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On the Minkowski measurability of self-similar fractals in $\mathbb{R}^d$

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Abstract: The question of Minkowski measurability of fractals is investigated for different situations by various authors, notably by M. Lapidus. In dimension one it is known that the attractor of an IFS consisting of similitudes (and satisfying a certain open set condition) is Minkowski measurable if and only if the IFS is of non-lattice type and it was conjectured that this would be true also in higher dimensions. Half of this conjecture was proved by Gatzouras in 2000, who showed that the attractor of an IFS (satisfying the open set condition) is Minkowski measurable if the IFS is of non-lattice type.

M. Lapidus and E. Pearse give in their recent work in 2010 a sketch of proof of this conjecture. We give in this work, under certain conditions needed for the application of the Lapidus-Pearse theory, a complete detailed proof of this conjecture, filling in the gaps and resolving the difficulties appearing in their sketch of proof. We also give an alternative proof of Gatzouras' theorem under the same restrictions and give an explicit formula for the Minkowski content.

Key words: Self-similar fractals, Minkowski measurability, tube formulas

1. Introduction

Let

$$F = \bigcup_{j=1}^{J} \varphi_j(F) =: \Phi(F) \subset \mathbb{R}^d$$

be a self-similar fractal, where $\varphi_j : \mathbb{R}^d \to \mathbb{R}^d$ are similitudes with scaling ratios $0 < r_j < 1$, $j = 1, 2, \ldots, J$, for $J \geq 2$. We assume the iterated function system (IFS) Φ to satisfy the open set condition, so that the Minkowski dimension D of F is given by the unique real root of the Moran equation $\sum_{j=1}^{J} r_j^D = 1$.

Let $F_\varepsilon = \{x \in \mathbb{R}^d \,|\, \mathrm{dist}(x, F) \leq \varepsilon\}$ and $V_F(\varepsilon)$ be the d-dimensional volume of F_ε. F is called Minkowski measurable if the limit

$$\mathcal{M}(F) := \lim_{\varepsilon \to 0^+} V_F(\varepsilon)\, \varepsilon^{D-d}$$

exists, is finite, and is different from zero. $\mathcal{M}(F)$ is then called the Minkowski content of F.

The IFS Φ is called of lattice type if the additive subgroup $\sum_{j=1}^{J} (\log r_j)\mathbb{Z}$ of $\mathbb{R}$ is discrete and otherwise (i.e. if this subgroup is dense in $\mathbb{R}$) of non-lattice type (see [9]). In the lattice case there is an r with

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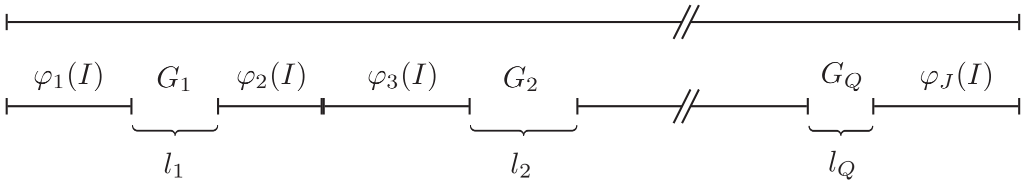

Figure 1. The convex hull $[F]=I$ and its first iteration.

$\log r_j = k_j \log r$, $k_j \in \mathbb{Z}^+$. This dichotomy is decisive for Minkowski measurability of fractals and it is known that in *dimension one* a self-similar fractal satisfying a certain open set condition is Minkowski measurable if and only if it is of non-lattice type (see [5], [7], [3], [8], and [9]). We now briefly recall the one-dimensional situation:

Let $d=1$ and I denote the convex hull of F, $I=[F]$. Assume that $D<1$ and the open set condition is satisfied for $\mathrm{int}(I)$. Then $\varphi_j(I)$ and $\varphi_k(I)$ are disjoint for $j \neq k$, except possibly at the endpoints. There will emerge $Q \leq J-1$ gaps on I, with lengths l_q, $q=1,2,\ldots,Q$ (see Figure 1). Then F is Minkowski measurable if and only if the IFS Φ is of non-lattice type, and in the case it is measurable the content is given by

$$\mathcal{M} = \frac{2^{1-D}\sum_{q=1}^{Q} l_q^D}{D(1-D)\sum_{j=1}^{J} r_j^D \log r_j^{-1}} \qquad \text{(see [9, p.262]).} \tag{1}$$

In the lattice case the fractal is not Minkowski measurable, but one can define an average Minkowski content by the formula

$$\mathcal{M}_{\mathrm{av}} = \lim_{T\to\infty} \frac{1}{\log T}\int_{1/T}^{1} \varepsilon^{-(1-D)} V_F(\varepsilon)\frac{d\varepsilon}{\varepsilon} \qquad \text{(see [9, p.257])}$$

and the formula (1) gives in this case the average Minkowski content.

Lapidus [5] conjectured that in higher dimensions the same relationship between Minkowski measurability and lattice dichotomy holds, that is, a self-similar fractal in $\mathbb{R}^d$ is Minkowski measurable if and only if it is of non-lattice type. Gatzouras proved in [4] that a non-lattice self-similar fractal in $\mathbb{R}^d$ (satisfying the open set condition) is Minkowski measurable. For the lattice case he conjectured independently of [5] that the Minkowski content does not exist. His method relied on renewal theory and Lapidus-van Frankenhuijsen [9, Remark 12.19] remarked that renewal theory was unlikely to solve this conjecture, but their approach by tube formulas, when extended to higher dimensions, would settle this issue. This extension is accomplished in [6], where a detailed exposition of the beautiful Lapidus–Pearse theory is given. In that paper the authors also give a sketch of proof of the above conjecture ([6, Corollary 8.5]), but they later realize that their "proof" is unsatisfactory, see [6, Remark 10.6 (Note added in proof)].

We give in this work, under certain conditions (see Section 2), a complete detailed proof of this conjecture, filling in the gaps and resolving the difficulties appearing in the above-mentioned sketch. Our main result (Theorem 2) contains also an alternative proof of the result of Gatzouras (under the same restrictions) via the Lapidus–Pearse theory, which yields a very explicit formula for the Minkowski content.

2. Preliminaries

In this section we recall the basics of the Lapidus–Pearse theory [6].

Let $C = [F]$ be the convex hull of F, for which we assume $\dim C = d$. Adopting the approach of Pearse and Winter [11] we want to put some additional conditions on the IFS Φ:

TSC (Tileset Condition): Φ satisfies the open set condition with $\operatorname{int} C$ as a feasible open set.

NTC (Nontriviality Condition): $\operatorname{int} C \nsubseteq \Phi(C) = \bigcup_{j=1}^{J} \varphi_j(C)$.

Now define $T_0 = \operatorname{int}(C) \setminus \Phi(C)$ and its iterates $T_n = \Phi^n(T_0)$, $n = 1, 2, 3, \ldots$ (see [10]). The tiling of the self-similar system is given by

$$\mathcal{T} := \{T_n\}_{n=0}^{\infty}.$$

Let $V_{T_n}^-(\varepsilon)$ denote the volume of the inner ε-neighborhood of T_n (i.e. $\{x \in \overline{T_n} \mid \operatorname{dist}(x, T_n^c) \leq \varepsilon\}$) and $V_{\mathcal{T}}^-(\varepsilon) := \sum_{n=0}^{\infty} V_{T_n}^-(\varepsilon)$.

Pearse and Winter prove in [11] the following implication: If the above conditions TSC and NTC hold, then the property $\partial C \subset F$ implies $V_F(\varepsilon) = V_{\mathcal{T}}^-(\varepsilon) + V_C(\varepsilon) - V_C(0)$. This is extremely important, because there are formulas available for $V_{\mathcal{T}}^-(\varepsilon)$ (see below, Theorem 1) and this relationship enables one to compute the true volume of the ε-neighborhood of the fractal. We will call this condition the Pearse–Winter condition:

PWC (Pearse–Winter Condition): $\partial C \subset F$.

To state the tube formula we need some additional assumptions and definitions. Assume that T_0 is the union of finitely many (connected) components, $T_0 = G_1 \cup G_2 \cup \cdots \cup G_Q$, called the generators of the tiling. We assume the generators to be monophase in the following sense of Lapidus and Pearse [6]: A bounded, open set $G \subset \mathbb{R}^d$ is called monophase if the volume $V_G^-(\varepsilon)$ of the inner ε-neighborhood of G admits an expression of the form

$$V_G^-(\varepsilon) = \textstyle\sum_{m=0}^{d-1} \kappa_m(G)\varepsilon^{d-m}, \qquad \text{for } \varepsilon < g, \tag{2}$$

where g denotes the inradius of G, i.e. supremum of the radii of the balls contained in G. For $\varepsilon \geq g$ we have $V_G^-(\varepsilon) = \text{volume}(G)$, which is denoted by $-\kappa_d(G)$, the negative sign being conventional [6].

Lapidus and Pearse introduce the following "scaling ζ-function":

Definition 1 *The scaling ζ-function of the self-similar fractal is defined by*

$$\zeta(s) = \sum_{n=0}^{\infty} \sum_{w \in W_n} r_w^s,$$

where W_n is the set of words $w = w_1 w_2 \cdots w_n$ of length n (with letters from $\{1, 2, \ldots, J\}$) and $r_w = r_{w_1} r_{w_2} \ldots r_{w_n}$.

The above series can be shown to converge absolutely to an analytic function for $\operatorname{Re}(s) > D$. A simple calculation shows that $\zeta(s)$ can be expressed as [9, Theorem 2.4]

$$\zeta(s) = \frac{1}{1 - \sum_{j=1}^{J} r_j^s} \quad \text{for } \operatorname{Re}(s) > D.$$

The right-hand side of the above equation is the meromorphic extension of $\zeta(s)$ to the whole complex plane. We will denote this extension also by $\zeta(s)$.

Definition 2 *The set* $\mathfrak{D} := \{\omega \in \mathbb{C} \mid \zeta(s)$ *has a pole at* $\omega\} = \{s \mid 1 - \sum_{j=1}^{J} r_j^s = 0\}$ *is called the set of complex dimensions of the self-similar fractal.*

Lapidus and Pearse define a second type of "ζ-function" associated with the tiling and related to the geometry of the monophase generators. We assume for simplicity that there is a single generator G (so that $T_0 = G$).

Definition 3 *The tubular* ζ*-function* $\zeta_{\mathcal{T}}(s,\varepsilon)$ *associated with the generator* G *is defined by*

$$\zeta_{\mathcal{T}}(s,\varepsilon) := \zeta(s)\,\varepsilon^{d-s}\sum_{m=0}^{d}\frac{g^{s-m}}{s-m}\kappa_m(G).$$

We now state the formula of Lapidus and Pearse for $V_{\mathcal{T}}^{-}(\varepsilon)$:

Theorem 1 (Tube formula for tilings of self-similar fractals, [6])

$$V_{\mathcal{T}}^{-}(\varepsilon) = \sum_{\omega\in\mathfrak{D}_{\mathcal{T}}} \operatorname{res}(\zeta_{\mathcal{T}}(s,\varepsilon);\omega),$$

where $\mathfrak{D}_{\mathcal{T}} = \mathfrak{D} \cup \{0,1,\ldots,d-1\}$.

Remark 1 *Lapidus and Pearse give in [6] a distributional proof for this formula. For a pointwise proof see [1].*

3. Main results

Our main result is the following theorem:

Theorem 2 *Let* $F = \Phi(F) = \bigcup_{j=1}^{J}\varphi_j(F)$ *be a self-similar fractal in* $\mathbb{R}^d$ *with* $\dim[F] = d$ *and the contractivity ratios of the similitudes* $\{\varphi_j\}$ *being* $\{r_j\}$.

We assume the tileset condition, the nontriviality condition, and the Pearse–Winter condition to hold (see TSC, NTC, and PWC in the former section) and we assume that the tiling has a single monophase generator G. *We additionally assume* $D > d-1$, *where* D *is the Minkowski dimension of* F.

Under these assumptions the following hold:

I. If the IFS Φ *is of non-lattice type, then* F *is Minkowski measurable with Minkowski content*

$$\begin{aligned}\mathcal{M}(F) = \operatorname{res}\left(\zeta_{\mathcal{T}}(s,\varepsilon)\,\varepsilon^{s-d}; D\right) &= \operatorname{res}\left(\zeta(s); D\right)\sum_{m=0}^{d}\frac{g^{D-m}}{D-m}\kappa_m(G)\\ &= \frac{\sum\limits_{m=0}^{d}\frac{g^{D-m}}{D-m}\kappa_m(G)}{\sum\limits_{j=1}^{J} r_j^D \log r_j^{-1}},\end{aligned}$$

II. If the IFS Φ is of lattice type then F is not Minkowski measurable. The average Minkowski content, which is defined by

$$\mathcal{M}_{\mathrm{av}} = \lim_{T\to\infty} \frac{1}{\log T} \int_{1/T}^{1} \varepsilon^{-(d-D)} V_F(\varepsilon) \frac{d\varepsilon}{\varepsilon},$$

exists and equals

$$\mathcal{M}_{\mathrm{av}}(F) = \operatorname{res}\left(\zeta_{\mathcal{T}}(s,\varepsilon)\, \varepsilon^{s-d}; D\right) = \frac{\sum_{m=0}^{d} \frac{g^{D-m}}{D-m} \kappa_m(G)}{\sum_{j=1}^{J} r_j^D \log r_j^{-1}}.$$

Remark 2 *In the case of multiple generators, one can define a total tubular zeta function by adding the tubular ζ-functions of the components and obtain a similar formula.*

Corollary 1 *If we specialize to the dimension $d = 1$, we obtain the formula (1): Each generator G_q $(q = 1, 2, \ldots, Q)$ is an interval of length l_q and*

$$V_{G_q}^{-}(\varepsilon) = \begin{cases} 2\varepsilon & , \text{ for } \varepsilon < g_q \\ l_q & , \text{ for } \varepsilon \geq g_q, \end{cases}$$

so that $\kappa_0(G_q) = 2$, $\kappa_1(G_q) = -l_q$ and the inradius g_q equals $l_q/2$. Since there are multiple generators we obtain the total tubular ζ-function by adding the tubular ζ-functions of each generator.

$$\zeta_{\mathcal{T}}(s,\varepsilon) = \sum_{q=1}^{Q} \zeta_{\mathcal{T}}^{q}(s,\varepsilon) = \sum_{q=1}^{Q} \zeta(s)\, \varepsilon^{1-s} \left(2\frac{g_q^s}{s} - l_q \frac{g_q^{s-1}}{s-1} \right) = \zeta(s)\, \varepsilon^{1-s} \frac{2^{1-s}}{s(1-s)} \sum_{q=1}^{Q} l_q^s,$$

and

$$\mathcal{M}(F) = \operatorname{res}\left(\zeta_{\mathcal{T}}(s,\varepsilon)\, \varepsilon^{s-1}; D\right) = \operatorname{res}\left(\zeta(s)\, \frac{2^{1-s}}{s(1-s)} \sum_{q=1}^{Q} l_q^s; D \right) = \frac{2^{1-D} \sum_{q=1}^{Q} l_q^D}{D(1-D) \sum_{j=1}^{J} r_j^D \log r_j^{-1}}.$$

Example 1 *Let $\triangle ABC$ be an acute triangle with corresponding sides a, b, and c. Let $\triangle A'B'C'$ be its orthic (pedal) triangle (see Figure 2a). The triangles $\triangle AC'B'$, $\triangle BA'C'$ and $\triangle CB'A'$ are scaled copies of the original triangle $\triangle ABC$ with scaling ratios $\cos A, \cos B$, and $\cos C$ (denoting the angles at the vertices A, B, C again with the same letter). Consider the collection of these maps as an iterated function system $\Phi = \{\varphi_j\}_{j=1}^{3}$ on $\mathbb{R}^2$ as indicated in Figure 2b. The associated self-similar fractal ("orthic fractal") is shown in Figure 2c (see [13]). The Minkowski dimension D is determined by $(\cos A)^D + (\cos B)^D + (\cos C)^D = 1$. This system satisfies the TSC, NTC, and PWC, and has a single generator $G = \triangle A'B'C'$.*

The volume of the inner ε-neighborhood of the generator G is given by

$$V_G(\varepsilon) = \begin{cases} \kappa_1(G)\varepsilon + \kappa_0(G)\varepsilon^2 & , \quad \text{for } \varepsilon \leq g \\ -\kappa_2(G) & , \quad \text{for } \varepsilon \geq g \end{cases}$$

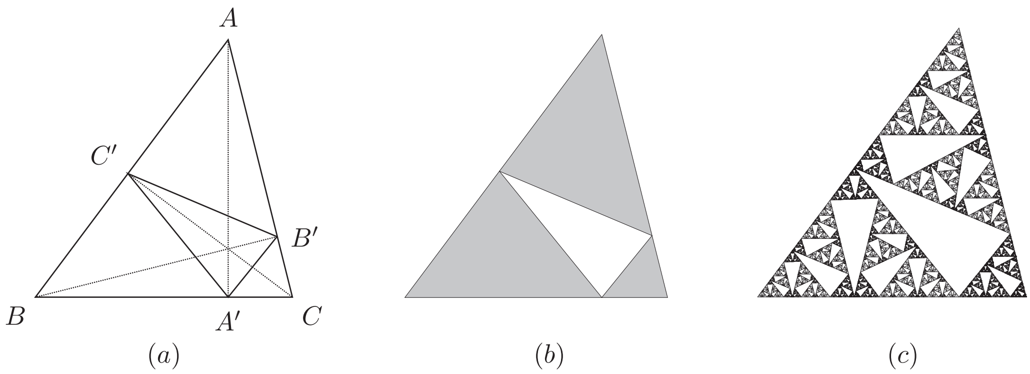

Figure 2. (a) An acute triangle $\triangle ABC$ with its orthic triangle $\triangle A'B'C'$, (b) the IFS, (c) its attractor.

where

$$g = \frac{4\text{Area}(\triangle ABC)\cos A\cos B\cos C}{a\cos A + b\cos B + c\cos C},$$

and

$$\begin{aligned}\kappa_0(G) &= -(\tan A + \tan B + \tan C),\\ \kappa_1(G) &= a\cos A + b\cos B + c\cos C,\\ \kappa_2(G) &= -2\cos A\cos B\cos C\,\text{Area}(\triangle ABC).\end{aligned}$$

Depending on the angles A, B, C, the orthic fractal may be of lattice or non-lattice type. If it is of non-lattice type, then by Theorem 2.I, its Minkowski content exists and is given by

$$-\frac{\left(\frac{g^D}{D}\kappa_0(G) + \frac{g^{D-1}}{D-1}\kappa_1(G) + \frac{g^{D-2}}{D-2}\kappa_2(G)\right)}{(\cos A)^D\log(\cos A) + (\cos B)^D\log(\cos B) + (\cos C)^D\log(\cos C)}.$$

If the orthic fractal is of lattice type then, by Theorem 2.II, the Minkowski content does not exist, but the average content exists and is given by the same expression.

4. Proof of Theorem 2

We consider first the more difficult non-lattice case (because in that case the distribution of the poles of the ζ-function could be utterly complicated). By the assumptions of the theorem we have

$$V_F(\varepsilon) = V_{\mathcal{T}}^{-}(\varepsilon) + V_C(\varepsilon) - V_C(0), \text{ where } C = [F].$$

We have to consider the limit behaviour of $V_F(\varepsilon)\,\varepsilon^{D-d}$ as ε tends to zero.

By the well-known Steiner formula, the volume of the ε-neighborhood of a bounded convex set in $\mathbb{R}^d$ can be expressed as a polynomial in ε [12]:

$$V_C(\varepsilon) = \textstyle\sum_{m=0}^{d} a_m\varepsilon^m \quad \text{ with } a_0 = V_C(0).$$

Hence, $\lim\limits_{\varepsilon\to 0^+}(V_C(\varepsilon) - V_C(0))\,\varepsilon^{D-d} = 0$ (by the assumption $D > d-1$). Thus, our concern will be the term $V_{\mathcal{T}}^{-}(\varepsilon)\,\varepsilon^{D-d}$.

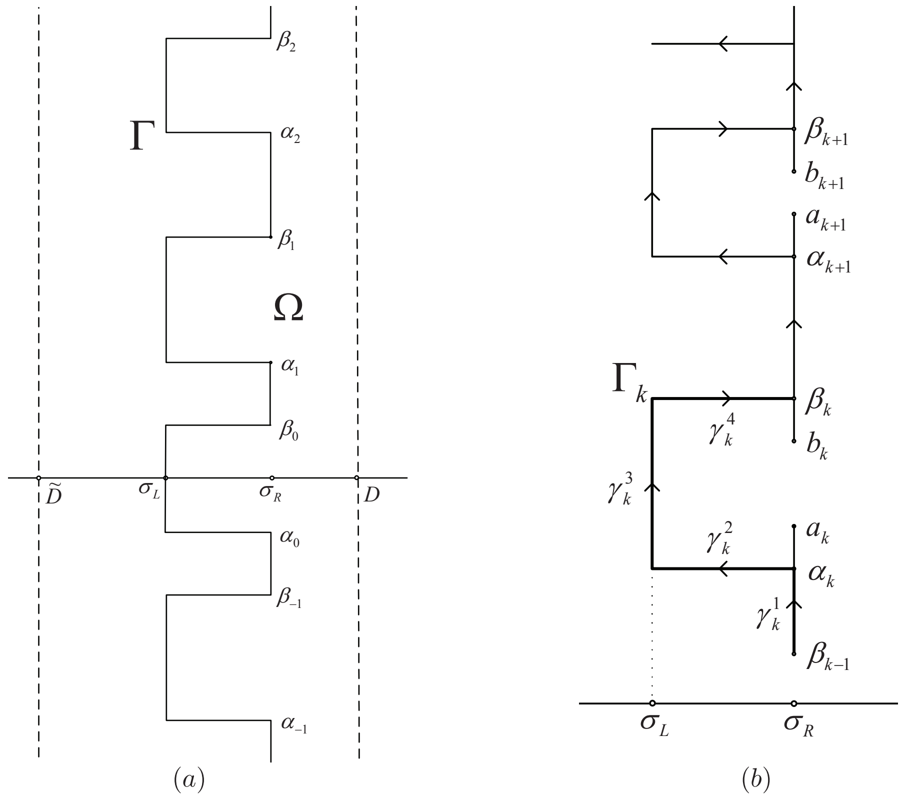

Figure 3. The path Γ.

To make the proof transparent, we will formulate several lemmas, whose proofs we defer to the next section. We first remark that by the nontriviality condition it holds $D < d$ ([11, Cor. 2.13]).

Lemma 3 *For any c satisfying $D < c < d$,*

$$V_{\mathcal{T}}^{-}(\varepsilon) = \frac{1}{2\pi \mathbf{i}} \int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty} \zeta_{\mathcal{T}}(s, \varepsilon) ds.$$

We choose c with $D < c < d$ and fix it throughout the paper. Now we want to convert this integral into an appropriate sum of residues of $\zeta_{\mathcal{T}}(s, \varepsilon)$ plus an integral on a path Γ (see Figure 3a) lying to the left of the line $\mathrm{Re}(s) = D$.

For the construction of this path Γ we need the following 2 lemmas. For convenience we assume that the contractivity ratios are ordered as

$$1 > r_1 \geq r_2 \geq \cdots \geq r_J > 0.$$

Lemma 4 *There exists $\widetilde{D} < D$ such that all the poles of $\zeta(s)$ in the strip $\{s \mid \widetilde{D} < \mathrm{Re}(s) < D\}$ are simple and the absolute values of the residues of $\zeta(s)$ at these poles are bounded by $1/\log r_1^{-1}$.*

Lemma 5 *There exist strictly increasing, real sequences* $\{\alpha_k\}_{k\in\mathbb{Z}}$ *and* $\{\beta_k\}_{k\in\mathbb{Z}}$ *with* $\alpha_k < \beta_k < \alpha_{k+1}$ *for all* k, $\alpha_0 < 0 < \beta_0$ *and*

$$\alpha_{k+1} - \alpha_k > \frac{\pi}{\log r_J^{-1}}, \qquad (k \in \mathbb{Z})$$

and there exist σ_L, σ_R *with* $\max\{\widetilde{D}, d-1\} < \sigma_L < \sigma_R < D$, *such that* $\zeta(s)$ *is uniformly bounded for all* $k \in \mathbb{Z}$ *on the (oriented) segments*

$$\begin{array}{ll} \gamma_k^1 := [\sigma_R + \mathbf{i}\beta_{k-1}, \sigma_R + \mathbf{i}\alpha_k], & \gamma_k^2 := [\sigma_R + \mathbf{i}\alpha_k, \sigma_L + \mathbf{i}\alpha_k], \\ \gamma_k^3 := [\sigma_L + \mathbf{i}\alpha_k, \sigma_L + \mathbf{i}\beta_k], & \gamma_k^4 := [\sigma_L + \mathbf{i}\beta_k, \sigma_R + \mathbf{i}\beta_k]. \end{array} \tag{3}$$

Let Γ_k be the concatenation of the segments γ_k^l, $l = 1,2,3,4$ and Γ be the path obtained by the concatenation of all Γ_k, $k \in \mathbb{Z}$ (see Figure 3b). Let Ω be the open region between Γ and the line $\mathrm{Re}(s) = D$. Then, by Lemma 4 and Lemma 5, there exists $K > 0$ such that

$$|\zeta(s)| \leq K, \text{ for all } s \in \Gamma$$

and

$$|\mathrm{res}(\zeta(s);\omega)| \leq K \text{ for all poles } \omega \in \Omega \text{ of } \zeta.$$

(As there are too many constants in the sequel, we will use the letter K for any of them, though they may differ in the appearing context.)

As $\zeta(s)$ is analytic in $\{s \,|\, \mathrm{Re}(s) > D\}$, all the poles of ζ lie in the half plane $\{s \,|\, \mathrm{Re}(s) \leq D\}$, and, by [9, Thm 2.17], D is the only pole of ζ with real part D. Now, the integral in Lemma 3 can be expressed as follows:

Lemma 6

$$\frac{1}{2\pi\mathbf{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \zeta_{\mathcal{T}}(s,\varepsilon) = \mathrm{res}(\zeta_{\mathcal{T}}(s,\varepsilon); D) + \sum_{\omega\in\Omega\cap\mathfrak{D}} \mathrm{res}(\zeta_{\mathcal{T}}(s,\varepsilon);\omega) + \frac{1}{2\pi\mathbf{i}} \int_{\Gamma} \zeta_{\mathcal{T}}(s,\varepsilon) ds.$$

The integral over Γ on the right-hand side above is absolutely convergent and can be estimated as follows:

Lemma 7 $\displaystyle\int_\Gamma |\zeta_{\mathcal{T}}(s,\varepsilon)||ds| = O(\varepsilon^{d-\sigma_R})$ *as* $\varepsilon \to 0^+$.

This means that we will get rid of this term in the evaluation of the limit $V_{\mathcal{T}}^-(\varepsilon)\varepsilon^{D-d}$ as $\varepsilon \to 0^+$: $O(\varepsilon^{d-\sigma_R})\varepsilon^{D-d} = o(1)$ since $\sigma_R < D$.

We have

$$\lim_{\varepsilon\to 0^+} \mathrm{res}(\zeta_{\mathcal{T}}(s,\varepsilon); D)\varepsilon^{D-d} = \frac{\sum_{m=0}^{d} \frac{g^{D-m}}{D-m}\kappa_m(G)}{\sum_{j=1}^{J} r_j^D \log r_j^{-1}},$$

where the numerator of the right-hand side is different from zero by Remark 4 below and the denominator is obviously non-zero. Therefore, the proof of the first part of Theorem 2 will be settled by the following lemma:

Lemma 8 $\lim_{\varepsilon\to 0^+} \varepsilon^{D-d} \sum_{\omega\in\Omega\cap\mathfrak{D}} \mathrm{res}(\zeta_{\mathcal{T}}(s,\varepsilon);\omega) = 0$.

Now we consider the lattice case. In this case the poles lie periodically on finitely many vertical lines (the rightmost being the line $\mathrm{Re}(s) = D$) and on each line they are separated by $p = 2\pi/\log r$ with $\log r$ being the generator of the group $\sum_{j=1}^{J}(\log r_j)\mathbb{Z}$ ([9, Thm 2.17]). Instead of the complicated Γ, we can use simply a vertical line $\mathrm{Re}(s) = \sigma < D$ (with σ sufficiently close to D so that $\zeta(s)$ has no poles in $\sigma < \mathrm{Re}(s) < D$). Applying the same procedures we can arrive at the formula

$$V_{\mathcal{T}}^-(\varepsilon) = \sum_{n=-\infty}^{\infty} \mathrm{res}(\zeta_{\mathcal{T}}(s,\varepsilon); D + \mathbf{i}np) + \frac{1}{2\pi\mathbf{i}} \int_{\sigma-\mathbf{i}\infty}^{\sigma+\mathbf{i}\infty} \zeta_{\mathcal{T}}(s,\varepsilon) ds.$$

As in Lemma 7, the integral on the right-hand side is $O(\varepsilon^{d-\sigma})$ as $\varepsilon \to 0^+$, so that we can omit this term. The non-real complex dimensions emerging on the line $\mathrm{Re}(s) = D$ will now cause oscillations and prevent the function $\varepsilon^{D-d}V_{\mathcal{T}}^-(\varepsilon)$ from having a limit as $\varepsilon \to 0^+$:

$$\begin{aligned} \varepsilon^{D-d}V_{\mathcal{T}}^-(\varepsilon) &= \frac{1}{\sum\limits_{j=1}^{J} r_j^D \log r_j^{-1}} \sum_{n\in\mathbb{Z}} \varepsilon^{-\mathbf{i}np} \sum_{m=0}^{d} \frac{g^{D+\mathbf{i}np-m}}{D+\mathbf{i}np-m} \kappa_m(G) \\ &=: \frac{1}{\sum\limits_{j=1}^{J} r_j^D \log r_j^{-1}} \sum_{n\in\mathbb{Z}} a_n \varepsilon^{-\mathbf{i}np} = \frac{1}{\sum\limits_{j=1}^{J} r_j^D \log r_j^{-1}} \sum_{n\in\mathbb{Z}} a_n e^{\mathbf{i}npx} \end{aligned}$$

by change of variable $x = -\log\varepsilon$. By Remark 3 below at most $d-1$ of a_n can vanish and by (6) $\sum_{n\in\mathbb{Z}}|a_n| < \infty$, so that the above Fourier series uniformly converges, is non-constant, and oscillates as $x \to \infty$ ($\varepsilon \to 0^+$). Thus a lattice fractal is never Minkowski measurable. However, the average Minkowski content always exists and can be calculated as follows (as $V_C(\varepsilon) - V_C(0)$ does not contribute):

$$\begin{aligned} \mathcal{M}_{\mathrm{av}} &= \lim_{T\to\infty} \frac{1}{\log T} \int_{1/T}^{1} \varepsilon^{D-d} V_F(\varepsilon) \frac{d\varepsilon}{\varepsilon} = \lim_{T\to\infty} \frac{1}{\log T} \int_{1/T}^{1} \varepsilon^{D-d} V_{\mathcal{T}}^-(\varepsilon) \frac{d\varepsilon}{\varepsilon} \\ &= \lim_{T\to\infty} \frac{1}{\log T} \frac{1}{\sum\limits_{j=1}^{J} r_j^D \log r_j^{-1}} \sum_{n\in\mathbb{Z}} \int_0^{\log T} a_n e^{\mathbf{i}npx} dx = \frac{a_0}{\sum\limits_{j=1}^{J} r_j^D \log r_j^{-1}}, \end{aligned}$$

where for the third equality we use the uniform convergence.

5. Proof of Lemmas

Proof [Proof of Lemma 3] Proof of a more general version of this lemma can be found in [2]. For the convenience of the reader we repeat the main steps below, omitting the justification of technical details. We have

$$V_{\mathcal{T}}^-(\varepsilon) = \sum_{n=0}^{\infty} V_{T_n}^-(\varepsilon) = \sum_{n=0}^{\infty} \sum_{w\in W_n} V_{r_w G}^-(\varepsilon),$$

where $r_w G$ is a copy of G scaled by r_w. Recall that $V_G^-(\varepsilon)$ is given as in (2). Since $V_{rG}^-(\varepsilon) = r^d V_G^-(\varepsilon/r)$, we can write

$$V_{rG}^-(\varepsilon) = \begin{cases} \sum_{m=0}^{d-1} \kappa_m(G) r^m \varepsilon^{d-m} & , \text{ for } \varepsilon < rg \\ -r^d \kappa_d(G) & , \text{ for } \varepsilon \geq rg. \end{cases}$$

We calculate the Mellin transform of $V_{\mathcal{T}}^-(\varepsilon)/\varepsilon^d$: The Mellin transform $\mathfrak{M}[f;s]$ of $f:(0,\infty)\to\mathbb{R}$ is given by $\mathfrak{M}[f;s] = \int_0^\infty f(x)x^{s-1}dx$. A routine calculation shows that

$$\mathfrak{M}\left[\frac{V_{rG}^-(\varepsilon)}{\varepsilon^d}; s\right] = r^s \sum_{m=0}^{d} \kappa_m(G) \frac{g^{s-m}}{s-m}, \qquad \text{for } d-1 < \mathrm{Re}(s) < d. \tag{4}$$

We then have for $D < \mathrm{Re}(s) < d$,

$$\begin{aligned} \mathfrak{M}\left[\frac{V_{\mathcal{T}}^-(\varepsilon)}{\varepsilon^d}; s\right] &= \int_0^\infty \sum_{n=0}^{\infty} \sum_{w\in W_n} V_{r_w G}^-(\varepsilon)\, \varepsilon^{s-d-1} d\varepsilon = \sum_{n=0}^{\infty} \sum_{w\in W_n} \int_0^\infty V_{r_w G}^-(\varepsilon)\, \varepsilon^{s-d-1} d\varepsilon \\ &= \sum_{n=0}^{\infty} \sum_{w\in W_n} r_w^s \left(\sum_{m=0}^{d} \kappa_m(G) \frac{g^{s-m}}{s-m} \right) = \zeta(s) \sum_{m=0}^{d} \kappa_m(G) \frac{g^{s-m}}{s-m}, \end{aligned}$$

where the assumption $\mathrm{Re}(s) > D$ is needed for the interchange of the sum and the integral (for details see [2]). Taking the inverse Mellin transform, we obtain

$$\frac{V_{\mathcal{T}}^-(\varepsilon)}{\varepsilon^d} = \frac{1}{2\pi \mathbf{i}} \int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty} \left(\zeta(s) \sum_{m=0}^{d} \kappa_m(G) \frac{g^{s-m}}{s-m} \right) \varepsilon^{-s} ds,$$

for any c satisfying $D < c < d$. Hence the claim is proved. □

In passing, we note the following results, which will be useful later.

Remark 3 *Putting $r=1$ in (4) gives*

$$\int_0^\infty V_G^-(\varepsilon)\, \varepsilon^{s-d-1} d\varepsilon = \sum_{m=0}^{d} \kappa_m(G) \frac{g^{s-m}}{s-m} = g^{s-d} \sum_{m=0}^{d} \kappa_m(G) \frac{g^{d-m}}{s-m} = g^{s-d} \frac{P(s)}{Q(s)}, \tag{5}$$

where $Q(s) = s(s-1)\cdots(s-d)$ is a polynomial of degree $d+1$. An important observation is that the degree of the polynomial $P(s)$ is at most $d-1$. This is a consequence of the continuity of $V_G^-(\varepsilon)$ at $\varepsilon = g$: $\sum_{m=0}^{d-1} \kappa_m(G) g^{d-m} = -\kappa_d(G)$.

Therefore, for any σ_1, σ_2 with $d-1 < \sigma_1 < \sigma_2 < d$, there exists $K > 0$ such that

$$\left| \sum_{m=0}^{d} \kappa_m(G) \frac{g^{s-m}}{s-m} \right| \leq \frac{K}{|s|^2} \qquad \textit{for } \sigma_1 \leq \mathrm{Re}(s) \leq \sigma_2. \tag{6}$$

Remark 4 *Putting $s = D$ in (5) gives*

$$\sum_{m=0}^{d} \kappa_m(G) \frac{g^{D-m}}{D-m} = \int_0^\infty V_G^-(\varepsilon)\, \varepsilon^{D-d-1} d\varepsilon, \tag{7}$$

which shows that $\sum_{m=0}^{d} \kappa_m(G) \frac{g^{D-m}}{D-m}$ *cannot be zero.*

Proof [Proof of Lemma 4] Recall that the contractivity ratios are assumed to be ordered as $1 > r_1 \geq r_2 \geq \cdots \geq r_J > 0$. Recall also that the Minkowski dimension D satisfies the Moran equation: $r_1^D + r_2^D + \cdots + r_J^D = 1$. We define $\widetilde{D}$ to be the unique real solution of the equation $r_1^{\widetilde{D}} + r_2^{\widetilde{D}} + \cdots + r_{J-1}^{\widetilde{D}} = 1$. It is clear that $\widetilde{D} < D$.

Let $f(s) = 1 - (r_1^s + r_2^s + \cdots + r_J^s)$, so that $\zeta(s) = 1/f(s)$. Let $s_0 = \sigma_0 + \mathbf{i}t_0$ be a zero of $f(s)$ in the strip $\{s \,|\, \widetilde{D} < \mathrm{Re}(s) < D\}$. We will first show that $\mathrm{Re}(r_j^{s_0}) \geq 0$, for all $j = 1, 2, \cdots, J$:

Since s_0 is a zero of f, we have $r_1^{s_0} + r_2^{s_0} + \ldots + r_J^{s_0} = 1$. Taking real parts, we obtain

$$\textstyle\sum_{j=1}^{J} \mathrm{Re}(r_j^{s_0}) = 1. \tag{8}$$

Suppose that for some j_0, we have $\mathrm{Re}(r_{j_0}^{s_0}) < 0$. Then,

$$\sum_{j=1}^{J} \mathrm{Re}(r_j^{s_0}) < \sum_{j=1, j\neq j_0}^{J} \mathrm{Re}(r_j^{s_0}) \leq \sum_{j=1, j\neq j_0}^{J} r_j^{\sigma_0} \leq \sum_{j=1}^{J-1} r_j^{\sigma_0} < \sum_{j=1}^{J-1} r_j^{\widetilde{D}} = 1.$$

This contradicts (8).

The nonnegativity of $\mathrm{Re}(r_j^{s_0})$ and (8) implies that

$$\mathrm{Re}(f'(s_0)) = \sum_{j=1}^{J} \log r_j^{-1}\, \mathrm{Re}(r_j^{s_0}) \geq \log r_1^{-1} \sum_{j=1}^{J} \mathrm{Re}(r_j^{s_0}) = \log r_1^{-1}.$$

Thus, the zero of $f(s)$ (and therefore the pole of $\zeta(s)$) at $s = s_0$ is simple. Moreover,

$$|\mathrm{res}(\zeta(s); s_0)| = \left| \frac{1}{f'(s_0)} \right| \leq \left| \frac{1}{\mathrm{Re}(f'(s_0))} \right| \leq \frac{1}{\log r_1^{-1}}.$$

□

Proof [Proof of Lemma 5] With $\widetilde{D}$ as in Lemma 4, choose $\sigma_L < D$ such that $\sigma_L > \max\{\widetilde{D}, d-1\}$. Let $r_1^{\sigma_L} + r_2^{\sigma_L} + \cdots + r_J^{\sigma_L} =: 1 + \lambda$. Let $0 < \psi < \frac{\pi}{2}$ be chosen such that

$$3\,\psi \frac{|\log r_J|}{|\log r_1|} < \sqrt{\frac{\lambda}{1+\lambda}} \tag{9}$$

and let

$$\mu := \frac{\psi^2 r_J^D}{8}. \tag{10}$$

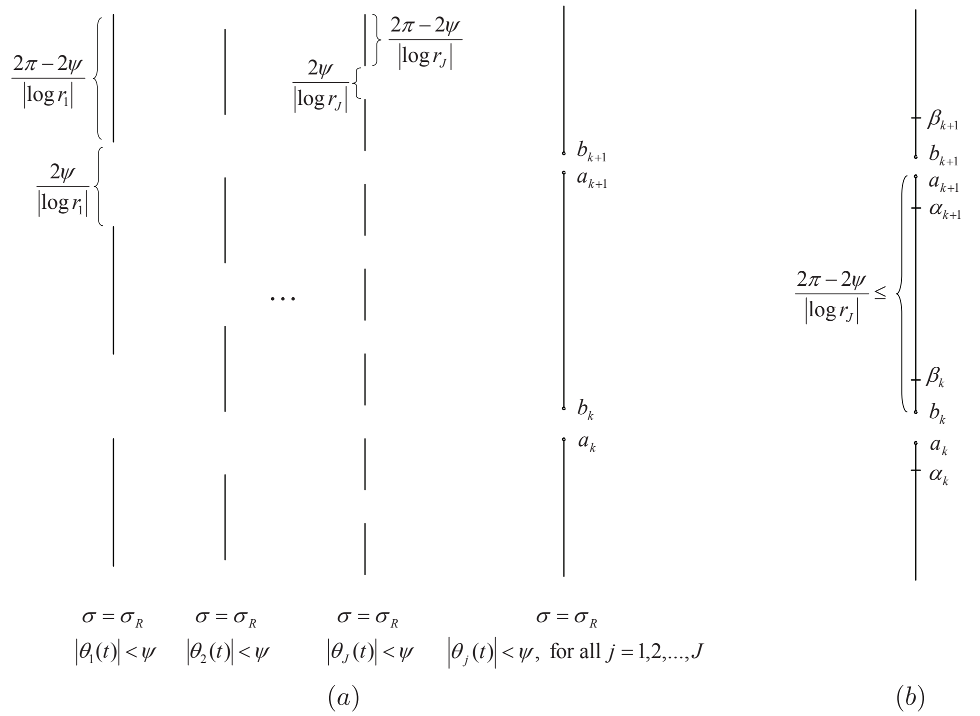

Figure 4. Construction of the sequences $\{a_k\}$, $\{b_k\}$, $\{\alpha_k\}$ and $\{\beta_k\}$.

Then there exists a unique real number σ_R such that $r_1^{\sigma_R} + r_2^{\sigma_R} + \cdots + r_J^{\sigma_R} = 1 + \mu$. Note that

$$\mu < \frac{\psi^2}{8} \leq \frac{\psi^2}{8} \frac{|\log r_J|^2}{|\log r_1|^2} < 9\psi^2 \frac{|\log r_J|^2}{|\log r_1|^2} < \frac{\lambda}{1+\lambda} < \lambda,$$

hence $\sigma_L < \sigma_R$.

For $s = \sigma + \mathbf{i}t$, let $-\pi \leq \theta_j(t) < \pi$ be the angle of r_j^s: $\theta_j(t) \equiv t \log r_j \pmod{2\pi}$.

To construct the sequences $\{\alpha_k\}$ and $\{\beta_k\}$, we first determine the points $s = \sigma_R + \mathbf{i}t$ on the line $\mathrm{Re}(s) = \sigma_R$ for which $|\theta_j(t)| < \psi$ for all $j = 1, 2, ..., J$.

The set $\{t \mid |\theta_j(t)| < \psi \text{ for all } j = 1, 2, \ldots, J\}$ is a union of countably many disjoint open intervals (see Figure 4a). That is

$$\{t \,|\, |\theta_j(t)| < \psi \text{ for all } j = 1, 2, \ldots, J\} = \bigcup_{k \in \mathbb{Z}} I_k =: \bigcup_{k \in \mathbb{Z}} (a_k, b_k),$$

where $a_k < b_k < a_{k+1}$ for $k \in \mathbb{Z}$ and $0 \in (a_0, b_0)$. Clearly, we have $|I_k| = b_k - a_k \leq 2\psi/|\log r_J|$ and $a_{k+1} - b_k \geq (2\pi - 2\psi)/|\log r_J|$.

We define $\alpha_k := a_k - 2\psi/|\log r_1|$ and $\beta_k := b_k + 2\psi/|\log r_1|$ (see Figure 4b). Clearly, $\alpha_k < \beta_k$. The inequality $\beta_k < \alpha_{k+1}$ follows from

$$2\frac{2\psi}{|\log r_1|} < \frac{2\pi - 2\psi}{|\log r_J|}.$$

This inequality is a consequence of the following inequalities (the second one being (9)):

$$\psi\left(\frac{2|\log r_J|}{|\log r_1|}+1\right)<3\,\psi\,\frac{|\log r_J|}{|\log r_1|}<\sqrt{\frac{\lambda}{1+\lambda}}<1<\pi.$$

Moreover, $\alpha_{k+1}-\alpha_k=a_{k+1}-a_k>a_{k+1}-b_k\geq(2\pi-2\psi)/|\log r_J|>\pi/|\log r_J|$.

We will prove that $\zeta(s)$ is uniformly bounded on the (oriented) segments γ_k^l as defined in (3), $(l=1,2,3,4$ and $k\in\mathbb{Z})$. This will follow from the following estimates (recall that $f(s)=1-(r_1^s+r_2^s+\cdots+r_J^s)$ and $\zeta(s)=1/f(s)$):

i) $\operatorname{Re}(f(s))\geq\mu$ for $s\in\gamma_k^1$,

ii) $\operatorname{Im}(f(s))\leq-\sin\psi$ for $s\in\gamma_k^2$,

iii) $\operatorname{Re}(f(s))\leq-\frac{\lambda}{2}$ for $s\in\gamma_k^3$,

iv) $\operatorname{Im}(f(s))\geq\sin\psi$ for $s\in\gamma_k^4$.

We begin with i): For $s=\sigma_R+\mathbf{i}t\in\gamma_k^1$, we have $|\theta_j(t)|\geq\psi$ for at least one $j=j_0$. Using (10) and the inequality $\cos\theta\leq1-\theta^2/4$ for $-\pi/2\leq\theta\leq\pi/2$, we get

$$\begin{aligned}\operatorname{Re}(r_{j_0}^s) &= r_{j_0}^{\sigma_R}\cos\theta_{j_0}(t)<r_{j_0}^{\sigma_R}\cos\psi\\ &\leq r_{j_0}^{\sigma_R}\left(1-\frac{\psi^2}{4}\right)=r_{j_0}^{\sigma_R}\left(1-\frac{2\mu}{r_J^D}\right)<r_{j_0}^{\sigma_R}-2\mu\frac{r_{j_0}^D}{r_J^D}\leq r_{j_0}^{\sigma_R}-2\mu.\end{aligned}$$

Hence,

$$\begin{aligned}\operatorname{Re}\left(\sum_{j=1}^J r_j^s\right) &= \operatorname{Re}(r_{j_0}^s)+\operatorname{Re}\left(\sum_{j=1,j\neq j_0}^J r_j^s\right)\\ &\leq r_{j_0}^{\sigma_R}-2\mu+\sum_{j=1,j\neq j_0}^J r_j^{\sigma_R}=\sum_{j=1}^J r_j^{\sigma_R}-2\mu=1+\mu-2\mu=1-\mu.\end{aligned}$$

Therefore $\operatorname{Re}(f(s))\geq\mu$.

We now prove ii): We first show that for each $j=1,2,\ldots,J$, $\theta_j(\alpha_k)$ satisfies $\psi\leq\theta_j(\alpha_k)\leq\sqrt{\frac{\lambda}{1+\lambda}}$. Fix $j\in\{1,2,\ldots,J\}$. By definition of a_k, we have $-\psi\leq\theta_j(a_k)\leq\psi$. Therefore, there exists $m\in\mathbb{Z}$ such that $2\pi m-\psi\leq a_k\log r_j\leq 2\pi m+\psi$. Then, since $\alpha_k=a_k-2\psi/|\log r_1|$,

$$2\pi m-\psi+\frac{2\psi}{|\log r_1|}|\log r_j|\leq\alpha_k\log r_j\leq 2\pi m+\psi+\frac{2\psi}{|\log r_1|}|\log r_j|.$$

Using (9) and noting that $|\log r_j|\geq|\log r_1|$, we obtain

$$2\pi m+\psi\leq\alpha_k\log r_j\leq 2\pi m+\sqrt{\frac{\lambda}{1+\lambda}}.$$

Therefore $\psi \le \theta_j(\alpha_k) \le \sqrt{\frac{\lambda}{1+\lambda}} < 1 < \frac{\pi}{2}$.

Now, for $s = \sigma + \mathbf{i}\alpha_k \in \gamma_k^2$, we have $\sigma_L \le \sigma \le \sigma_R < D$ and

$$\mathrm{Im}(f(s)) = -\sum_{j=1}^{J} r_j^{\sigma} \sin\theta_j(\alpha_k) \le -\sin\psi \sum_{j=1}^{J} r_j^{\sigma} \le -\sin\psi \sum_{j=1}^{J} r_j^{D} = -\sin\psi.$$

We now prove iii): Reasoning as we did in the proof of part ii, it can be easily shown that $-\sqrt{\frac{\lambda}{1+\lambda}} \le \theta_j(\beta_k) \le -\psi$ and, for $\alpha_k \le t \le \beta_k$, we have $-\sqrt{\frac{\lambda}{1+\lambda}} \le \theta_j(t) \le \sqrt{\frac{\lambda}{1+\lambda}}$, for every $j \in \{1,2,\dots,J\}$. For $s \in \gamma_k^3$, we have $s = \sigma_L + \mathbf{i}t,\ \alpha_k \le t \le \beta_k$. Noting that $\cos\theta \ge 1-\theta^2/2$,

$$\begin{aligned}\mathrm{Re}(f(s)) = 1 - \sum_{j=1}^{J} r_j^{\sigma_L}\cos\theta_j(t) &\le 1 - \cos\left(\sqrt{\frac{\lambda}{1+\lambda}}\right)\sum_{j=1}^{J} r_j^{\sigma_L} \\ &\le 1 - \left(1 - \frac{1}{2}\frac{\lambda}{1+\lambda}\right)(1+\lambda) = -\frac{\lambda}{2}.\end{aligned}$$

Finally, we prove the case iv). For $s \in \gamma_k^4$, we have $s = \sigma + \mathbf{i}\beta_k,\ \sigma_L \le \sigma \le \sigma_R < D$ and $-\sqrt{\frac{\lambda}{1+\lambda}} \le \theta_j(\beta_k) \le -\psi$. Then

$$\mathrm{Im}(f(s)) = -\sum_{j=1}^{J} r_j^{\sigma}\sin\theta_j(\beta_k) \ge -\sin(-\psi)\sum_{j=1}^{J} r_j^{\sigma} \ge \sin\psi \sum_{j=1}^{J} r_j^{D} = \sin\psi.$$

□

Proof [Proof of Lemma 6] By [9, Theorem 3.26], there exists an increasing sequence $\{\rho_n\}_{n=1}^{\infty}$ tending to infinity such that $\zeta(s)$ is uniformly bounded on the lines $\mathrm{Im}(s) = \pm\rho_n$. That is, there exists $K > 0$ such that

$$|\zeta(s)| \le K, \quad \mathrm{Im}(s) = \pm\rho_n,\ n = 1,2,\dots.$$

By the residue theorem,

$$\begin{aligned}\frac{1}{2\pi\mathbf{i}}\int_{c-\mathbf{i}\rho_n}^{c+\mathbf{i}\rho_n} \zeta_{\mathcal{T}}(s,\varepsilon)ds &= \frac{1}{2\pi\mathbf{i}}\int_{L_n}\zeta_{\mathcal{T}}(s,\varepsilon)ds + \frac{1}{2\pi\mathbf{i}}\int_{\Gamma_n}\zeta_{\mathcal{T}}(s,\varepsilon)ds + \frac{1}{2\pi\mathbf{i}}\int_{L_n'}\zeta_{\mathcal{T}}(s,\varepsilon)ds \\ &+ \mathrm{res}\left(\zeta_{\mathcal{T}}(s,\varepsilon);D\right) + \sum_{\omega\in\Omega_{\rho_n}\cap\mathfrak{D}} \mathrm{res}\left(\zeta_{\mathcal{T}}(s,\varepsilon);\omega\right)\end{aligned} \tag{11}$$

where $L_n = \{s \mid \mathrm{Im}(s) = \rho_n\} \cap \overline{\Omega},\ L_n' = \{s \mid \mathrm{Im}(s) = -\rho_n\} \cap \overline{\Omega},\ \Gamma_n = \{s \mid -\rho_n \le \mathrm{Im}(s) \le \rho_n\} \cap \Gamma$ with appropriate orientations and $\Omega_{\rho_n} = \{s \mid -\rho_n < \mathrm{Im}(s) < \rho_n\} \cap \Omega$.

Using (6) we obtain (for fixed ε)

$$\left|\int_{L_n}\zeta_{\mathcal{T}}(s,\varepsilon)ds\right| \le \int_{L_n}|\zeta_{\mathcal{T}}(s,\varepsilon)||ds| \le K\int_{L_n}\frac{K}{|s|^2}|ds| \le \frac{K}{\rho_n^2}(c-\sigma_L) \to 0,$$

as $n \to \infty$, since the length of L_n is at most $(c - \sigma_L)$ and $\rho_n \to \infty$ as $n \to \infty$. Similarly, $\lim_{n\to\infty} \int_{L'_n} \zeta_{\mathcal{T}}(s,\varepsilon)ds = 0$.

It will be shown in the next lemma that the integral of $\zeta_{\mathcal{T}}$ over Γ absolutely converges, and so letting $n \to \infty$ in (11) gives the desired result. □

Proof [Proof of Lemma 7] Let $R_1 < 0 < R_2$ and let $\Gamma_{R_1}^{R_2}$ be the part of Γ that lies in the strip $\{s \,|\, R_1 \le \mathrm{Im}(s) \le R_2\}$. We will show that, for some $K > 0$ (independent of R_1 and R_2),

$$\int_{\Gamma_{R_1}^{R_2}} |\zeta_{\mathcal{T}}(s,\varepsilon)||ds| \le K\, \varepsilon^{d-\sigma_R}, \qquad \text{for } \varepsilon < 1.$$

Suppose $\Gamma_{R_1}^{R_2}$ have non-empty intersection with Γ_k only for $k_0 \le k \le k_1$. Then

$$\int_{\Gamma_{R_1}^{R_2}} |\zeta_{\mathcal{T}}(s,\varepsilon)||ds| \le \sum_{k=k_0}^{k_1} \int_{\Gamma_k} |\zeta_{\mathcal{T}}(s,\varepsilon)||ds| = \sum_{k=k_0}^{k_1} \sum_{l=1}^{4} \int_{\gamma_k^l} |\zeta_{\mathcal{T}}(s,\varepsilon)||ds|.$$

Recall that, $|\zeta(s)| \le K$ for $s \in \Gamma$ (by Lemma 5) and, by (6),

$$\left| \sum_{m=0}^{d} \kappa_m(G) \frac{g^{s-m}}{s-m} \right| \le \frac{K}{|s|^2} \le \frac{K}{\sigma_L^2 + |\mathrm{Im}s|^2} \qquad \text{for } \sigma_L \le \mathrm{Re}(s) \le \sigma_R.$$

Now, for $s \in \gamma_k^1$, we have $s = \sigma_R + \mathbf{i}t$, $\beta_{k-1} \le t \le \alpha_k$, and

$$\int_{\gamma_k^1} |\zeta_{\mathcal{T}}(s,\varepsilon)||ds| \le K\, \varepsilon^{d-\sigma_R} \int_{\beta_{k-1}}^{\alpha_k} \frac{K}{\sigma_L^2 + t^2} dt.$$

Similarly, $\int_{\gamma_k^3} |\zeta_{\mathcal{T}}(s,\varepsilon)||ds| \le K\varepsilon^{d-\sigma_R} \int_{\alpha_k}^{\beta_k} dt/(\sigma_L^2 + t^2)$. Therefore,

$$\sum_{k=k_0}^{k_1} \int_{\gamma_k^1+\gamma_k^3} |\zeta_{\mathcal{T}}(s,\varepsilon)||ds| \le K\varepsilon^{d-\sigma_R} \int_{\beta_{k_0-1}}^{\beta_{k_1}} \frac{dt}{\sigma_L^2 + t^2} \le K\varepsilon^{d-\sigma_R}. \tag{12}$$

By Lemma 5, $\alpha_{k+1} - \alpha_k \ge \frac{\pi}{\log r_J^{-1}} =: C$, $\alpha_0 < 0$ and $\alpha_1 > 0$. This implies $|\alpha_k| \ge (|k|-1)C$ and $|\beta_k| \ge (|k|-1)C$ for all $|k| \ge 2$.

Now, for $s \in \gamma_k^2$, we have $s = \sigma + \mathbf{i}\alpha_k, \sigma_L \le \sigma \le \sigma_R$ and so, for $|k| \ge 2$,

$$\int_{\gamma_k^2} |\zeta_{\mathcal{T}}(s,\varepsilon)||ds| \le K\varepsilon^{d-\sigma_R} \int_{\gamma_k^2} \frac{|ds|}{|s|^2} \quad \le \quad K\varepsilon^{d-\sigma_R} \frac{K}{|\alpha_k|^2}(\sigma_R - \sigma_L) \le \frac{K\varepsilon^{d-\sigma_R}}{(|k|-1)^2 C^2}.$$

A similar inequality holds when γ_k^2 is replaced with γ_k^4. Hence

$$\sum_{k=k_0}^{k_1} \int_{\gamma_k^2+\gamma_k^4} |\zeta_{\mathcal{T}}(s,\varepsilon)||ds| \le K\varepsilon^{d-\sigma_R}. \tag{13}$$

Combining (12) and (13) we get the desired result. □

Proof [Proof of Lemma 8] Let

$$h(\varepsilon) := \varepsilon^{D-d} \sum_{\omega\in\Omega\cap\mathfrak{D}} \mathrm{res}(\zeta_{\mathcal{T}}(s,\varepsilon);\omega) = \sum_{\omega\in\Omega\cap\mathfrak{D}} \varepsilon^{D-\omega}\mathrm{res}(\zeta(s);\omega) \sum_{m=0}^{d} \kappa_m(G)\frac{g^{\omega-m}}{\omega-m}.$$

By Lemma 4 and (6)

$$|h(\varepsilon)| \leq \sum_{\omega\in\Omega\cap\mathfrak{D}} \frac{1}{|\log r_1|}\varepsilon^{D-\mathrm{Re}(\omega)}\frac{K}{|\omega|^2} = K\sum_{\omega\in\Omega\cap\mathfrak{D}} \frac{\varepsilon^{D-\mathrm{Re}(\omega)}}{|\omega|^2}. \tag{14}$$

Let $\Omega_n = \{s \,|\, -n < \mathrm{Im}(s) < n\} \cap \Omega$, $n = 1, 2, \ldots$. By [9, Theorem 3.6], there exists $C, M > 0$ such that the number of poles of $\zeta(s)$ in the strip satisfies

$$Cn - M \leq \#\left(\mathfrak{D} \cap \{-n < \mathrm{Im}(s) < n\}\right) \leq Cn + M$$

for all $n = 1, 2, \ldots$. Let $\Pi_1 = \Omega_1$ and $\Pi_n = \Omega_n \setminus \Omega_{n-1}$ for $n \geq 2$. Then

$$\#(\mathfrak{D}\cap\Pi_n) = \#(\mathfrak{D}\cap\Omega_n)\setminus(\mathfrak{D}\cap\Omega_{n-1}) \leq Cn + M - (C(n-1) - M) = C + 2M$$

for $n \geq 2$. For $n = 1$, the above inequality clearly holds. Let $\eta > 0$ be arbitrary. We will show that $|h(\varepsilon)| < \eta$ for sufficiently small ε: By (14), for any $n \geq 1$,

$$|h(\varepsilon)| \leq K\sum_{\omega\in\Omega_n\cap\mathfrak{D}} \frac{\varepsilon^{D-\mathrm{Re}(\omega)}}{|\omega|^2} + K\sum_{\omega\in(\Omega\setminus\Omega_n)\cap\mathfrak{D}} \frac{\varepsilon^{D-\mathrm{Re}(\omega)}}{|\omega|^2}. \tag{15}$$

Now, for $\varepsilon < 1$

$$\sum_{\omega\in(\Omega\setminus\Omega_n)\cap\mathfrak{D}} \frac{\varepsilon^{D-\mathrm{Re}(\omega)}}{|\omega|^2} = \sum_{k=n+1}^{\infty}\sum_{\omega\in\Pi_k\cap\mathfrak{D}} \frac{\varepsilon^{D-\mathrm{Re}(\omega)}}{|\omega|^2} \leq \sum_{k=n+1}^{\infty}\frac{C+2M}{(k-1)^2},$$

since $\varepsilon^{D-\mathrm{Re}(\omega)} < 1$ and for $\omega \in \Pi_k, (k \geq 2)$, we have $|\omega|^2 \geq |\mathrm{Im}(\omega)|^2 \geq (k-1)^2$. Because of the convergence of the series $\sum_{k=2}^{\infty}\frac{1}{(k-1)^2}$, there exists n_0 such that the second term on the right-hand side of (15) is less than $\eta/2$ for $n = n_0$. To deal with the first term, note that the set $\Omega_{n_0}\cap\mathfrak{D}$ has finitely many elements. Let $\delta := \min\{D - \mathrm{Re}(\omega) \,|\, \omega \in \Omega_{n_0}\cap\mathfrak{D}\}$. Recall that all the poles of ζ, except the one at $s = D$, have real part less than D (see [9, Theorem 2.17]). Therefore, $\delta > 0$ and

$$K\sum_{\omega\in\Omega_{n_0}\cap\mathfrak{D}} \frac{\varepsilon^{D-\mathrm{Re}(\omega)}}{|\omega|^2} \leq K\varepsilon^{\delta}\sum_{\omega\in\Omega_{n_0}\cap\mathfrak{D}}\frac{1}{|\omega|^2} \leq K\varepsilon^{\delta}$$

which is less than $\eta/2$ when ε is sufficiently small. □

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