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Quasinormability and diametral dimension

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Abstract: Two versions of diametral dimension are shown to coincide for quasinormable Fréchet spaces. The diametral dimension is determined by a single bounded subset in certain cases.

Key words and phrases: Diametral dimension, Fréchet spaces, Köthe spaces

1. Introduction

The set $\Delta(E)$ of sequences (ξ_n) such that for each neighborhood U of zero of a locally convex space E there is another such neighborhood with $\lim \xi_n d_n(V,U) = 0$, where $d_n(V,U)$ is the n-th diameter of V with respect to U, is called the *diametral dimension* of E. ([3], [6], [7], [8]). Another version is the set $\Delta_b(E)$ of all sequences (ξ_n) such that for each neighborhood U and each bounded subset B we have $\lim \xi_n d_n(B,U) = 0$. $\Delta_b(E)$ is less frequently used than $\Delta(E)$. We always have $c_0 \subset \Delta(E) \subset \Delta_b(E)$. In [6] Mitiagin claimed that $\Delta(E) = \Delta_b(E)$ holds for every Fréchet space (F-space) E, referring for the proof to a forthcoming joint paper. However, there is an example of a Köthe space $\lambda(A)$, which is a Montel space but fails to be a Schwartz space. In this case we have

$$\Delta(\lambda(A)) = c_0 \subset \ell_\infty \subset \Delta_b(\lambda(A)).$$

On the other hand, if E is a locally convex space with a bounded subset that is not precompact, we have $\Delta(E) = \Delta_b(E) = c_0$.

We recall that a Fréchet-Montel space (FM-space) is a Fréchet-Schwartz space (FS-space) if and only if E is quasinormable [3]. There is an extensive literature concerning quasinormability (cf. [1]). We want to single out a remarkable result of Meise and Vogt [5], which states than an F-space is quasinormable if and only if it is isomorphic to a quotient space of the complete tensor product $\ell^1(I)\tilde{\otimes}_\pi\lambda(A)$, where I is a suitable index set and $\lambda(A)$ a suitable Köthe-Schwartz space.

We recall the definition of the n-th diameter

$$d_n(A,B) = \inf\inf\{\alpha > 0 : A \subset \alpha B + L\}$$

where A and B are subsets of a locally convex space E with $A \subset \rho B$ for some $\rho > 0$. The second infimum is taken over all subspaces L of E with dimension not exceeding $n \in N$.

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Proposition 1 *If E is a quasinormable metrisable space, then $\Delta(E) = \Delta_b(E)$.*

Proof For $(\xi_n) \in \Delta_b(E)$ we find $\delta_n \geq \delta_{n+1} > 0$ with $\lim \xi_n \delta_n = 0$. By quasinormability we can choose a base of absolutely convex, closed neighborhoods $U_1, \supset U_2 \supset \cdots$ such that for each k and $\delta_n > 0$ there is a bounded subset $B_{k,n}$ with

$$U_{k+1} \subset B_{k,n} + \delta_n U_k.$$

In particular,

$$U_{n+1} \subset B_{k,n} + \delta_n U_n$$

holds for each $n \geq k$. To see how much of $B_{k,n}$ we need in the above inclusion we observe that for each $x \in U_{n+1}$ there is a $b \in B_{k,n}$ with $\|x - b\|_n \leq \delta_n$. This means if we replace $B_{k,n}$ with

$$B_{k,n} \cap (1 + \delta_0) U_n$$

the above inclusion still holds. Therefore, we will assume without loss of generality that

$$B_{k,n} \subset (1 + \delta_0) U_n$$

in the above inclusion.

This implies that

$$B_k = \bigcup_{n=k}^{\infty} B_{k,n}$$

is a bounded set and therefore for all $n \geq k$ we obtain

$$U_{k+1} \subset B_k + \delta_n U_k.$$

Using the definition of the n-th diameter, from the above inclusion for $n \geq k$ we get

$$d_n(U_{k+1}, U_k) \leq d_n(B_k, U_k) + \delta_n.$$

Hence

$$\lim \xi_n d_n(U_{k+1}, U_k) = 0.$$

□

Our result implies that for an F-space we have $\Delta(E) \neq \Delta_b(E)$ if and only if E is a Montel but not a Schwartz space.

In certain cases it is sufficient to consider a single bounded subset of E to determine $\Delta(E)$. We will call an absolutely convex bounded subset B of an F-space E a *prominent set* if $\lim \xi_n d_n(B, U_k) = 0$ for every k implies $(\xi_n) \in \Delta(E)$. If E has a prominent set B then since

$$\Delta(E) = \{(\xi_n) : \lim \xi_n d_n(B, U_k) = 0, \quad k = 1, 2, ...\}$$

the diametral dimension as a space of sequences is an F-space itself. For an exponent sequence $0 < \alpha_n \leq \alpha_{n+1} \leq \cdots$ with $\lim \alpha_n = \infty$ the unit ball B_1 of ℓ_1 is a prominent set of the finite-type power series space $\Lambda_1(\alpha)$ (cf. for example [6], [8]). We will generalize this result in what follows.

Following [10], we call a Banach space $(\ell, \| \ \|)$ of scalar sequences an *admissible space* if $\|e_n\| = 1$ and for $a \in \ell_\infty$, $x \in \ell$ we have $ax = (a_n x_n) \in \ell$ and $\|ax\| \leq \|a\|_\infty \|x\|$. As usual e_n is that sequence with 1 as the n-th term and zero elsewhere. The classical sequence spaces $\ell_p, 1 \leq p \leq \infty$ and c_0 are the most well-known examples of admissible spaces.

Let A be a Köthe set and $\lambda^\ell(A)$ be the space of all sequences $x = (x_n)$ such that $(x_n a_n^k) \in \ell$ for each k. Equipped with the seminorms, $\|x\|_k = \|a^k x\|$, $\lambda^\ell(A)$ is an F-space. $\lambda^{\ell_1}(A)$ is of course the usual Köthe space $\lambda(A)$. In fact the spaces $\lambda^{\ell_p}(A), \lambda^{c_0}(A), 1 \leq p \leq \infty$, are also quite well known.

A Köthe space $\lambda(A)$ is called a *smooth sequence space of finite type* [8] (or a G_1-space) if $0 < a_{n+1}^k \leq a_n^k$ and for each k there is a j with $(a_n^k/(a_n^j)^2) \in \ell_\infty$.

Proposition 2 *Let $\lambda(A)$ be a G_1-space and ℓ an admissible space with closed unit ball B_ℓ. Then B_ℓ is a prominent subset of $\lambda^\ell(A)$.*

Proof Let

$$U_k = \{(x_n) \in \lambda^\ell(A) : \|(x_n a_n^k)\| \leq 1\}.$$

By [10], Prop. 1, we have the basic inequality

$$\inf\left\{\frac{a_i^k}{a_i^j} : i \leq n\right\} \leq d_n(U_j, U_k) \leq \sup\left\{\frac{a_i^k}{a_i^j} : i \geq n\right\}$$

We note that both sides of this inequality are independent of ℓ. With the same argument we can easily show

$$d_n(B_\ell, U_k) = a_n^k.$$

Now for k given we choose j such that for some $\rho > 0$ we have $a_n^k \leq \rho (a_n^j)^2$ for all $n \in N$. From the above inequality we obtain

$$d_n(U_j, U_k) \leq \rho \sup\{a_i^j : i \geq n\} = \rho a_n^j.$$

Therefore,

$$d_n(U_j, U_k) \leq \rho d_n(B_\ell, U_j).$$

This shows

$$\Delta(\lambda^\ell(A)) = \Delta_b(\lambda^\ell(A)) = \{(\xi_n) : \lim \xi_n d_n(B_\ell, U_j) = 0 \text{ for all } j \in N\}.$$

□

Of course, a finite type power series space $\Lambda_1(\alpha)$ is a G_1-space. In this special case the closed unit ball B_p of ℓ_p or B_0 of c_0 is a prominent subset of $\Lambda_1^{\ell_p}(\alpha)$, or of $\Lambda_1^{c_0}(\alpha)$.

We will now give a necessary and sufficient condition for a bounded subset to be prominent.

Proposition 3 *Let B be a bounded subset of an F-space E. B is a prominent set if and only if for each k there is a p and $\rho > 0$ such that*

$$d_n(U_p, U_k) \leq \rho \, d_n(B, U_p)$$

for all $n \in N$.

Proof Sufficiency follows immediately from definitions of $\Delta(E)$ and $\Delta_b(E)$. Let us now assume B is a prominent subset of E. In this case, the diametral dimension is

$$\Delta(E) = \lambda^{c_0}(B_E)$$

where

$$B_E = \{(d_n(B, U_k)) : k = 1, 2, \ldots\}$$

and so $\Delta(E)$ is itself an F-space. On the other hand, from the definition of $\Delta(E)$, for a given k we have

$$\Delta(E) \subset \cup_{p \geq k}\{(\xi_n) : \lim \xi_n d_n(U_p, U_k) = 0\}$$

The right-hand side of the above inclusion is an LB-space and the canonical inclusion map is sequentially closed. Therefore, by the Grothendieck factorization theorem we can find $m > k$ such that

$$\Delta(E) \subset \{(\xi_n) : \lim \xi_n d_n(U_m, U_k) = 0\}.$$

This implies the existence of j and $\rho > 0$ with

$$d_n(U_m, U_k) \leq \rho\, d_n(B, U_j), \quad n \in N.$$

Finally we choose $p = \max\{m, j\}$. □

Let $\lambda(A)$ now be a *smooth sequence space of infinite type* [8]. This means $0 < a_n^k \leq a_{n+1}^k \leq \ldots$ and for each k there is a_j with $((a_n^k)^2/a_n^j) \in \ell_\infty$. To avoid the trivial case $\lambda(A) = \ell_1$ we will also assume $\lim_{n\to\infty} a_n^k = \infty$ for every k. This implies that $\lambda(A)$ is a Schwartz space, and so

$$\Delta(\lambda(A)) = \Delta_b(\lambda(A))$$

by Proposition 1. Let B be a prominent subset of $\lambda(A)$. Since a bounded set, which contains a prominent subset, is itself prominent, we can assume without loss of generality [2] that

$$B = \{(\xi_n) : \Sigma|\xi_n|a_n \leq 1\}$$

where (a_n) is some sequence such that for each k there is a $\rho_k > 0$ with $a_n^k \leq \rho_k a_n$ for all $n \in N$. By Prop. 3. for each k there is a $\rho > 0$ and $p \geq k$ with

$$d_n(U_p, U_k) \leq \rho\, d_n(B, U_p).$$

We choose m so that $((a_n^p)^3/a_n^m) \in \ell_\infty$. By the basic inequality

$$d_n(B, U_p) \leq \inf\{a_i^p/a_i : i \geq n\}$$

but

$$\frac{a_i^p}{a_i} \leq c\frac{a_i^m}{(a_i^p)^2 a_i} \leq \frac{c\rho_m}{(a_i^p)^2}$$

for some constant $c > 0$.

Applying the left-hand side of the basic inequality we have

$$\frac{a_0^k}{a_n^p} \leq d_n(U_p, U_k).$$

Hence $(a_n^p) \in \ell_\infty$, which is a contradiction. So in contrast to Prop. 2 we have the following result.

Proposition 4 *A smooth sequence space of infinite type that is also a Schwartz space has no prominent subset.*

In particular, an infinite type power series space $\Lambda_\infty(\alpha)$ has no prominent subset although $\Delta(\Lambda_\infty(\alpha)) = \Delta_b(\Lambda_\infty(\alpha))$.

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