Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2013) 37: 852 – 862

doi:10.3906/mat-1109-14

Complete cotorsion pairs in the category of complexes

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Received: 09.09.2011 • **Accepted:** 09.09.2012 • **Published Online:** 26.08.2013 • **Printed:** 23.09.2013

Abstract: In this paper, we study completeness of cotorsion pairs in the category of complexes of R-modules. Let $(\mathcal{A},\mathcal{B})$ be a cotorsion pair in R-Mod. It is shown that the cotorsion pairs $(\widetilde{\mathcal{A}},\mathrm{dg}\widetilde{\mathcal{B}})$ and $(\overline{\mathcal{A}},\overline{\mathcal{A}}^{\perp})$ are complete if $\mathcal{A}$ is closed under pure submodules and cokernels of pure monomorphisms, where in Gillespie's definitions $\widetilde{\mathcal{A}}$ is the class of exact complexes with cycles in $\mathcal{A}$ and $\mathrm{dg}\widetilde{\mathcal{B}}$ is the class of complexes X with components in $\mathcal{B}$ such that the complex $\mathcal{H}om(A,X)$ is exact for every complex $A\in\widetilde{\mathcal{A}}$; and $\overline{\mathcal{A}}$ is the class of all complexes with components in $\mathcal{A}$. Furthermore, they are perfect. As an application, we get that every complex over a right coherent ring has a Gorenstein flat cover, which generalizes the well-known results on the existence of Gorenstein flat covers.

Key words: Complete, cotorsion pair, cover, Gorenstein flat complex

1. Introduction and preliminaries

In this paper, R denotes a ring with unity, R-Mod denotes the category of left R-modules, and $\mathcal{C}(R)$ denotes the abelian category of complexes of left R-modules. A complex

$$\cdots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} \cdots$$

of left R-modules will be denoted (C,δ) or C. Given a left R-module M, we will denote by $\mathrm{D}^n(M)$ the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{id} M \longrightarrow 0 \longrightarrow \cdots$$

with the M in the n and $(n-1)$-th position. Also, by $\mathrm{S}^n(M)$ we mean the complex with M in the n-th place and 0 in the other places, and the character module $M^+=\mathrm{Hom}(M,\mathbb{Q}/\mathbb{Z})$. Given a complex C and an integer i, Σ^iC denotes the complex such that $(\Sigma^iC)_n=C_{n-i}$ and whose boundary operators are $(-1)^i\delta^C_{n-i}$. The n-th homology module of C is the module $\mathrm{H}_n(C)=\mathrm{Z}_n(C)/\mathrm{B}_n(C)$, where $\mathrm{Z}_n(C)=\mathrm{Ker}(\delta^C_n)$, $\mathrm{B}_n(C)=\mathrm{Im}(\delta^C_{n+1})$; we set $\mathrm{C}_n(C)=\mathrm{Coker}(\delta^C_{n+1})$.

Throughout the paper we use both the subscript notation for complexes and the superscript notation. When we use superscripts for a complex we will use subscripts to distinguish complexes. For example, if $(K^i)_{i\in I}$ is a family of complexes, then K^i_n denotes the n-th component of the complex K^i.

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2000 *AMS Mathematics Subject Classification:* 18G35, 55U15, 03C35.

For objects C and D of $\mathcal{C}(R)$, $\mathrm{Hom}(C,D)$ is the abelian group of morphisms from C to D in $\mathcal{C}(R)$ and $\mathrm{Ext}^i(C,D)$ for $i \geq 1$ will denote the groups we get from the right derived functor of Hom. $\mathcal{H}om(C,D)$ denotes the complex of abelian groups with n-th component $\mathcal{H}om(C,D)_n$ and boundary operator

$$\delta_n((\varphi_i)_{i\in\mathbb{Z}}) = (\delta^D_{n+i}\varphi_i - (-1)^n\varphi_{i-1}\delta^C_i)_{i\in\mathbb{Z}}.$$

It is easy to see that $\mathrm{Hom}(C,D) = Z_0(\mathcal{H}om(C,D))$. We recall the notations introduced in [5]. Let $\underline{\mathrm{Hom}}(C,D) = Z(\mathcal{H}om(C,D))$, we then see that $\underline{\mathrm{Hom}}(C,D)$ can be made into a complex with $\underline{\mathrm{Hom}}(C,D)_n$ the abelian group of morphisms from C to $\Sigma^{-n}D$ and with boundary operator given by $\delta_n(f) : C \longrightarrow \Sigma^{-(n-1)}D$ with $\delta_n(f)_m = (-1)^n\delta^D f_m$, $\forall m \in \mathbb{Z}$ for $f \in \underline{\mathrm{Hom}}(C,D)_n$, and note that the new functor $\underline{\mathrm{Hom}}(C,D)$ will have right derived functors whose values will be complexes. These values should certainly be denoted $\underline{\mathrm{Ext}}^i(C,D)$. It is not hard to see that $\underline{\mathrm{Ext}}^i(C,D)$ is the complex

$$\cdots \longrightarrow \mathrm{Ext}^i(C,\Sigma^{-(n+1)}D) \longrightarrow \mathrm{Ext}^i(C,\Sigma^{-n}D) \longrightarrow \mathrm{Ext}^i(C,\Sigma^{-(n-1)}D) \longrightarrow \cdots$$

with boundary operators induced by the boundary operators of D. Also we mean by $C^+ = \underline{\mathrm{Hom}}(C, D^0(\mathbb{Q}/\mathbb{Z}))$ the complex

$$\cdots \longrightarrow \mathrm{Hom}(C_{-1},\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}(C_0,\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}(C_1,\mathbb{Q}/\mathbb{Z}) \longrightarrow \cdots .$$

If X is a complex of right R-modules and Y is a complex of left R-modules, the tensor product of X and Y is the complex of abelian groups $X \otimes Y$ with $(X\otimes Y)_n = \oplus_{t\in\mathbb{Z}}(X_t \otimes_R Y_{n-t})$ and $\delta(x\otimes y) = \delta^X_t(x)\otimes y + (-1)^t x \otimes \delta^Y_{n-t}(y)$, $\forall x \in X^t, y \in Y^{n-t}$. Define $X\overline{\otimes}Y$ to be $\frac{X\otimes Y}{B(X\otimes Y)}$. Then with the maps

$$\tfrac{(X\otimes Y)_n}{B_n(X\otimes Y)} \longrightarrow \tfrac{(X\otimes Y)_{n-1}}{B_{n-1}(X\otimes Y)}, \quad x\otimes y \mapsto \delta_X(x)\otimes y,$$

where $x \otimes y$ is used to denote the coset in $\frac{(X\otimes Y)_n}{B_n(X\otimes Y)}$, we get a complex of abelian groups.

Let $\mathcal{A},\mathcal{B}$ be classes of objects in an abelian category $\mathcal{D}$. Let D be an object of $\mathcal{D}$. We recall the definition introduced in [2]. A morphism $f : D \to B$ is called a $\mathcal{B}$-preenvelope of D if $B \in \mathcal{B}$ and $\mathrm{Hom}(B,B') \to \mathrm{Hom}(D,B') \to 0$ is exact for all $B' \in \mathcal{B}$. If, moreover, any morphism $g : B \to B$ such that $gf = f$ is an automorphism of B then $f : D \to B$ is called a $\mathcal{B}$-envelope. A monomorphism $\alpha : D \longrightarrow B$ with $B \in \mathcal{B}$ is said to be a special $\mathcal{B}$-preenvelope of D if $\mathrm{Coker}(\alpha) \in {}^{\perp}\mathcal{B}$, where ${}^{\perp}\mathcal{B} = \{A \in \mathcal{D} : \mathrm{Ext}^1(A,B) = 0 \text{ for all } B \in \mathcal{B}\}$. Dually we have the concepts of a (special) $\mathcal{B}$-precover and a $\mathcal{B}$-cover. A pair of classes of objects $(\mathcal{A},\ \mathcal{B})$ is called a cotorsion pair (or cotorsion theory) [15, 20] if $\mathcal{A}^{\perp} = \mathcal{B}$ and ${}^{\perp}\mathcal{B} = \mathcal{A}$, where $\mathcal{A}^{\perp} = \{B \in \mathcal{D} : \mathrm{Ext}^1(A,B) = 0 \text{ for all } A \in \mathcal{A}\}$. A cotorsion pair $(\mathcal{A},\ \mathcal{B})$ is called hereditary if whenever $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is exact with $A, A'' \in \mathcal{A}$ then A' is also in $\mathcal{A}$. A cotorsion pair $(\mathcal{A},\ \mathcal{B})$ is called complete if every $D \in \mathcal{D}$ has a special $\mathcal{B}$-preenvelope and a special $\mathcal{A}$-precover. A cotorsion pair $(\mathcal{A},\ \mathcal{B})$ is called perfect if every $D \in \mathcal{D}$ has a $\mathcal{B}$-envelope and an $\mathcal{A}$-cover. A cotorsion pair $(\mathcal{A},\ \mathcal{B})$ is said to be cogenerated by a set X if $X^{\perp} = \mathcal{A}^{\perp}$. It is well known that a perfect cotorsion pair is complete, but the converse may be false in general. In [1], Eklof and Trlifaj proved that a cotorsion pair $(\mathcal{A},\ \mathcal{B})$ in R-Mod is complete when it is cogenerated by a set. This result actually holds in any Grothendieck category with enough projectives, as Hovey proved in [17]. For unexplained concepts and notations, we refer the reader to [4, 5, 6, 11, 15, 21].

In [12], Gillespie introduced the following definition.

Definition 1.1 *([12, Definition 3.3]) Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair on an abelian category $\mathcal{C}$. Let X be a complex.*

(1) X is called an $\mathcal{A}$ complex if it is exact and $\mathrm{Z}_n(X) \in \mathcal{A}$ for all $n \in \mathbb{Z}$.

(2) X is called a $\mathcal{B}$ complex if it is exact and $\mathrm{Z}_n(X) \in \mathcal{B}$ for all $n \in \mathbb{Z}$.

(3) X is called a dg-$\mathcal{A}$ complex if $X_n \in \mathcal{A}$ for each $n \in \mathbb{Z}$, and $\mathcal{H}om(X, B)$ is exact whenever B is a $\mathcal{B}$ complex.

(4) X is called a dg-$\mathcal{B}$ complex if $X_n \in \mathcal{B}$ for each $n \in \mathbb{Z}$, and $\mathcal{H}om(A, X)$ is exact whenever A is a $\mathcal{A}$ complex.

In particular, if $(\mathcal{A}, \mathcal{B}) = (\mathrm{Proj}, R\text{-}Mod)$, then $\mathcal{A}$ complexes and dg-$\mathcal{A}$ complexes is just projective complexes and DG-projective complexes, respectively. If $(\mathcal{A}, \mathcal{B}) = (R\text{-}Mod, \mathrm{Inj})$, then $\mathcal{B}$ complexes and dg-$\mathcal{B}$ complexes are just injective complexes, and DG-injective complexes respectively.

We denote the class of $\mathcal{A}$ complexes by $\widetilde{\mathcal{A}}$ and the class of dg-$\mathcal{A}$ complexes by $\mathrm{dg}\widetilde{\mathcal{A}}$. Similarly, the class of $\mathcal{B}$ complexes is denoted by $\widetilde{\mathcal{B}}$ and the class of dg-$\mathcal{B}$ complexes by $\mathrm{dg}\widetilde{\mathcal{B}}$. In [12], it was shown that $(\widetilde{\mathcal{A}}, \mathrm{dg}\widetilde{\mathcal{B}})$ and $(\mathrm{dg}\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ are cotorsion pairs in $\mathcal{C}(R)$ if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in R-Mod, and proven that $(\mathcal{A}, \mathcal{B})$ is hereditary if and only if $(\widetilde{\mathcal{A}}, \mathrm{dg}\widetilde{\mathcal{B}})$ is hereditary if and only if $(\mathrm{dg}\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ is hereditary. In [13] and [14], it was considered the question of whether or not the induced cotorsion pairs are complete when the original cotorsion pair is complete, and shown that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in an abelian category $\mathcal{C}$ can induce two natural homological model structures on $\mathrm{Ch}(\mathscr{C})$ under certain conditions.

In section 2 of this article, the completeness of the cotorsion pair $(\widetilde{\mathcal{A}}, \mathrm{dg}\widetilde{\mathcal{B}})$ is studied. It is given a sufficient condition such that the cotorsion pair $(\widetilde{\mathcal{A}}, \mathrm{dg}\widetilde{\mathcal{B}})$ is complete. As some applications, we get that every complex over a right coherent ring has a $\widetilde{\mathcal{GF}}$-cover, every complex has a $\widetilde{\mathcal{F}_n}$-cover, and every complex has a $\widetilde{\mathcal{MF}}$-cover, where $\mathcal{GF}$, $\mathcal{F}_n$, and $\mathcal{MF}$ respectively denote the classes of all Gorenstein flat left R-modules, all left R-modules with flat dimension less than or equal to a fixed nonnegative integer n, and all min-flat left R-modules.

Section 3 is devoted to studying complexes in the class $\overline{\mathcal{A}}^{\perp}$, and completeness of the cotorsion pair $(\overline{\mathcal{A}}, \overline{\mathcal{A}}^{\perp})$. We prove that a complex C is in $\overline{\mathcal{A}}^{\perp}$ if and only if C_n is in $\mathcal{A}^{\perp}$ for all $n \in \mathbb{Z}$ and $\mathcal{H}om(G, C)$ is exact for any $G \in \overline{\mathcal{A}}$, and $(\overline{\mathcal{A}}, \overline{\mathcal{A}}^{\perp})$ is complete if $\mathcal{A}$ is closed under pure submodules and cokernels of pure monomorphisms. As an application, we get that every complex over a right coherent ring has a Gorenstein flat cover, which generalizes Theorem 5.4.8 in [11] and Theorem 2.12 in [10].

2. $\widetilde{\mathcal{A}}$-covers of complexes

First are given some characterizations of $\mathcal{A}$ complexes and $\mathcal{B}$ complexes.

Lemma 2.1 *([14, Lemma 4.2]) Let $\mathscr{C}$ be an abelian category, $\mathrm{Ch}(\mathscr{C})$ be the category of complexes on $\mathscr{C}$. For each object $C \in \mathscr{C}$ and X, $Y \in \mathrm{Ch}(\mathscr{C})$, we have the following isomorphisms.*

(1) If X is an exact complex, then $\mathrm{Ext}^1_{\mathscr{C}}(\mathrm{C}_n(X), C) \cong \mathrm{Ext}^1_{\mathrm{Ch}(\mathscr{C})}(X, \mathrm{S}^n(C))$.

(2) If Y is an exact complex, then $\mathrm{Ext}^1_{\mathscr{C}}(C, \mathrm{Z}_n(Y)) \cong \mathrm{Ext}^1_{\mathrm{Ch}(\mathscr{C})}(\mathrm{S}^n(C), Y)$.

Proposition 2.2 *Let $(\mathcal{A},\mathcal{B})$ be a cotorsion pair in R-Mod. Then the following assertions are equivalent.*

(1) C is an $\mathcal{A}$ complex.

(2) For every dg-$\mathcal{B}$ complex G, $\mathrm{Ext}^1(C,G)=0$.

(3) For every bounded above complex G with each component in $\mathcal{B}$, $\mathrm{Ext}^1(C,G)=0$.

(4) For every bounded complex G with each component in $\mathcal{B}$, $\mathrm{Ext}^1(C,G)=0$.

(5) For any $B\in\mathcal{B}$, and any $n\in\mathbb{Z}$, $\mathrm{Ext}^1(C,\mathrm{S}^n(B))=0$.

Proof (1) $\Rightarrow$ (2) It follows from the proof of [12, Proposition 3.6].

(2) $\Rightarrow$ (3) It is clear since every bounded above complex with components in $\mathcal{B}$ is dg-$\mathcal{B}$ complex (see [12, Lemma 3.4(2)]).

(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (1) First, we show that C is exact. Let $f_n : C_n/\mathrm{B}_n(C) \longrightarrow I$ be an injective homomorphism, where I is an injective module. Then the induced morphism of complexes $f : C \longrightarrow \mathrm{S}^n(I)$ follows as

$$\begin{array}{lccccccccc} C= & \cdots \longrightarrow & C_{n+1} & \xrightarrow{\delta_{n+1}} & C_n & \xrightarrow{\delta_n} & C_{n-1} & \longrightarrow \cdots \\ & & \downarrow & & \downarrow{\scriptstyle f_n\eta} & & \downarrow & \\ \mathrm{S}^n(I)= & \cdots \longrightarrow & 0 & \longrightarrow & I & \longrightarrow & 0 & \longrightarrow \cdots, \end{array}$$

where $\eta : C_n \longrightarrow C_n/\mathrm{B}_n(C)$ is the natural epimorphism. We get that f is homotopic to zero since $\mathrm{Ext}^1(C,\mathrm{S}^n(I))=0$. Let $\{S_n\}_{n\in\mathbb{Z}}$ be homotopy, then $S_{n-1}\delta_n = f_n\eta$. Thus $\mathrm{Z}_n(C)\subseteq \mathrm{B}_n(C)$, and so C is an exact complex. Next it is proven that $\mathrm{Z}_n(C)\in\mathcal{A}$. By Lemma 2.1, $\mathrm{Ext}^1_R(\mathrm{C}_n(C),B)\cong \mathrm{Ext}^1(C,\mathrm{S}^n(B))$ for any $B\in\mathcal{B}$. But $\mathrm{Ext}^1(C,\mathrm{S}^n(B))=0$, so $\mathrm{Ext}^1_R(\mathrm{C}_n(C),B)=0$. Thus $\mathrm{C}_n(C)\in\mathcal{A}$. Since $\mathrm{Z}_n(C)\cong \mathrm{C}_{n+1}(C)$, we have $\mathrm{Z}_n(C)\in\mathcal{A}$. Therefore C is $\mathcal{A}$ complex. □

Proposition 2.3 *Let $(\mathcal{A},\mathcal{B})$ be a cotorsion pair in R-Mod. Then the following assertions are equivalent.*

(1) C is a $\mathcal{B}$ complex.

(2) For every dg-$\mathcal{A}$ complex G, $\mathrm{Ext}^1(G,C)=0$.

(3) For every bounded below complex G with each component in $\mathcal{A}$, $\mathrm{Ext}^1(G,C)=0$.

(4) For every bounded complex G with each component in $\mathcal{A}$, $\mathrm{Ext}^1(G,C)=0$.

(5) For any $A\in\mathcal{A}$, and any $n\in\mathbb{Z}$, $\mathrm{Ext}^1(\mathrm{S}^n(A),C)=0$.

Proof (1) $\Rightarrow$ (2) It follows from the proof of [12, Proposition 3.6].

(2) $\Rightarrow$ (3) is clear since every bounded below complex with each component in $\mathcal{A}$ is dg-$\mathcal{A}$ complex (see [12, Lemma 3.4(1)]).

(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (1) Note that $\mathcal{H}om(\mathrm{S}^0(R),C)\cong C$ for any complex, we obtain that $\mathrm{H}^n(C)\cong \mathrm{Ext}^1(\mathrm{S}^{1-n}(R),C)$ by [12, Lemma 2.1]. Since ${}_RR\in\mathcal{A}$, it follows that C is an exact complex by the assumption. By Lemma 2.1, $\mathrm{Ext}^1_R(A,\mathrm{Z}_n(C))\cong \mathrm{Ext}^1(\mathrm{S}^n(A),C)$ for all $A\in\mathcal{A}$, which implies that $\mathrm{Z}_n(C)\in\mathcal{B}$ for all $n\in\mathbb{Z}$. Thus C is $\mathcal{B}$ complex. □

According to [11], a short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathcal{C}(R)$ is called pure if the sequence $0 \longrightarrow F\overline{\otimes}A \longrightarrow F\overline{\otimes}B$ is exact for any (or finitely presented) complex F of right R-modules. Equivalently, $\underline{\mathrm{Hom}}(F,B) \longrightarrow \underline{\mathrm{Hom}}(F,C) \longrightarrow 0$ is surjective for all finitely presented complex F of left R-modules. A subcomplex $S \subset C$ is pure if $0 \longrightarrow S \longrightarrow C \longrightarrow C/S \longrightarrow 0$ is a pure exact sequence.

Lemma 2.4 *([12, Lemma 4.6]) Let $|R| \leq \aleph$, where $\aleph$ is some infinite cardinal. Then for any $C \in \mathcal{C}(R)$ and any element $x \in C$ (by this we mean $x \in C_n$ for some n), there exists a pure subcomplex $P \subseteq C$ with $x \in P$ and $|P| \leq \aleph$.*

Lemma 2.5 *Suppose S, T and C are complexes of left R-modules such that $S \subseteq T \subseteq C$. If S is pure in C and T/S is pure in C/S, then T is pure in C.*

Proof Let D be any complex of right R-modules. Then we get the following commutative diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & D\overline{\otimes}S & \longrightarrow & D\overline{\otimes}T & \longrightarrow & D\overline{\otimes}T/S & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D\overline{\otimes}S & \longrightarrow & D\overline{\otimes}C & \longrightarrow & D\overline{\otimes}C/S & \longrightarrow & 0 \end{array}$$

where all of the maps are the obvious ones. Thus $0 \to D\overline{\otimes}T \to D\overline{\otimes}C$ is exact, and so T is pure in C. □

Note that the similar result holds in R-Mod.

Lemma 2.6 *If $0 \to A \to B \to C \to 0$ is pure exact in $\mathcal{C}(R)$, then $0 \to \mathrm{Z}_n(A) \to \mathrm{Z}_n(B) \to \mathrm{Z}_n(C) \to 0$ is pure exact in R-Mod for all $n \in \mathbb{Z}$.*

Proof By the hypothesis, we have an exact sequence $0 \to \mathrm{Z}_n(A) \to \mathrm{Z}_n(B) \to \mathrm{Z}_n(C) \to 0$ in R-Mod. Let P be any finitely presented module, and $f : P \to \mathrm{Z}_n(C)$ be any R-homomorphism. We define $\alpha : \mathrm{S}^n(P) \longrightarrow C$ as

$$\begin{array}{rccccccccc} \mathrm{S}^n(P) = & \cdots & \longrightarrow & 0 & \longrightarrow & P & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & \downarrow & & \downarrow \lambda f & & \downarrow & & \\ C = & \cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots, \end{array}$$

where $\lambda : \mathrm{Z}_n(C) \longrightarrow C_n$ is the natural inclusion. Since $\mathrm{S}^n(P)$ is a finitely presented complex, there exists $\beta : \mathrm{S}^n(P) \longrightarrow B$ such that the diagram

$$\begin{array}{ccccccccc} & & & & & & \mathrm{S}^n(P) & & \\ & & & & & \swarrow^{\beta} & \downarrow \alpha & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

commutes. Thus

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{\beta_n} & \downarrow{\scriptstyle \lambda f} & & \\ B_n & \longrightarrow & C_n & \longrightarrow & 0 \end{array}$$

commutes. Since β is a morphism of complexes from $S^n(P)$ to B, we get $\delta_n^B \beta_n = 0$, and so $\mathrm{Im}(\beta_n) \subseteq \mathrm{Z}_n(B)$, which imply that $\beta_n : P \longrightarrow \mathrm{Z}_n(B)$ and

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{\beta_n} & \downarrow{\scriptstyle f} & & \\ \mathrm{Z}_n(B) & \longrightarrow & \mathrm{Z}_n(C) & \longrightarrow & 0 \end{array}$$

commutes. □

Lemma 2.7 *If $\mathcal{A}$ is closed under pure submodules and cokernels of pure monomorphisms, then $\widetilde{\mathcal{A}}$ is closed under pure subcomplexes and cokernels of pure monomorphisms.*

Proof Suppose $0 \to S \to C \to C/S \to 0$ is a pure exact sequence in $\mathcal{C}(R)$ with $C \in \widetilde{\mathcal{A}}$. Then $0 \to (C/S)^+ \to C^+ \to S^+ \to 0$ is split, and so S^+ and $(C/S)^+$ are exact, which implies that S and C/S are exact. By Lemma 2.6, $\mathrm{Z}_n(S)$ and $\mathrm{Z}_n(C/S)$ are in $\mathcal{A}$ for all $n \in \mathbb{Z}$. Therefore, S and C/S are in $\widetilde{\mathcal{A}}$. □

Next we prove that $(\widetilde{\mathcal{A}}, \mathrm{dg}\widetilde{\mathcal{B}})$ is complete under additional conditions. The method of proof is learned from [12, Proposition 4.9].

Theorem 2.8 *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in R-Mod. If $\mathcal{A}$ is closed under pure submodules and cokernels of pure monomorphisms, then the cotorsion pair $(\widetilde{\mathcal{A}}, \mathrm{dg}\widetilde{\mathcal{B}})$ is complete. Furthermore, it is perfect.*

Proof Suppose $G \in \widetilde{\mathcal{A}}$, and $|R| \leq \aleph$ for some infinite cardinal $\aleph$. We will show that G is equal to the union of a continuous chain $(P^\alpha)_{\alpha<\lambda}$ of pure subcomplexes of G with $|P^0| \leq \aleph$ and $|P^{\alpha+1}/P^\alpha| \leq \aleph$ for all α.

Set $T = \coprod_{n \in \mathbb{Z}} G_n$. We may well order the set T so that for some ordinal λ,

$$T = \{x_0, x_1, x_2, ..., x_\alpha, ...\}_{\alpha<\lambda}.$$

For x_0, use Lemma 2.4 to find a pure subcomplex $P^1 \subseteq G$ containing x_0 with $|P^1| \leq \aleph$. Then G/P^1 is in $\widetilde{\mathcal{A}}$ by Lemma 2.7. Now $\overline{x_1} \in G/P^1$. Therefore we can find a pure subcomplex $P^2/P^1 \subseteq G/P^1$ containing $\overline{x_1}$ such that $|P^2/P^1| \leq \aleph$. Then $(G/P^1)/(P^2/P^1) \cong G/P^2$ is in $\widetilde{\mathcal{A}}$. By Lemma 2.5, we get P^2 is pure. Note that $P^1 \subseteq P^2$ and x_0, $x_1 \in P^2$. In general, given any ordinal α, and having constructed pure subcomplexes $P^1 \subseteq P^2 \subseteq ... \subseteq P^\alpha$ where $x_\gamma \in P^\alpha$ for all $\gamma < \alpha$, we find a pure subcomplex $P^{\alpha+1} \subseteq G$ as follows: $\overline{x_\alpha} \in G/P^\alpha$, so by Lemma 2.4 we can find a pure subcomplex $P^{\alpha+1}/P^\alpha \subseteq G/P^\alpha$ containing $\overline{x_\alpha}$ such that $|P^{\alpha+1}/P^\alpha| \leq \aleph$. Thus $(G/P^\alpha)/(P^{\alpha+1}/P^\alpha) \cong G/P^{\alpha+1}$ is in $\widetilde{\mathcal{A}}$, whence $P^{\alpha+1}$ is pure. We now have $P^1 \subseteq P^2 \subseteq ... \subseteq P^\alpha \subseteq P^{\alpha+1}$ and $x_0, x_1, ..., x_\alpha \in P^{\alpha+1}$. For the case when α is a limit ordinal we just define $P^\alpha = \bigcup_{\gamma<\alpha} P^\gamma$. Then as we noted above, P^α is pure, and $x_\gamma \in P^\alpha$ for all $\gamma < \alpha$. This construction gives us the directed continuous chain $(P^\alpha)_{\alpha<\lambda}$.

If C is a complex such that $\mathrm{Ext}^1(P^0, C) = 0$ and $\mathrm{Ext}^1(P^{\alpha+1}/P^{\alpha}, C) = 0$ whenever $\alpha + 1 < \lambda$, then $\mathrm{Ext}^1(G, C) = 0$ by [12, Lemma 4.5]. Let X be a set of representatives of all complexes $C \in \widetilde{\mathcal{A}}$ with $|C| \leq \aleph$. Then $\widetilde{\mathcal{A}}^{\perp} = X^{\perp}$. That is, $(\widetilde{\mathcal{A}}, \mathrm{dg}\widetilde{\mathcal{B}})$ is cogenerated by X. Thus $(\widetilde{\mathcal{A}}, \mathrm{dg}\widetilde{\mathcal{B}})$ is complete.

Since $\mathcal{A}$ is closed under direct sums, $\mathcal{A}$ is closed under direct limits by [15, Corollary 1.2.7]. Thus the cotorsion pair $(\widetilde{\mathcal{A}}, \mathrm{dg}\widetilde{\mathcal{B}})$ is automatically perfect. □

According to [3], a module M is called Gorenstein flat if there exists an exact sequence in R-Mod

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow \cdots$$

of flat R-modules such that $M = \mathrm{Ker}(F_0 \to F_{-1})$ and that remains exact whenever $E \otimes -$ is applied for any injective right R-module E. Let $\mathcal{GF}$ denote the class of all Gorenstein flat left R-modules. In [7, Theorem 3.1.9] (also, see [10]), it was proven that over a right coherent ring $(\mathcal{GF}, \mathcal{GF}^{\perp})$ is a perfect and hereditary cotorsion pair. By Theorem 2.8, we get the following corollary.

Corollary 2.9 *Every complex over a right coherent ring has a* $\widetilde{\mathcal{GF}}$*-cover.*

Proof By [7, Corollary 2.1.9], we have that $\mathcal{GF}$ is closed under direct limits. Thus it is enough to prove that $\mathcal{GF}$ is closed under pure submodules and cokernels of pure monomorphisms. Suppose $0 \to P \to M \to M/P \to 0$ is pure exact in R-Mod with $M \in \mathcal{GF}$. Then $0 \to (M/P)^+ \to M^+ \to P^+ \to 0$ is split, and $M^+ \in \mathcal{GI}$ by [16, Theorem 3.6], where $\mathcal{GI}$ denotes the class of Gorenstein injective modules. Thus $(M/P)^+$ and P^+ are in $\mathcal{GI}$ by [16, Theorem 2.6], which implies that M/P and P are in $\mathcal{GF}$. □

The symbol $\mathcal{F}_n$ stands for the class of all left R-modules with flat dimension less than or equal to a fixed nonnegative integer n. In [19, Theorem 3.4], it was proven that $(\mathcal{F}_n, \mathcal{F}_n^{\perp})$ is a perfect and hereditary cotorsion pair. Note that $\mathcal{F}_n$ is closed under pure submodules, cokernels of pure monomorphisms and direct limits. Thus we have the following result.

Corollary 2.10 *Every complex has a* $\widetilde{\mathcal{F}_n}$*-cover.*

A left R-module M is called min-flat [18] if $\mathrm{Tor}_1(R/I, M) = 0$ for each simple right ideal I. Let $\mathcal{MF}$ denote the class of all min-flat left R-modules. In [18, Theorem 3.4], it was proven that $(\mathcal{MF}, \mathcal{MF}^{\perp})$ is a perfect cotorsion pair. Note that $\mathcal{MF}$ is closed under pure submodules, cokernels of pure monomorphisms and direct limits.

Corollary 2.11 *Every complex has a* $\widetilde{\mathcal{MF}}$*-cover.*

Remark 2.12 *It is well known that the class of modules closed under pure submodules and cokernels of pure monomorphisms is Kaplansky class (see [8, Definition 2.1] and [9, Proposition 3.2.2]). In [13], Gillespie has considered the completeness of the cotorsion pair* $(\widetilde{\mathcal{A}}, dg\widetilde{\mathcal{B}})$ *in the condition of Kaplansky classes in a locally* k*-presentable Grothendieck category. But Theorem 2.8 is not a particular case of Theorem 4.12 in [13]. For example, in general the cotorsion pair* $(\mathcal{MF}, \mathcal{MF}^{\perp})$ *is not hereditary. Thus* $\mathcal{MF}$ *does not satisfy condition 4 of Theorem 4.12 in [13].*

3. $\overline{\mathcal{A}}$-covers of complexes

Let $\mathcal{A}$ be the class of R-modules and $\overline{\mathcal{A}}$ denote the class of all complexes with each component in $\mathcal{A}$.

Lemma 3.1 *([12, Lemma 3.1]) Let $\mathscr{C}$ be abelian category, $\mathrm{Ch}(\mathscr{C})$ be the category of complexes on $\mathscr{C}$. For each object $C \in \mathscr{C}$ and $X,\ \ Y \in \mathrm{Ch}(\mathscr{C})$, we have the following isomorphisms.*

(1) $\mathrm{Ext}^1_{\mathscr{C}}(X_n, C) \cong \mathrm{Ext}^1_{\mathrm{Ch}(\mathscr{C})}(X, \mathrm{D}^{n+1}(C))$.

(2) $\mathrm{Ext}^1_{\mathscr{C}}(C, Y_n) \cong \mathrm{Ext}^1_{Ch(\mathscr{C})}(\mathrm{D}^n(C), Y)$.

Proposition 3.2 *Let C be a complex. Then C is in $\overline{\mathcal{A}}^{\perp}$ if and only if C_n is in $\mathcal{A}^{\perp}$ for all $n \in \mathbb{Z}$ and $\mathcal{H}om(G, C)$ is exact for any $G \in \overline{\mathcal{A}}$.*

Proof $\Rightarrow$) Suppose (C, δ) is in $\overline{\mathcal{A}}^{\perp}$. By Lemma 3.1, we have $\mathrm{Ext}^1(F, C_n) \cong \mathrm{Ext}^1(\mathrm{D}^n(F), C)$ for each $F \in \mathcal{A}$. But $\mathrm{Ext}^1(\mathrm{D}^n(F), C) = 0$, so $\mathrm{Ext}^1(F, C_n) = 0$. Therefore, C_n is in $\mathcal{A}^{\perp}$.

For any $G \in \overline{\mathcal{A}}$, $\mathcal{H}om(G, C)$ is exact if and only if for each n each map of complexes $f : G \to \Sigma^{-n}C$ is homotopic to 0 if and only if for each n and each map of complexes $f : G \to \Sigma^{-n}C$ the sequence $0 \longrightarrow \Sigma^{-n}C \longrightarrow \mathrm{M}(f) \longrightarrow \Sigma^{-1}G \longrightarrow 0$ splits if and only if for each n and each map of complexes $f : G \to \Sigma^{-n}C$ the sequence $0 \longrightarrow C \longrightarrow \Sigma^{-n}\mathrm{M}(f) \longrightarrow \Sigma^{n-1}G \longrightarrow 0$ splits where $\mathrm{M}(f)$ denotes the mapping cone of f. Since G is in $\overline{\mathcal{A}}$, $\Sigma^{n-1}G$ is also in $\overline{\mathcal{A}}$. By the hypothesis, $\mathrm{Ext}^1(\Sigma^{n-1}G, C) = 0$. So the sequence $0 \longrightarrow C \longrightarrow \Sigma^{-n}\mathrm{M}(f) \longrightarrow \Sigma^{n-1}G \longrightarrow 0$ splits, and so $\mathcal{H}om(G, C)$ is an exact complex.

$\Leftarrow$) Suppose C_n is in $\mathcal{A}^{\perp}$ for all $n \in \mathbb{Z}$ and $\mathcal{H}om(G, C)$ is exact for any $G \in \overline{\mathcal{A}}$. Any exact sequence $0 \longrightarrow C \longrightarrow W \longrightarrow G \longrightarrow 0$ of complexes with $G \in \overline{\mathcal{A}}$ splits at the module level. So this sequence is isomorphic to $0 \longrightarrow C \longrightarrow \mathrm{M}(f) \longrightarrow G \longrightarrow 0$, where $f : \Sigma^1 G \to C$ is a map of complexes. Since $\mathcal{H}om(\Sigma^1 G, C)$ is exact, the sequence $0 \longrightarrow C \longrightarrow \mathrm{M}(f) \longrightarrow G \longrightarrow 0$ splits in $\mathcal{C}(R)$ by [11, Lemma 2.3.2]. So $0 \longrightarrow C \longrightarrow W \longrightarrow G \longrightarrow 0$ also splits. □

Remark 3.3 *If ${}_RR \in \mathcal{A}$, $C \in \overline{\mathcal{A}}^{\perp}$, then C is exact by $\mathrm{H}^n(C) \cong \mathrm{Ext}^1(\underline{R}[1-n], C)$ for all $n \in \mathbb{Z}$.*

Proposition 3.4 *If (C, δ) is in $\overline{\mathcal{A}}^{\perp}$, then $\mathrm{Z}_n(C)$ is in $\mathcal{A}^{\perp}$ for all $n \in \mathbb{Z}$.*

Proof For any $F \in \mathcal{A}$, it is enough to prove that $\mathrm{Ext}^1(F, \mathrm{Z}_n(C)) = 0$. Consider the exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow F \longrightarrow 0$ with P a projective module. It yields an exact sequence of complexes

$$0 \longrightarrow \mathrm{S}^n(K) \longrightarrow \mathrm{S}^n(P) \longrightarrow \mathrm{S}^n(F) \longrightarrow 0.$$

By the hypothesis, $\mathrm{Ext}^1(\mathrm{S}^n(F), C) = 0$. So $\mathrm{Hom}(\mathrm{S}^n(P), C) \longrightarrow \mathrm{Hom}(\mathrm{S}^n(K), C) \longrightarrow 0$ is exact. Let $f : K \to \mathrm{Z}_n(C)$ be an R-homomorphism. We define $\alpha_n : K \to C_n$ as $\alpha_n = \lambda f$ where λ is the inclusion map and $\alpha_i = 0$ for $i \neq n$. In this way we obtain a map of complexes $\alpha : \mathrm{S}^n(K) \to C$. Then there exists $\beta : \mathrm{S}^n(P) \to C$ such that the diagram

$$\begin{array}{ccc} \mathrm{S}^n(K) & \longrightarrow & \mathrm{S}^n(P) \\ {\scriptstyle\alpha}\downarrow & \swarrow {\scriptstyle\beta} & \\ C & & \end{array}$$

commutes. Hence we have the commutative diagram

$$\begin{array}{ccc} K & \longrightarrow & P \\ {\scriptstyle \lambda f}\downarrow & \swarrow {\scriptstyle \beta_n} & \\ C_n & & \end{array}$$

Since β is a morphism of complexes from $S^n(P)$ to C, we obtain $\delta_n\beta_n = 0$, which implies that $\mathrm{Im}\beta_n \subseteq \mathrm{Z}_n(C)$. So we define $g : P \to \mathrm{Z}_n(C)$ as $g = \beta_n$. Thus $\mathrm{Hom}(P, Z_n(C)) \longrightarrow \mathrm{Hom}(K, \mathrm{Z}_n(C)) \longrightarrow 0$ is exact. On the other hand, we have an exact sequence $\mathrm{Hom}(P, \mathrm{Z}_n(C)) \longrightarrow \mathrm{Hom}(K, \mathrm{Z}_n(C)) \longrightarrow \mathrm{Ext}^1(F, \mathrm{Z}_n(C)) \longrightarrow 0$. Therefore, $\mathrm{Ext}^1(F, \mathrm{Z}_n(C)) = 0$. □

Lemma 3.5 *If G is in $\mathcal{A}^{\perp}$, then $\mathrm{D}^n(G)$ is in $\overline{\mathcal{A}}^{\perp}$ for all $n \in \mathbb{Z}$.*

Proof By Lemma 3.1, we have $\mathrm{Ext}^1(F_{n-1}, G) \cong \mathrm{Ext}^1(F, \mathrm{D}^n(G))$ for each $F \in \overline{\mathcal{A}}$. But $\mathrm{Ext}^1(F_{n-1}, G) = 0$, so $\mathrm{Ext}^1(F, \mathrm{D}^n(G)) = 0$. Therefore, $\mathrm{D}^n(G)$ is in $\mathcal{A}^{\perp}$. □

Proposition 3.6 *If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in R-Mod, then $(\overline{\mathcal{A}}, \overline{\mathcal{A}}^{\perp})$ is a cotorsion pair in $\mathcal{C}(R)$.*

Proof It follows from Proposition 3.2 in [14]. □

Lemma 3.7 *If $0 \to A \to B \to C \to 0$ is pure exact in $\mathcal{C}(R)$, then $0 \to A_n \to B_n \to C_n \to 0$ is pure exact in R-Mod for all $n \in \mathbb{Z}$.*

Proof Suppose P is a finitely presented module and $f : P \to C_n$. Then we have a commutative diagram

$$\begin{array}{ccccccccc} & & & & & & \mathrm{D}^n(P) & & \\ & & & & & {\scriptstyle \beta}\swarrow & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0, \end{array}$$

since $\mathrm{D}^n(P)$ is a finitely presented complex, where $\alpha : \mathrm{D}^n(P) \to C$ follows as

$$\begin{array}{rcccccccccc} \mathrm{D}^n(P) = & \cdots \longrightarrow & 0 & \longrightarrow & P & \longrightarrow & P & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \delta_n^C f} & & \downarrow & & \\ C = & \cdots \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & C_{n-2} & \longrightarrow & \cdots, \end{array}$$

Thus

$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle \beta_n}\swarrow & \downarrow{\scriptstyle f} & & \\ B_n & \longrightarrow & C_n & \longrightarrow & 0 \end{array}$$

commutes. That is, $0 \to A_n \to B_n \to C_n \to 0$ is pure. □

Lemma 3.8 *If $\mathcal{A}$ is closed under pure submodules and cokernels of pure monomorphisms, then $\overline{\mathcal{A}}$ is closed under pure subcomplexes and cokernels of pure monomorphisms.*

Proof It follows from Lemma 3.7. □

Based on the preceding results, we get the following theorem by analogy with the proof of Theorem 2.8.

Theorem 3.9 *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in R-Mod. If $\mathcal{A}$ is closed under pure submodules and cokernels of pure monomorphisms, then the cotorsion pair $(\overline{\mathcal{A}}, \overline{\mathcal{A}}^{\perp})$ is complete. Furthermore, it is perfect.*

In [11], García Rozas defined Gorenstein flat complexes and characterized such complexes over Gorenstein rings. A complex C is called Gorenstein flat if there exists an exact sequence of complexes $\cdots \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$ such that each F^i is flat, $C = \mathrm{Ker}(F^0 \to F^1)$ and the sequence remains exact when $I\overline{\otimes}-$ is applied to it for any injective complex I. It was proven that every complex over a commutative Gorenstein ring has a Gorenstein flat cover [11, Theorem 5.4.8]. We will show that the same result holds if R is a right coherent ring.

The following lemma is due to Yang [22, Theorem 5].

Lemma 3.10 *Let R be a right coherent ring, C a complex. Then C is Gorenstein flat if and only if C_n is Gorenstein flat in R-Mod for all $n \in \mathbb{Z}$.*

According to the above lemma, it is shown that over a right coherent ring the class of Gorenstein flat complexes coincides with $\overline{\mathcal{GF}}$. Thus we get the following corollary.

Corollary 3.11 *Every complex over a right coherent ring has a Gorenstein flat cover.*

According to [10, Theorem 2.12], all left modules over a right coherent ring have Gorenstein flat covers. Corollary 3.11 shows that the corresponding result holds in the category of complexes of R-modules, and generalizes Theorem 5.4.8 in [11].

Analogously, we have the following two corollaries.

Corollary 3.12 *Every complex has a $\overline{\mathcal{F}}_n$-cover.*

Corollary 3.13 *Every complex has a $\overline{\mathcal{MF}}$-cover.*

Acknowledgement

The authors would like to thank the referee for valuable suggestions and helpful corrections. This paper was supported by the National Natural Science Foundation of China (No. 11201377; 11261050).

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