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Remarks on the paper "On some new inequalities for convex functions" by M. Tunç

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Abstract: In this note, we slightly generalize Theorem 2 in the paper by M. Tunç and point out that the assumption of Theorem 3 is not sufficient.

A misuse of the term 'mean' is also discussed.

Key words: Convex function, mean

In the paper [3] the author proves the following theorem:

Theorem 1 (Theorem 2, [3]) *If $f,g:[a,b]\to\mathbb{R}$ are convex, then*

$$\begin{aligned}&\frac{1}{(b-a)^2}\int_a^b (b-x)[f(a)g(x)+f(x)g(a)]dx\\ &+\frac{1}{(b-a)^2}\int_a^b (x-a)[f(b)g(x)+f(x)g(b)]dx\\ &\le \frac{M(a,b)}{3}+\frac{N(a,b)}{6}+\frac{1}{b-a}\int_a^b f(x)g(x)dx,\end{aligned}\tag{1}$$

where $M(a,b)=f(a)g(a)+f(b)g(b), N(a,b)=f(a)g(b)+f(b)g(a)$.

In fact, this theorem can be restated as follows:

Theorem 2 *If $f,g:[a,b]\to\mathbb{R}$ are of the same convexity (i.e. both convex or both concave), then* (1) *holds. If f and g are of opposite convexity, then* (1) *is reversed.*

Proof Since for $a<x<b$ we have

$$x=\frac{b-x}{b-a}a+\frac{x-a}{b-a}b,$$

the inequality

$$\left(f(x)-\frac{b-x}{b-a}f(a)-\frac{x-a}{b-a}f(b)\right)\left(g(x)-\frac{b-x}{b-a}g(a)-\frac{x-a}{b-a}g(b)\right)\ge 0\tag{2}$$

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holds if f and g are of the same convexity, else (2) is reversed.

Integrating the above inequality over the interval $[a,b]$, we obtain the desired result. □

Theorem 3 in [3] requires correction.

Theorem 3 *Let $f,g:[a,b]\to\mathbb{M}$ be convex, nonnegative functions. Then*

$$\frac{1}{b-a}\int_a^b \left(f(\tfrac{a+b}{2})g(x)+g(\tfrac{a+b}{2})f(x)\right)dx \quad\leq\quad \frac{1}{2(b-a)}\int_a^b f(x)g(x)dx \;+\; \frac{M}{12} \;+\; \frac{N}{6} \;+\; f(\tfrac{a+b}{2})g(\tfrac{a+b}{2}), \quad (3)$$

(M and N being as in Theorem 1).

The original version does not contain the nonnegativity assumption, but then it is easy to produce a counterexample: let g be convex and $f(x)\equiv -1$. Then the inequality (3) becomes

$$\frac{1}{b-a}\int_a^b g(x)dx \geq \frac{g(a)+g(b)}{2}$$

- obviously opposite to the right-hand side of the Hermite-Hadamard inequality.

As Theorem 3 is not valid in the general case, we cannot trust Proposition 5 in [3] in the case $a,b<0$ (especially because x^n is not convex in the interval $(-\infty,0)$ for odd n).

We feel obliged to comment on the use of the term 'mean' in section 3: in the mathematical literature the word 'mean' denotes a function taking values between the extremities of its argument(s). The attempt to extend the definition of the geometric, arithmetic, logarithmic, and generalised logarithmic means is only partially successful.

$$\begin{aligned} A(a,b) &= \frac{a+b}{2},\ G(a,b)=\sqrt{ab},\ L(a,b)=\frac{b-a}{\ln|b|-\ln|a|}, \\ L_n(a,b) &= \left(\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right)^{\frac{1}{n}},\ K(a,b)=\sqrt{\frac{a^2+b^2}{2}} \end{aligned} \qquad (4)$$

define means for positive a,b. Clearly, the expressions above make sense for some other arguments, but usually their values do not lie between the arguments: $G(-1,-4)=2$, $L(e^2,-e)=e^2+e$, $\lim_{a\to-\infty} L_2(a,1)=\infty$ etc., thus calling them 'means' should be regarded as a mistake.

Tracing back the cited literature for the source of this misuse, we see that the process started in the paper by Dragomir and Agarwal ([1]), the mistake was reproduced by Kirmaci ([2]) and, consequently, by Tunç.

It is worth noting that the extended logarithmic means L_n and power means $M_n(a,b)=A(a^n,b^n)^{1/n}$ can be extended to the real line in the case of positive real exponents. To this end, let $f_n(x)=\mathrm{sgn}(x)|x|^n$. Then f_n is a strictly increasing, odd function and we can define

$$L_n(a,b)=f_n^{-1}\left(\frac{\int_a^b f_n(t)dt}{b-a}\right) \quad\text{and}\quad M_n(a,b)=f_n^{-1}\left(\frac{f_n(a)+f_n(b)}{2}\right).$$

Both definitions match the original ones for positive arguments and define means (in the case of odd natural n, L_n matches the original definition). Unfortunately the power functions (with the exception of $n=1$) cannot be extended to a bijection preserving the convexity.

Clearly this method cannot be applied to negative exponents.

References

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