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Slant submersions from almost product Riemannian manifolds

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Abstract: In this paper, we define the concept of almost product Riemannian submersion between almost product Riemannian manifolds. We introduce slant submersions from almost product Riemannian manifolds onto Riemannian manifolds. We give examples and investigate the geometry of foliations that arise from the definition of a Riemannian submersion. We also find necessary and sufficient conditions for a slant submersion to be totally geodesic.

Key words: Riemannian submersion, almost product Riemannian submersion, almost product Riemannian manifold, slant submersion

1. Introduction

Given a $C^\infty-$submersion π from a Riemannian manifold (M, g) onto a Riemannian manifold (B, g'), there are several kinds of submersions according to the conditions on it: e.g., Riemannian submersion ([8], [14]), slant submersion ([15],[16]), almost Hermitian submersion [18], or quaternionic submersion [10]. As we know, Riemannian submersions are related to physics and have their applications in the Yang–Mills theory ([4],[19]), Kaluza–Klein theory ([3],[11]), supergravity and superstring theories ([12],[13]), etc. On the other hand, the geometry of slant submanifolds was initiated by B.Y. Chen as a generalization of both holomorphic and totally real submanifolds in complex geometry ([5],[6]). Slant submanifolds of almost product manifolds were studied in [17] and [1].

Riemannian submersions between almost Hermitian manifolds were studied by Watson in [18] under the name of holomorphic submersions. One of the main results of this notion is that vertical and horizontal distributions are invariant under almost complex structure. He showed that if the total manifold is a Kähler manifold, the base manifold is also a Kähler manifold. Recently, Şahin [16] introduced slant submersions from almost Hermitian manifolds to Riemannian manifolds. He showed that the geometry of slant submersions is quite different from holomorphic submersions. Indeed, although every holomorphic submersion is harmonic, slant submersions may not be harmonic. The paper is organized as follows. In Section 2 we recall some notions needed for this paper. In section 3 we introduce the notion of almost product Riemannian submersions. We obtain that if M is a locally product Riemannian manifold, then B is also a locally product manifold. In section 4, we give the definition of slant Riemannian submersions and provide examples. We also investigate the geometry of leaves of the distributions. Finally, we give necessary and sufficient conditions for such submersions to be totally geodesic.

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2. Preliminaries

In this section, we define almost product Riemannian manifolds, recall the notion of Riemannian submersions between Riemannian manifolds, and give a brief review of basic facts of Riemannian submersions.

Let M be an m-dimensional manifold with a tensor F of type $(1,1)$ such that

$$F^2 = I, (F \neq I).$$

Then, we say that M is an almost product manifold with almost product structure F. We put

$$P = \frac{1}{2}(I+F), \quad Q = \frac{1}{2}(I-F).$$

Then we get

$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0, \quad F = P - Q.$$

Thus P and Q define 2 complementary distributions P and Q. We easily see that the eigenvalues of F are $+1$ or -1.

If an almost product manifold M admits a Riemannian metric g such that

$$g(FX, FY) = g(X, Y) \tag{1}$$

for any vector fields X and Y on M, then M is called an almost product Riemannian manifold, denoted by (M, g, F).

Denote the Levi–Civita connection on M with respect to g by ∇. Then, M is called a locally product Riemannian manifold if F is parallel with respect to ∇, i.e.

$$\nabla_X F = 0, X \in \Gamma(TM)[20].$$

Let (M, g) and (B, g') be 2 Riemannian manifolds. A surjective $C^\infty-$map $\pi : M \to B$ is a $C^\infty-$submersion if it has maximal rank at any point of M. Putting $\mathcal{V}_x = ker\pi_{*x}$, for any $x \in M$, we obtain an integrable distribution $\mathcal{V}$, which is called vertical distribution and corresponds to the foliation of M determined by the fibers of π. The complementary distribution $\mathcal{H}$ of $\mathcal{V}$, determined by the Riemannian metric g, is called horizontal distribution. A $C^\infty-$submersion $\pi : M \to B$ between 2 Riemannian manifolds (M, g) and (B, g') is called a Riemannian submersion if, at each point x of M, π_{*x} preserves the length of the horizontal vectors. A horizontal vector field X on M is said to be basic if X is $\pi-$related to a vector field X' on B. It is clear that every vector field X' on B has a unique horizontal lift X to M and X is basic.

We recall that the sections of $\mathcal{V}$, respectively $\mathcal{H}$, are called the vertical vector fields, respectively horizontal vector fields. A Riemannian submersion $\pi : M \to B$ determines 2 $(1,2)$ tensor fields T and A on M, by the following formulas:

$$T(E, F) = T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF \tag{2}$$

and

$$A(E, F) = A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF \tag{3}$$

for any $E, F \in \Gamma(TM)$, where v and h are the vertical and horizontal projections (see [7]). From (2) and (3), one can obtain

$$\nabla_U W = T_U W + \hat{\nabla}_U W; \tag{4}$$

$$\nabla_U X = T_U X + h(\nabla_U X); \tag{5}$$

$$\nabla_X U = v(\nabla_X U) + A_X U; \tag{6}$$

$$\nabla_X Y = A_X Y + h(\nabla_X Y), \tag{7}$$

for any $X, Y \in \Gamma((ker\pi_*)^\perp)$, $U, W \in \Gamma(ker\pi_*)$. Moreover, if X is basic then

$$h(\nabla_U X) = h(\nabla_X U) = A_X U. \tag{8}$$

We note that for $U, V \in \Gamma(ker\pi_*)$, $T_U V$ coincides with the second fundamental form of the immersion of the fiber submanifolds and for $X, Y \in \Gamma((ker\pi_*)^\perp)$, $A_X Y = \frac{1}{2}v[X, Y]$ reflecting the complete integrability of the horizontal distribution $\mathcal{H}$. It is known that A is alternating on the horizontal distribution: $A_X Y = -A_Y X$, for $X, Y \in \Gamma((ker\pi_*)^\perp)$, and T is symmetric on the vertical distribution: $T_U V = T_V U$, for $U, V \in \Gamma(ker\pi_*)$.

We now recall the following result which will be useful for later.

Lemma 2.1 *(See [7],[14]). If $\pi : M \to B$ is a Riemannian submersion and X, Y basic vector fields on M, $\pi-$related to X' and Y' on B, then we have the following properties:*

1. *$h[X, Y]$ is a basic vector field and $\pi_* h[X, Y] = [X', Y'] \circ \pi$;*
2. *$h(\nabla_X Y)$ is a basic vector field $\pi-$related to $(\nabla'_{X'} Y')$, where ∇ and ∇' are the Levi–Civita connection on M and B;*
3. *$[E, U] \in \Gamma(ker\pi_*)$, for any $U \in \Gamma(ker\pi_*)$ and for any basic vector field E.*

Let (M, g_M) and (N, g_N) be Riemannian manifolds and $\pi : M \to N$ is a smooth map. Then the second fundamental form of π is given by

$$(\nabla\pi_*)(X, Y) = \nabla_{\pi_* X}\pi_* Y - \pi_*(\nabla_X Y) \tag{9}$$

for $X, Y \in \Gamma(TM)$, where we denote conveniently by ∇ the Levi–Civita connections of the metrics g_M and g_N. Recall that π is said to be harmonic if $trace(\nabla\pi_*) = 0$ and π is called a totally geodesic map if $(\nabla\pi_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$ [2]. It is known that the second fundamental form is symmetric.

3. Almost product Riemannian submersions

In this section, we define the notion of almost product Riemannian submersion. We now define the almost product map, which is similar to the notion of almost complex map between 2 almost Hermitian manifolds. The results given in this section can be found in [9].

Definition 3.1 *Let M and B be almost product Riemannian manifolds with almost product structures F and F', respectively. A mapping $\pi : M \to B$ is said to be an almost product map if $\pi_* \circ F = F' \circ \pi_*$. By using the above definition, we are ready to give the following notion.*

Definition 3.2 *Let (M, F, g) and (B, F', g') be almost product Riemannian manifolds. A Riemannian submersion $\pi : M \to B$ is called an almost product Riemannian submersion if π is an almost product map, i.e. $\pi_* \circ F = F' \circ \pi_*$.*

By using the almost product map, we have the following result.

Proposition 3.1 *Let $\pi : (M, F, g) \to (B, F', g')$ be an almost product Riemannian submersion from an almost product manifold M onto an almost product manifold B, and let X be a basic vector field on M, $\pi-$related to X' on B. Then, FX is also a basic vector field $\pi-$related to $F'X'$.*

The next proposition shows that an almost product submersion puts some restrictions on the distributions $\mathcal{V}$ and $\mathcal{H}$.

Proposition 3.2 *Let $\pi : (M, F, g) \to (B, F', g')$ be an almost product Riemannian submersion from an almost product manifold M onto an almost product manifold B. Then, the horizontal and vertical distributions are $F-$ invariant.*

Proof For any vertical vector field U, we have $\pi_*(FU) = F'(\pi_* U) = 0$, and thus FU is vertical. Obviously, for any horizontal vector field X and any vertical vector field U, we get $g(FX, U) = g(X, FU) = 0$, which implies that FX is horizontal.

In the sequel, we show that the base manifold is a locally product manifold if the total manifold is a locally product manifold. □

Theorem 3.1 *Let (M, F, g) be a locally product manifold and (B, F', g') be an almost product manifold. Suppose that $\pi : (M, F, g) \to (B, F', g')$ be an almost product Riemannian submersion. Then (B, F', g') is a locally product Riemannian manifold.*

Proof For $X', Y' \in \Gamma(TB)$ such that $\pi_* X = X', \pi_* Y = Y'$, where $X, Y \in \Gamma(TM)$, since M is a locally product manifold, for $X, Y \in \Gamma(\mathcal{H})$, we have

$$0 = (\nabla_X F)Y = \nabla_X FY - F\nabla_X Y.$$

Then, by using $\pi_* F = F'\pi_*$, we get

$$\pi_*((\nabla_X F)Y) = \pi_*(\nabla_X FY) - F'\pi_*(\nabla_X Y).$$

On the other hand, from Proposition 3.1, we know that if X is $\pi-$related to X', then FX is $\pi-$related to $F'X'$. Also, from Lemma 2.1, it follows that $h(\nabla_X FY)$ and $h(\nabla_X Y)$ are $\pi-$related to $\nabla'_{X'} F'Y'$ and $\nabla'_{X'} Y'$. Thus, we have

$$\pi_*((\nabla_X F)Y) = \nabla'_{X'} F'Y' - F'\nabla'_{X'} Y'.$$

Hence

$$\pi_*((\nabla_X F)Y) = (\nabla'_{X'} F')Y' = 0,$$

which proves the assertion. □

As the fibers of an almost product submersion are an invariant submanifold of M with respect to F, we have the following.

Corollary 3.1 *Let $\pi : (M, F, g) \to (B, F', g')$ be an almost product submersion from a locally product Riemannian manifold M onto an almost product manifold B. Then, the fibers are locally product manifolds.*

4. Slant submersions

Definition 4.1 *Let π be a Riemannian submersion from an almost product Riemannian manifold (M_1, g_1, F_1) onto a Riemannian manifold (M_2, g_2). If for any nonzero vector $X \in (ker\pi_*)$; $p \in M_1$, the angle $\theta(X)$ between FX and the space $(ker\pi_*)$ is a constant, i.e. it is independent of the choice of the point $p \in M_1$ and choice of the tangent vector X in $(ker\pi_*)$, then we say that π is a slant submersion. In this case, the angle θ is called the slant angle of the slant submersion.*

It is known that the distribution $(ker\pi_*)$ is integrable for a Riemannian submersion between Riemannian manifolds. In fact, its leaves are $\pi^{-1}(p)$, $p \in M_1$, i.e. fibers. Thus, it follows from the above definition that the fibers of a slant submersion are slant submanifolds of M_1, for slant submanifolds (see [17]).

We first give some examples of slant submersions.

Example 4.1 *Define a map $\pi : R^4 \to R^2$ by*

$$\pi(x_1, x_2, x_3, x_4) = (\frac{x_1 + x_2}{\sqrt{2}}, \frac{x_3 + x_4}{\sqrt{2}}).$$

Then, the kernel of π_* is

$$\mathcal{V} = ker\pi_* = Span\{V_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, V_2 = -\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\},$$

and the horizontal distribution is spanned by

$$\mathcal{H} = (ker\pi_*)^{\perp} = Span\{X = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, Y = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\}.$$

Hence, we have

$$g(X, X) = g(Y, Y) = 2, g'(\pi_* X, \pi_* X) = g'(\pi_* Y, \pi_* Y) = 2.$$

Thus, π is a Riemannnian submersion. Moreover, we can easily obtain that π satisfies

$$\pi_* FX = F'\pi_* X$$

and

$$\pi_* FY = F'\pi_* Y.$$

Then, π is an almost product Riemannian submersion.

Thus the map π is a slant submersion with slant angle $\theta = 0$.

Example 4.2 *Every antiinvariant Riemannian submersion from an almost product Riemannian manifold onto a Riemannian manifold is a slant submersion with $\theta = \frac{\pi}{2}$.*

Example 4.3 *Consider the following Riemannian submersion given by*

$$\begin{aligned} \pi : R^4 &\to R^2 \\ (x_1, ..., x_4) &\to (\frac{x_1 - x_2}{\sqrt{2}}, x_4). \end{aligned}$$

Then π is a slant submersion with slant angle $\theta = \frac{\pi}{4}$.

Example 4.4 *Define a map* $\pi : R^4 \to R^2$ *by*

$$\pi(x_1, ..., x_4) = (x_2, x_1 \sin\alpha - x_4 \cos\alpha),$$

where $0 < \alpha < \frac{\pi}{2}$. *Then the map* π *is a slant submersion with the slant angle* $\theta = \alpha$.

Example 4.5 *Define a map* $\pi : R^4 \to R^2$ *by*

$$\pi(x_1, ..., x_4) = (x_1 \cos\alpha - x_2 \sin\alpha, x_3 \sin\beta - x_4 \cos\beta).$$

Then the map π *is a slant submersion with the slant angle* θ *with* $\cos\theta = |\sin(\alpha + \beta)|$.

Let π be a Riemannian submersion from an almost product Riemannian manifold M_1 with the structure (g_1, F) onto a Riemannian manifold (M_2, g_2). Then for $X \in \Gamma(ker\pi_*)$, we write

$$FX = \phi X + \omega X, \tag{10}$$

where ϕX and ωX are vertical and horizontal parts of FX. From Eqs. (1) and (10), one can easily see that

$$g_1(X, \phi Y) = g_1(\phi X, Y), \tag{11}$$

for any $X, Y \in \Gamma(ker\pi_*)$.

Also, for $Z \in \Gamma((ker\pi_*)^\perp)$, we have

$$FZ = BZ + CZ, \tag{12}$$

where BZ and CZ are vertical and horizontal component of FZ. From Eqs. (1) and (12), one can easily see that

$$g_1(Z_1, CZ_2) = g_1(CZ_1, Z_2), \tag{13}$$

for any $Z_1, Z_2 \in \Gamma((ker\pi_*)^\perp)$.

We define the covariant derivatives of ϕ and ω as follows:

$$(\nabla_X \phi)Y = \hat{\nabla}_X \phi Y - \phi \hat{\nabla}_X Y \tag{14}$$

and

$$(\nabla_X \omega)Y = h\nabla_X \omega Y - \omega \hat{\nabla}_X Y \tag{15}$$

for $X, Y \in \Gamma(ker\pi_*)$, where $\hat{\nabla}_X Y = v\nabla_X Y$. Then we easily have

Lemma 4.1 *Let* (M_1, g_1, F) *be a locally product Riemannian manifold and* (M_2, g_2) *a Riemannian manifold. Let* $\pi : (M_1, g_1, F) \to (M_2, g_2)$ *be a slant submersion. Then we get*

$$\begin{aligned} \hat{\nabla}_X \phi Y + T_X \omega Y &= \phi \hat{\nabla}_X Y + BT_X Y \\ T_X \phi Y + h\nabla_X \omega Y &= \omega \hat{\nabla}_X Y + CT_X Y \end{aligned}$$

for any $X, Y \in \Gamma(ker\pi_*)$.

Let π be a slant submersion from an almost product Riemannian manifold (M_1, g_1, F_1) onto a Riemannian manifold (M_2, g_2) with the slant angle $\theta \in (0, \frac{\pi}{2})$; then we say that ω is parallel with respect to the Levi–Civita connection ∇ on $(ker\pi_*)$ if its covariant derivative with respect to ∇ vanishes, i.e. we have

$$(\nabla_X \omega)Y = h\nabla_X \omega Y - \omega \hat{\nabla}_X Y = 0 \tag{16}$$

for $X, Y \in \Gamma(ker\pi_*)$.

Invariant and antiinvariant submanifolds are particular classes of slant submanifolds with slant angles $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant submanifold that is neither an invariant nor antiinvariant submanifold is called a proper slant submanifold([1]).

Theorem 4.1 *Let π be a Riemannian submersion from an almost product Riemannian manifold (M_1, g_1, F) onto a Riemannian manifold (M_2, g_2). Then π is a proper slant submersion if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$\phi^2 X = \lambda X$$

for $X \in \Gamma(ker\pi_*)$. If π is a proper slant submersion, then $\lambda = \cos^2\theta$.

Proof For any nonzero $X \in \Gamma(ker\pi_*)$, we can write

$$\cos\theta(X) = \frac{\|\phi X\|}{\|FX\|}, \tag{17}$$

where $\theta(X)$ is the slant angle. By using Eqs. (11), (17), and (1), we get

$$\begin{aligned} g_1(\phi^2 X, X) &= g_1(\phi X, \phi X) \\ &= \cos^2\theta(X) g_1(FX, FX) \\ &= \cos^2\theta(X) g_1(X, X) \end{aligned} \tag{18}$$

for all $X \in \Gamma(ker\pi_*)$. Since g_1 is Riemannian metric, from Eq. (18) we have

$$\phi^2 X = \cos^2\theta(X) X, \quad X \in \Gamma(ker\pi_*). \tag{19}$$

Let $\lambda = \cos^2\theta$. Then it is obvious that $\lambda \in [0, 1]$.

Conversely, let us assume that there exists a constant $\lambda \in [0, 1]$ such that $\phi^2 = \lambda I$ is satisfied. From Eqs. (10), (11), and (1) we get

$$\begin{aligned} \cos\theta(X) &= \frac{g_1(FX, \phi X)}{\|FX\|\|\phi X\|} \\ &= \frac{\lambda g_1(FX, FX)}{\|FX\|\|\phi X\|}, \end{aligned}$$

for all $X \in \Gamma(ker\pi_*)$. Thus we have

$$cos\theta(X) = \frac{\lambda\|FX\|}{\|\phi X\|}.$$

Since $cos\theta(X) = \frac{\|\phi X\|}{\|FX\|}$, then by using the last equation we obtain $cos^2\theta(X) = \lambda$, which implies that $\theta(X)$ is a constant and π is a proper slant submersion. If π is a proper slant submersion, then $\lambda = \cos^2\theta$. □

From Theorem 4.1 and Eq. (10) we have the following result.

Lemma 4.2 *Let π be a slant submersion from an almost product Riemannian manifold (M_1, g_1, F_1) onto a Riemannian manifold (M_2, g_2) with slant angle $\theta \in (0, \frac{\pi}{2})$. Then, for any $X, Y \in \Gamma(ker\pi_*)$, we have*

$$g_1(\phi X, \phi Y) = \cos^2\theta g_1(X, Y) \tag{20}$$

$$g_1(\omega X, \omega Y) = \sin^2\theta g_1(X, Y). \tag{21}$$

Proposition 4.1 *Let π be a slant submersion from a locally product Riemannian manifold onto a Riemannian manifold with the slant angle $\theta \in (0, \frac{\pi}{2})$. If ω is parallel with respect to ∇ on $(ker\pi_*)$, then we have*

$$T_{\phi X}\phi X = \cos^2\theta T_X X \tag{22}$$

for $X \in (ker\pi*)$.

Proof If ω is parallel, then from Lemma 4.1 we have $CT_X Y = T_X\phi Y$ for $X, Y \in (ker\pi*)$. Interchanging the role of X and Y, we get $CT_Y X = T_Y\phi X$. Thus we have

$$CT_X Y - CT_Y X = T_X\phi Y - T_Y\phi X.$$

Since T is symmetric, we derive $T_X\phi Y = T_Y\phi X$. Then substituting Y by ϕX, we get $T_X\phi^2 X = T_{\phi X}\phi X$. Finally, using Theorem 4.1, we obtain Eq. (22). □

We now investigate the geometry of the leaves of distributions $(ker\pi_*)$ and $(ker\pi_*)^\perp$.

Theorem 4.2 *Let π be a slant submersion from a locally product Riemannian manifold (M_1, g_1, F_1) onto a Riemannian manifold (M_2, g_2) with slant angle $\theta \in (0, \frac{\pi}{2})$. Then the distribution $(ker\pi_*)$ defines a totally geodesic foliation on M_1 if and only if*

$$g_1(h\nabla_X\omega\phi Y, Z) = -g_1(h\nabla_X\omega Y, CZ) - g_1(T_X\omega Y, BZ)$$

for $X, Y \in \Gamma(ker\pi_*)$ and $Z \in \Gamma((ker\pi_*)^\perp)$.

Proof For $X, Y \in \Gamma(ker\pi_*)$ and $Z \in \Gamma((ker\pi_*)^\perp)$, from Eqs. (1) and (10) we have

$$g_1(\nabla_X Y, Z) = g_1(\nabla_X\phi Y, FZ) + g_1(\nabla_X\omega Y, FZ).$$

Using Eqs. (1),(10), and (12) we get

$$\begin{aligned} g_1(\nabla_X Y, Z) &= g_1(\nabla_X\phi^2 Y, Z) + g_1(\nabla_X\omega\phi Y, Z) \\ &+ g_1(\nabla_X\omega Y, BZ) + g_1(\nabla_X\omega Y, CZ). \end{aligned}$$

Then from Eq. (5) and Theorem 4.1 we obtain

$$\begin{aligned} g_1(\nabla_X Y, Z) &= \cos^2\theta g_1(\nabla_X Y, Z) + g_1(h\nabla_X\omega\phi Y, Z) \\ &+ g_1(T_X\omega Y, BZ) + g_1(h\nabla_X\omega Y, CZ). \end{aligned}$$

Hence, we have

$$\begin{aligned}\sin^2\theta g_1(\nabla_X Y, Z) &= g_1(h\nabla_X \omega\phi Y, Z) \\ &+ g_1(T_X\omega Y, BZ) + g_1(h\nabla_X \omega Y, CZ),\end{aligned}$$

which proves the assertion. □

Theorem 4.3 *Let π be a slant submersion from a locally product Riemannian manifold (M_1, g_1, F_1) onto a Riemannian manifold (M_2, g_2) with slant angle $\theta \in (0, \frac{\pi}{2})$. Then the following conditions are equivalent:*

(a) the distribution $((ker\pi_*)^\perp)$ defines a totally geodesic foliation on M_1,

(a) $g_1(h\nabla_{Z_1} Z_2, \omega\phi X) = -g_1(h\nabla_{Z_1} CZ_2 + A_{Z_1} BZ_2, \omega X)$

for $X \in \Gamma(ker\pi_*)$ and $Z_1, Z_2 \in \Gamma((ker\pi_*)^\perp)$.

Proof For $X \in \Gamma(ker\pi_*)$ and $Z_1, Z_2 \in \Gamma((ker\pi_*)^\perp)$, we have

$$\begin{aligned}g_1(\nabla_{Z_1} Z_2, X) &= g_1(\nabla_{Z_1} FZ_2, FX) \\ &= g_1(\nabla_{Z_1} FZ_2, \phi X) + g_1(\nabla_{Z_1} FZ_2, \omega X) \\ &= \cos^2\theta g_1(\nabla_{Z_1} Z_2, X) + g_1(\nabla_{Z_1} Z_2, \omega\phi X) \\ &+ g_1(h\nabla_{Z_1} CZ_2, \omega X) + g_1(A_{Z_1} BZ_2, \omega X)\end{aligned}$$

so that

$$\begin{aligned}\sin^2\theta g_1(\nabla_{Z_1} Z_2, X) &= g_1(\nabla_{Z_1} Z_2, \omega\phi X) \\ &+ g_1(h\nabla_{Z_1} CZ_2 + A_{Z_1} BZ_2, \omega X).\end{aligned}$$

Hence, we get $(a) \Leftrightarrow (b)$.

Finally we give necessary and sufficient conditions for a slant submersion with slant angle $\theta \in (0, \frac{\pi}{2})$ to be totally geodesic. Recall that a differentiable map π between Riemannian manifolds (M_1, g_1) and (M_2, g_2) is called a totally geodesic map if $(\nabla\pi_*)(X, Y) = 0$ for all $X, Y \in \Gamma(TM_1)$. □

Theorem 4.4 *Let π be a slant submersion from a locally product Riemannian manifold (M_1, g_1, F_1) onto a Riemannian manifold (M_2, g_2) with slant angle $\theta \in (0, \frac{\pi}{2})$. Then π is totally geodesic if and only if*

$$g_1(h\nabla_X \omega\phi Y, Z) = -g_1(h\nabla_X \omega Y, CZ) - g_1(T_X\omega Y, CZ)$$

and

$$g_1(h\nabla_{Z_1} \omega\phi X, Z_2) = g_1(A_{Z_1} BZ_2 + h\nabla_{Z_1} CZ_2, \omega X)$$

for $Z, Z_1, Z_2 \in \Gamma((ker\pi_*)^\perp)$ and $X, Y \in \Gamma(ker\pi_*)$.

Proof First of all, since π is a Riemannian submersion, we have

$$(\nabla\pi_*)(Z_1, Z_2) = 0$$

for $Z_1, Z_2 \in \Gamma((ker\pi_*)^\perp)$.

For $X, Y \in \Gamma(ker\pi_*)$ and $Z, Z_1, Z_2 \in \Gamma((ker\pi_*)^\perp)$, from Eqs. (1) and (10) we have

$$g_2((\nabla\pi_*)(X, Y), \pi_* Z) = -g_1(\nabla_X F\phi Y, Z) - g_1(\nabla_X \omega Y, FZ).$$

Using Eqs. (10) and (12) we get

$$\begin{aligned} g_2((\nabla\pi_*)(X, Y), \pi_* Z) &= -g_1(\nabla_X \phi^2 Y, Z) - g_1(\nabla_X \omega\phi Y, Z) \\ &- g_1(\nabla_X \omega Y, BZ) - g_1(\nabla_X \omega Y, CZ). \end{aligned}$$

Then Theorem 3.1 and Eqs. (4) and (5) imply that

$$\begin{aligned} g_2((\nabla\pi_*)(X, Y), \pi_* Z) &= -\cos^2\theta g_1(\nabla_X Y, Z) - g_1(h\nabla_X \omega\phi Y, Z) \\ &- g_1(T_X \omega Y, BZ) - g_1(h\nabla_X \omega Y, CZ). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \sin^2\theta g_2((\nabla\pi_*)(X, Y), \pi_* Z) &= -g_1(h\nabla_X \omega\phi Y, Z) - g_1(T_X \omega Y, BZ) \\ &- g_1(h\nabla_X \omega Y, CZ). \end{aligned} \tag{23}$$

Similarly, we get

$$\begin{aligned} \sin^2\theta g_2((\nabla\pi_*)(X, Z_1), \pi_* Z_2) &= g_1(A_{Z_1} BZ_2 + h\nabla_{Z_1} CZ_2, \omega X) \\ &- g_1(h\nabla_{Z_1} \omega\phi X, Z_2). \end{aligned} \tag{24}$$

Then the proof follows from Eqs. (23) and (24). □

Remark The geometry of almost product submersions is different from slant submersions defined on almost product manifolds. For instance, the fibers of almost product submersions are almost product submanifolds, but the fibers of slant submersions are slant submanifolds of almost product manifolds.

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