots, e^{nm}\},$$

that is,

$$S_- \subset \text{span}\{e^1, e^2, \ldots, e^{nm}\}. \tag{2.13}$$

Conversely, if $\mathbf{f} \in \text{span}\{e^1, e^2, \ldots, e^{nm}\}$, then $\mathbf{f} = \sum_{i=1}^{m} \sum_{j=1}^{n} f_i^{[j-1]}(a) e^{i+(j-1)m}$, which implies $[f, \theta_i]_n(\infty) = 0$ $(i = 1, 2, \ldots, 2d - nm)$, that is, $\mathbf{f} \in \mathbf{S}_-$, thus

$$\text{span}\{e^1, e^2, \ldots, e^{nm}\} \subset \mathbf{S}_-. \tag{2.14}$$

Equations (2.13) and (2.14) imply $\mathbf{S}_- = \text{span}\{e^1, e^2, \ldots, e^{nm}\}$.

Similarly, $\mathbf{S}_+ = \text{span}\{f^1, f^2, \ldots, f^{2d-nm}\}$. Therefore, Theorem 2 holds. □

Theorem 4 $\mathbf{L}$ *is a* k*-grade* $(0 \le k \le d-[\frac{nm+1}{2}])$ *complete Lagrangian subspace in* $\mathbf{S}$ *if and only if there exist* a_{ij}, $b_{it} \in \mathbb{C}$ $(i=1,2,\ldots,d; j=1,2,\ldots,nm; t=1,2,\ldots,2d-nm)$*, such that*

$$\begin{aligned}\mathbf{L} &= span\{a_{11}e^1 + a_{12}e^2 + \cdots + a_{1,nm}e^{nm} + b_{11}f^1 + b_{12}f^2 + \cdots + b_{1,2d-nm}f^{2d-nm},\\ &\ldots, a_{d1}e^1 + a_{d2}e^2 + \cdots + a_{d,nm}e^{nm} + b_{d1}f^1 + b_{d2}f^2 + \cdots + b_{d,2d-nm}f^{2d-nm}\},\end{aligned} \tag{2.15}$$

and (i) rank $A = [\frac{nm+1}{2}] + k$*, rank* $B = d - [\frac{nm}{2}] + k$*, where* $A = (a_{ij})_{d\times nm}$ *and* $B = (b_{it})_{d\times(2d-nm)}$*;*
(ii) $\alpha_i H \alpha_j^* = 0$ $(1 \le i,j \le d)$*,where* $\alpha_i = (a_{i1}, \ldots, a_{i,nm}, b_{i1}, b_{i2}, \ldots, b_{i,2d-nm})$ *and* H *defined in Theorem 1.*

Proof (Necessity) For all $\mathbf{f}, \mathbf{g} \in \mathbf{L}$, there exist s_{1i}, s_{2i} $(i=1,2,\ldots,d) \in \mathbb{C}$, such that

$$\begin{aligned}\mathbf{f} &= \textstyle\sum_{i=1}^d s_{1i}(a_{i1}e^1 + \cdots + a_{i,nm}e^{nm} + b_{i1}f^1 + \cdots + b_{i,2d-nm}f^{2d-nm})\\ &= (\textstyle\sum_{i=1}^d s_{1i}a_{i1})e^1 + \cdots + (\sum_{i=1}^d s_{1i}a_{i,nm})e^{nm} + (\sum_{i=1}^d s_{1i}b_{i1})f^1 + \cdots\\ &\quad + (\textstyle\sum_{i=1}^d s_{1i}b_{i,2d-nm})f^{2d-nm},\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}\mathbf{g} &= (\textstyle\sum_{i=1}^d s_{2i}a_{i1})e^1 + \cdots + (\sum_{i=1}^d s_{2i}a_{i,nm})e^{nm} + (\sum_{i=1}^d s_{2i}b_{i1})f^1 + \cdots\\ &\quad + (\textstyle\sum_{i=1}^d s_{2i}b_{i,2d-nm})f^{2d-nm}.\end{aligned} \tag{2.17}$$

By Theorem 1 and (ii), we obtain

$$\begin{aligned}[\mathbf{f}:\mathbf{g}] &= (\textstyle\sum_{i=1}^d s_{1i}a_{i,1}, \ldots, \sum_{i=1}^d s_{1i}a_{i,nm}, \sum_{i=1}^d s_{1i}b_{i1}, \ldots, \sum_{i=1}^d s_{1i}b_{i,2d-nm})H\\ &\quad (\textstyle\sum_{i=1}^d s_{2i}a_{i1}, \ldots, \sum_{i=1}^d s_{2i}a_{i,nm}, \sum_{i=1}^d s_{2i}b_{i1}, \ldots, \sum_{i=1}^d s_{2i}b_{i,2d-nm})^*\\ &= (s_{11}, \ldots, s_{1d})(A|B)H\begin{pmatrix} A^* \\ B^* \end{pmatrix}(s_{21}, \ldots, s_{2d})^* = 0,\end{aligned} \tag{2.18}$$

which implies $[\mathbf{L}:\mathbf{L}] = 0$, that is, $\mathbf{L}$ is a Lagrangian subspace in $\mathbf{S}$.

With the theory of matrices and (i), there exist matrices $\widetilde{A}_{([\frac{nm}{2}]-k)\times nm}$, $\widetilde{B}_{(d-[\frac{nm+1}{2}]-k)\times(2d-nm)}$, $C_{(2k+[\frac{nm+1}{2}]-[\frac{nm}{2}])\times nm}$, $D_{(2k+[\frac{nm+1}{2}]-[\frac{nm}{2}])\times(2d-nm)}$ satisfying rank $\widetilde{A} = [\frac{nm}{2}]-k$, rank $\widetilde{B} = d-[\frac{nm+1}{2}]-k$, rank C = rank $D = 2k+[\frac{nm+1}{2}]-[\frac{nm}{2}]$, rank $\begin{pmatrix} \widetilde{A} \\ C \end{pmatrix} = [\frac{nm+1}{2}]+k$, rank $\begin{pmatrix} D \\ \widetilde{B} \end{pmatrix} = d - [\frac{nm}{2}] + k$, such that $(A|B)$ is equivalent to

$$\begin{pmatrix} \widetilde{A}_{([\frac{nm}{2}]-k)\times nm} & 0_{([\frac{nm}{2}]-k)\times(2d-nm)} \\ C_{(2k+[\frac{nm+1}{2}]-[\frac{nm}{2}])\times nm} & D_{(2k+[\frac{nm+1}{2}]-[\frac{nm}{2}])\times(2d-nm)} \\ 0_{(d-[\frac{nm+1}{2}]-k)\times nm} & \widetilde{B}_{(d-[\frac{nm+1}{2}]-k)\times(2d-nm)} \end{pmatrix}. \tag{2.19}$$

From (2.19), we see that rank$(A|B) = d$ which implies dim $\mathbf{L} = d$, thus by Lemma 4, we see that $\mathbf{L}$ is a complete Lagrangian subspace in $\mathbf{S}$. Next we give the fact that $\mathbf{L}$ is k-grade.

By (2.15) and (2.19), we see that there only exist $[\frac{nm}{2}]-k$ linearly independent vectors f_r $(1 \le r \le [\frac{nm}{2}]-k)$ in $\mathbf{L}$ such that $[f_r, \theta_i]_n(\infty) = 0$ $(1 \le r \le [\frac{nm}{2}]-k; i=1,2,\ldots,2d-nm)$, that is, f_r $(1 \le r \le [\frac{nm}{2}]-k) \in \mathbf{S}_-$, which implies

$$\dim \mathbf{L} \cap \mathbf{S}_- = [\frac{nm}{2}] - k. \tag{2.20}$$

Similarly, there only exist $d-[\frac{nm+1}{2}]-k$ linearly independent vectors g_s $(1\le s\le d-[\frac{nm+1}{2}]-k)$ in $\mathbf{L}$ such that $g_s^{[i]}(a)=0$ $(1\le s\le d-[\frac{nm+1}{2}]-k; i=0,1,2,\dots,n-1)$, that is, $g_s(1\le s\le d-[\frac{nm+1}{2}]-k)\in\mathbf{S}_+$, which implies

$$\dim \mathbf{L}\cap\mathbf{S}_+ = d-[\frac{nm+1}{2}]-k. \tag{2.21}$$

Together with Definition 5, (2.20) and (2.21), we obtain

$$k=d-[\frac{nm+1}{2}]-\ \dim \mathbf{L}\cap\mathbf{S}_+ = [\frac{nm}{2}]-\ \dim\mathbf{L}\cap\mathbf{S}_-,$$

thus $\mathbf{L}$ is k-grade.

(Sufficiency) Since $\mathbf{L}$ is a k-grade complete Lagrangian subspace in $\mathbf{S}$,

$$\dim\mathbf{L}=d, \dim\mathbf{L}\cap\mathbf{S}_-=[\frac{nm}{2}]-k, \dim\mathbf{L}\cap\mathbf{S}_+=d-[\frac{nm+1}{2}]-k \text{ and } [\mathbf{L}:\mathbf{L}]=0. \tag{2.22}$$

Since $\mathbf{S}=\operatorname{span}\{e^1,\dots,e^{nm},f^1,\dots,f^{2d-nm}\}$ and $\dim\mathbf{L}=d$, there exist $a_{ij},b_{it}\in\mathbb{C}$ $(i=1,2,\dots,d; j=1,2,\dots,nm; t=1,2,\dots,2d-nm)$, such that

$$\begin{aligned}\mathbf{L}\ &=\operatorname{span}\{a_{11}e^1+a_{12}e^2+\cdots+a_{1,nm}e^{nm}+b_{11}f^1+b_{12}f^2+\cdots+b_{1,2d-nm}f^{2d-nm},\\ &\dots,a_{d1}e^1+a_{d2}e^2+\cdots+a_{d,nm}e^{nm}+b_{d1}f^1+b_{d2}f^2+\cdots+b_{d,2d-nm}f^{2d-nm}\},\end{aligned} \tag{2.23}$$

by (2.23) and $[\mathbf{L}:\mathbf{L}]=0$, it is verified that (ii) is true.

By (2.22) and (2.23), we see that (i) is true. This completes the proof. □

Corollary 1 $\mathbf{L}$ *is a k-grade $(0\le k\le d-[\frac{nm+1}{2}])$ complete Lagrangian subspace in S if and only if there exist $a_{ij},b_{it}\in\mathbb{C}$ $(i=1,2,\dots,d; j=1,2,\dots,nm; t=1,2,\dots,2d-nm)$, such that*

$$\begin{aligned}\mathbf{L}\ &=\{\mathbf{f}\in\mathbf{S}|\exists s_i(i=1,2,\dots,d)\in\mathbb{C},(f(a)^t,f^{[1]}(a)^t,\dots,f^{[n-1]}(a)^t)^t=\\ &A^t(s_1,s_2,\dots,s_d)^t,\ ([f,\theta_1]_n(\infty),\dots,[f,\theta_{2d-nm}]_n(\infty))^t=B^t(s_1,\dots,s_d)^t\},\end{aligned}$$

and (i) rank $A=[\frac{nm+1}{2}]+k$, rank $B=d-[\frac{nm}{2}]+k$, where $A=(a_{ij})_{d\times nm}$ and $B=(b_{it})_{d\times(2d-nm)}$;
(ii) $\alpha_iH\alpha_j^*=0$ $(1\le i,j\le d)$, where $\alpha_i=(a_{i1},\dots,a_{i,nm},b_{i1},b_{i2},\dots,\ b_{i,2d-nm})$ and H defined in Theorem 1.

Proof (Sufficiency) For all $\mathbf{f}\in\mathbf{L}$, by Theorem 4, there exist s_i $(i=1,2,\dots,d)\in\mathbb{C}$, such that

$$\begin{aligned}\mathbf{f}\ &=\textstyle\sum_{i=1}^d s_i(a_{i1}e^1+\cdots+a_{i,nm}e^{nm}+b_{i1}f^1+\cdots+b_{i,2d-nm}f^{2d-nm})\\ &=(\textstyle\sum_{i=1}^d s_ia_{i1})e^1+\cdots+(\sum_{i=1}^d s_ia_{i,nm})e^{nm}+(\sum_{i=1}^d s_ib_{i1})f^1+\cdots\\ &\quad+(\textstyle\sum_{i=1}^d s_ib_{i,2d-nm})f^{2d-nm}.\end{aligned} \tag{2.24}$$

By (2.6), we obtain

$$\begin{gathered}\textstyle\sum_{i=1}^d s_ia_{i1}=f_1(a),\dots,\sum_{i=1}^d s_ia_{i,m}=f_m(a),\dots,\\ \textstyle\sum_{i=1}^d s_ia_{i,nm-m+1}=f_1^{[n-1]}(a),\dots,\sum_{i=1}^d s_ia_{i,nm}=f_m^{[n-1]}(a);\\ \textstyle\sum_{i=1}^d s_ib_{i1}=[f,\theta_1]_n(\infty),\dots,\sum_{i=1}^d s_ib_{i,2d-nm}=[f,\theta_{2d-nm}]_n(\infty),\end{gathered} \tag{2.25}$$

that is,

$$\begin{array}{l}(f(a)^t, f^{[1]}(a)^t, \ldots, f^{[n-1]}(a)^t)^t = A^t(s_1, s_2, \ldots, s_d)^t, \\ ([f,\theta_1]_n(\infty), \ldots, [f,\theta_{2d-nm}]_n(\infty))^t = B^t(s_1, \ldots, s_d)^t.\end{array} \tag{2.26}$$

Obviously, (i) and (ii) hold.

(Necessity) For arbitrary $\mathbf{f} \in \mathbf{L}$, equation (2.26) implies that (2.25) holds. By (2.6), we see that (2.24) is true. From Theorem 4, we get that $\mathbf{L}$ is a k-grade complete Lagrangian subspace in $\mathbf{S}$. This completes the proof of Corollary. □

4. The case with two singular endpoints

Theorem 4 can be generalized to the case when $l(y)$ is singular at the endpoint a. For this we need Kodaira's deficiency index formula for vector-valued symmetric differential operators.

Let T_0 be the minimal operator associated with $l(y)$ and $\mathcal{D}(T_0)$ is the domain of T_0. Choose c to be a fixed point between a and ∞, and write T_0^- and T_0^+ as the minimal operators generated by $l(y)$ in $L_r^2(a,c]$ and $L_r^2[c,\infty)$, respectively; $\mathcal{D}(T_0^-)$ and $\mathcal{D}(T_0^+)$ are the domains associated with them. We use T_1, T_1^- and T_1^+ to denote the maximal operators generated in $L_r^2(a,\infty), L_r^2(a,c]$ and $L_r^2[c,\infty)$ by $l(y)$, respectively; $\mathcal{D}(T_1), \mathcal{D}(T_1^-), \mathcal{D}(T_1^+)$ are the domains associated with them. Denote the deficiency indices of T_0^- and T_0^+ as (d_1^+, d_1^-) and (d_2^+, d_2^-), respectively, then we see from Theorem 4.3 of [16] that

$$[\frac{nm+1}{2}] \le d_i^+, d_i^- \le nm \ (i = 1, 2). \tag{3.1}$$

Letting (d^+, d^-) be the deficiency index of T_0, we have the following Kodaira's formula.

Lemma 7 *(Kodaira's formula, Theorem 4.2 of [16])*

$$d^+ = d_1^+ + d_2^+ - nm, d^- = d_1^- + d_2^- - nm.$$

According to the definition of deficiency index, equations $l(y) = iy$ and $l(y) = -iy$ have d_1^+ linearly independent solutions $\phi_1, \ldots, \phi_{d_1^+}$ in $L_r^2(a,c]$ and d_1^- linearly independent solutions $\phi_{d_1^+ + 1}, \ldots, \phi_{d_1^+ + d_1^-}$ in $L_r^2(a,c]$, respectively.

Similarly, equations $l(y) = iy$ and $l(y) = -iy$ have d_2^+ linearly independent solutions $\psi_1, \ldots, \psi_{d_2^+}$ in $L_r^2[c,\infty)$ and d_2^- linearly independent solutions $\psi_{d_2^+ + 1}, \ldots, \psi_{d_2^+ + d_2^-}$ in $L_r^2[c,\infty)$, respectively.

Denote

$$N_1 \triangleq d_1^+ + d_1^- - nm, \ N_2 \triangleq d_2^+ + d_2^- - nm. \tag{3.2}$$

By Lemma 1, there exist ϕ_i $(i = 1, 2, \ldots, N_1)$ and ψ_i $(i = 1, 2, \ldots, N_2)$ as above satisfying

$$\text{rank } J^- = N_1, \quad \text{rank } J^+ = N_2, \tag{3.3}$$

where $J^- = ([\phi_r, \phi_s]_n(a))_{1 \le r,s \le N_1} = \text{diag}\{-iE_{q_1}, iE_{p_1}\}$ $(p_1 + q_1 = N_1, p_1 \ge 0, q_1 \ge 0)$,
$J^+ = ([\psi_r, \psi_s]_n(\infty))_{1 \le r,s \le N_2} = \text{diag}\{-iE_{q_2}, iE_{p_2}\}$ $(p_2 + q_2 = N_2, p_2 \ge 0, q_2 \ge 0)$ and $(J^-)^* = -J^-$, $(J^+)^* = -J^+$.

Assume $d_1^+ + d_2^+ - nm = d_1^- + d_2^- - nm = d$, i.e., $d^+ = d^- = d$, it is well known from the general operator theory that the minimal operator associated with $l(y)$ can be extended to a self-adjoint differential operator in $L_r^2(a,\infty)$. Similar to the case with a finite regular endpoint a in Section 2, we obtain following results.

Lemma 8 *The complex vector space*

$$\mathbf{S} = \mathcal{D}(T_1)/\mathcal{D}(T_0),$$

with the skew-Hermitian from $[\cdot:\cdot]$*, as in (1.11), is a complex symplectic space and dim* $\mathbf{S} = 2d$.

Lemma 9 *Suppose the linearly subspace of* $\mathbf{S}$

$$\mathbf{S}_- = \{y \in \mathbf{S} \mid [y,\psi_i]_n(\infty) = 0, i = 1,2,\ldots,N_2\},$$

$$\mathbf{S}_+ = \{y \in \mathbf{S} \mid [y,\phi_i]_n(a) = 0, i = 1,2,\ldots,N_1\},$$

then $\mathbf{S} = \mathbf{S}_- \oplus \mathbf{S}_+$*, and dim* $\mathbf{S}_- = N_1$*, dim* $\mathbf{S}_+ = N_2$.

Lemma 10 (Balanced intersection principle) *For each complete Lagrangian space* $\mathbf{L}$ *in* $\mathbf{S}$*, then*

$$0 \le \left[\frac{N_2}{2}\right] - dim\ L \cap S_+ = \left[\frac{N_1}{2}\right] - dim\ L \cap S_- \le \min\left\{\left[\frac{N_2}{2}\right], \left[\frac{N_1}{2}\right]\right\} \triangleq \nu.$$

Lemma 11 *Let* $def(T_0^-) = (d_1^+, d_1^-)$, $def(T_0^+) = (d_2^+, d_2^-)$, $N_1 = d_1^+ + d_1^- - nm$, $N_2 = d_2^+ + d_2^- - nm$. *For all* $\alpha_1, \alpha_2, \ldots, \alpha_{N_1}$, $\beta_1, \beta_2, \ldots, \beta_{N_2} \in \mathbb{C}$, *there exists* $y \in \mathcal{D}(T_1)$, *such that*

$$[y,\phi_r]_n(a) = \alpha_r\ (r = 1,2,\ldots,N_1), [y,\psi_s]_n(\infty) = \beta_s\ (s = 1,2,\ldots,N_2), \tag{3.4}$$

where ϕ_r, ψ_s *is defined as above.*

Proof By von Neumann's decomposition in [10], for all $y \in \mathcal{D}(T_1)$, y has unique representation

$$y = \begin{cases} y_0 + \sum_{i=1}^m \sum_{j=1}^n d_{ij}\chi_{ij} + \sum_{k=1}^{N_1} c_k\phi_k & (y_0 \in \mathcal{D}(T_0^-), x \in (a,c]), \\ y_0' + \sum_{i=1}^m \sum_{j=1}^n d_{ij}'\chi_{ij}' + \sum_{k=1}^{N_2} c_s'\Psi_s & (y_0' \in \mathcal{D}(T_0^+), x \in [c,\infty)), \end{cases}$$

where $\chi_{ij} \in \mathcal{D}(T_1^-)$ satisfy

$$\chi_{ij}^{[k-1]}(a) = \begin{cases} 0_{m\times 1} & \text{for } j \ne k; \\ e_i & \text{for } j = k, \end{cases} \ \chi_{ij}(t) = 0 \text{ for all } t \le a-1\ (1 \le i \le m;\ 1 \le j \le n)$$

and $\chi_{ij}' \in \mathcal{D}(T_1^+)$ satisfy

$$\chi_{ij}^{[k-1]}(a)' = \begin{cases} 0_{m\times 1} & \text{for } j \ne k; \\ e_i & \text{for } j = k, \end{cases} \ \chi_{ij}(t) = 0 \text{ for all } t \ge a+1\ (1 \le i \le m;\ 1 \le j \le n),$$

and $d_{ij}, d_{ij}', c_1, \ldots, c_{N_1}, c_1', \ldots, c_{N_2}' \in \mathbb{C}$. Choose

$$\begin{pmatrix} c_1 \\ \vdots \\ c_{N_1} \end{pmatrix} = -(J^-)^t \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{N_1} \end{pmatrix}, \begin{pmatrix} c_1' \\ \vdots \\ c_{N_2}' \end{pmatrix} = -(J^+)^t \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{N_2} \end{pmatrix}.$$

Then $y \in \mathcal{D}(T_1)$, and it is easy to verify that y satisfies (3.4), and so the results follow. □

From Lemma 8, we see that dim $\mathbf{S} = 2d$, so the complex symplectic space $\mathbf{S}$ is linearly isomorphic to $\mathbb{C}^{2d}$. We can use the customary unit basis vectors in $\mathbb{C}^{2d}$,

$$e^1 = (1, 0, \ldots, 0)^t, e^2 = (0, 1, 0, \ldots, 0)^t, \ldots, e^{N_1} = (\overbrace{0, \ldots, 0, 1}^{N_1}, 0, \ldots, 0)^t,$$
$$f^1 = (\overbrace{0, \ldots, 0, 1}^{N_1+1}, 0, \ldots, 0)^t, f^2 = (\overbrace{0, \ldots, 0, 1}^{N_1+2}, 0, \ldots, 0)^t, f^{N_2} = (0, \ldots, 0, 1)^t,$$

so

$$\mathbf{S} = \text{span}\{e^1, e^2, \ldots, e^{N_1}, f^1, f^2, \ldots, f^{N_2}\}.$$

From Lemma 2, we can introduce corresponding coordinates in $\mathbf{S}$ by the convenient choice

$$\begin{aligned}\mathbf{f} &= ([f, \phi_1]_n(a), \ldots, [f, \phi_{N_1}]_n(a), [f, \psi_1]_n(\infty), \ldots, [f, \psi_{N_2}]_n(\infty)) \\ &= [f, \phi_1]_n(a)e^1 + \cdots + [f, \phi_{N_1}]_n(a)e^{N_1} + [f, \psi_1]_n(\infty)f^1 \\ &\quad + [f, \psi_{N_2}]_n(\infty)f^{N_2},\end{aligned} \tag{3.5}$$

where $\mathbf{f} = \{f + \mathcal{D}(T_0)\}$, for $f \in \mathcal{D}(T_1)$.

In terms of these coordinates, the symplectic form $[\cdot : \cdot]$ in $\mathbf{S}$ can be expressed as the following form.

Theorem 5 *For* $\mathbf{f}, \mathbf{g} \in \mathbf{S}$*, we have* $[\mathbf{f} : \mathbf{g}] = \mathbf{f} H' \mathbf{g}^*$*, where*

$$H' = \begin{pmatrix} -J^- & 0_{N_1 \times N_2} \\ 0_{N_2 \times N_1} & J^+ \end{pmatrix}_{2d \times 2d},$$

J^- *and* J^+ *defined in (3.3).*

Theorem 6 *A complete Lagrangian subspace in* $\mathbf{S}$ *is* 0*-grade, or* 1*-grade,...,or* ν*-grade, where* $\nu = \min\{[\frac{N_1}{2}], [\frac{N_2}{2}]\}$.

Theorem 7 *For* $\mathbf{S}_-$ *and* $\mathbf{S}_+$ *defined in Lemma 9, we have*

$$\mathbf{S}_- = span\{e^1, e^2, \ldots, e^{N_1}\}, \mathbf{S}_+ = span\{f^1, f^2, \ldots, f^{N_2}\}.$$

Theorem 8 $\mathbf{L}$ *is a* k*-grade* $(0 \le k \le \nu)$ *complete Lagrangian subspace in* $\mathbf{S}$ *if and only if there exist* $a'_{ij}, b'_{is} \in \mathbb{C}(i = 1, 2, \ldots, d; j = 1, 2, \ldots, N_1; s = 1, 2, \ldots, N_2)$*, such that*

$$\begin{aligned}\mathbf{L} &= span\{a'_{11}e^1 + \cdots + a'_{1,N_1}e^{N_1} + b'_{11}f^1 + \cdots + b'_{1,N_2}f^{N_2}, \ldots, \\ &\quad a'_{d,1}e^1 + \cdots + a'_{d,N_1}e^{N_1} + b'_{d,1}f^1 + \cdots + b'_{d,N_2}f^{N_2}\},\end{aligned}$$

and
(i) rank $A' = d - [\frac{N_2}{2}] + k$*, rank* $B' = d - [\frac{N_1}{2}] + k$*;*
(ii) $\alpha_i H' \alpha_j^* = 0$ $(1 \le i, j \le d)$*, where* $\alpha_i = (a'_{i1}, \ldots, a'_{i,N_1}, b'_{i1}, \ldots, b'_{i,N_2})$ $(1 \le i \le d)$*, and* H' *defined in Theorem 5 and* $A' = (a'_{ij})_{d \times N_1}, B' = (b'_{is})_{d \times N_2}$.

Corollary 2 $\mathbf{L}$ *is a k-grade ($0 \le k \le \nu$) complete Lagrangian subspace in* $\mathbf{S}$ *if and only if there exist* $a'_{ij}, b'_{it} \in \mathbb{C}$ *($i = 1, 2, \ldots, d; j = 1, 2, \ldots, N_1; t = 1, 2, \ldots, N_2$), such that*

$$\begin{aligned}\mathbf{L} \;&= \{f \in S | \exists s_i (i = 1, 2, \ldots, d) \in \mathbb{C}, ([f, \phi_1]_n(a), \ldots, [f, \phi_{N_1}]_n(a))^t = \\ &A'^t(s_1, s_2, \ldots, s_d)^t, ([f, \psi_1]_n(\infty), \ldots, [f, \psi_{N_2}]_n(\infty))^t = B^t(s_1, \ldots, s_d)^t\},\end{aligned}$$

and
(i) rank $A' = d - [\frac{N_2}{2}] + k$*, rank* $B' = d - [\frac{N_1}{2}] + k$*;*
(ii) $\alpha_i H' \alpha_j^* = 0 (1 \le i, j \le d)$*, where* $\alpha_i = (a'_{i1}, \ldots, a'_{i,N_1}, b'_{i1}, \ldots, b'_{i,N_2})$ $(1 \le i \le d)$*, and* H' *defined in Theorem 5 and* $A' = (a'_{ij})_{d \times N_1}, B' = (b'_{is})_{d \times N_2}$.

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