Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2013) 37: 739 – 746
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doi:10.3906/mat-1111-43

Radical operations on the multiplicative lattice

Esra ŞENGELEN SEVİM*
İstanbul Bilgi University, Department of Mathematics, Dolapdere, İstanbul, Turkey

Received: 29.11.2011 • **Accepted:** 18.07.2012 • **Published Online:** 26.08.2013 • **Printed:** 23.09.2013

Abstract: The purpose of this paper is to introduce interesting and useful properties of quasi-radical and radical operations on the elements of a multiplicative lattice.

Key words: Multiplicative lattice, radical operations, quasi-radical operations

1. Introduction

By a multiplicative lattice, we mean a complete lattice L, with least element 0 and compact greatest element I, on which there is defined a commutative, associative, completely join distributive product for which I is a multiplicative identity. Multiplicative lattices have been studied extensively by E. W. Johnson and C. Jayaram, see [2-7].

Throughout this paper, L denotes a multiplicative lattice. An element $a \in L$ is said to be proper if $a < I$. An element $p < I$ in L is said to be prime if $ab \leq p$ implies $a \leq p$ or $b \leq p$. We denote the set of prime elements in L by $Spec(L)$. An element $I^* < I$ in L is said to be maximal if $I^* < x \leq I$ implies $x = I$. It is easily seen that maximal elements are prime.

If a is an element of a multiplicative lattice L, we define

$$\sqrt{a} = \bigvee \{t \in L | t^n \leq a \text{ for some natural number n}\}.$$

In this paper we explain the concept of an operation F on the elements of a multiplicative lattice L and define the F-radical of an element. We shall also define the concepts of F-radical and F-prime elements, as well as the F-prime spectrum of the multiplicative lattice. We will also state some natural properties in Relations 2.1. Moreover, we explain the concept of quasi-radical operations on the elements of a multiplicative lattice. Quasi-radical operations have been studied for commutative rings with identity by A. Benhissi, M. Rosenlund, and D. Laksov, see [1], [10], [11] and [12]. We will show that if a is an element in a multiplicative lattice L then $F(a) = \sqrt{F(a)} = F(\sqrt{a})$ for any quasi-radical operation F on the elements of L. Furthermore, we explain the concept of a radical operation F on the elements in a multiplicative lattice and show that any radical operation F on the elements in a multiplicative lattice is quasi-radical. Finally, we state the theorem, which shows that a quasi-radical operation satisfying certain condition must be radical. Many of the interesting radical operations have been studied by D. Laksov, J-J. Risler and G. Strengle, see [8], [9], [11] and [13].

*Correspondence: esra.sengelen@bilgi.edu.tr
2010 *AMS Mathematics Subject Classification:* 06B23, 06B75.

2. Operation in multiplicative lattice

We now define the concept of an operation F on the elements in a multiplicative lattice and define the F-radical of an element in a multiplicative lattice. We further define the concept of F-radical, F-prime elements and F-prime spectrum of the multiplicative lattice. We state some properties in Relation 2.1, for operations on the elements in a multiplicative lattice, and show some implications regarding their interconnections in Proposition 2.1.

We begin with the following definitions.

Definition 2.1 *An operation F on the elements of L is a correspondence that to every element a in L associates an element $F(a)$ in L.*

Here onward, unless otherwise stated, F denotes an operation on the elements of a multiplicative lattice L.

Definition 2.2 *(i). For an element a of L, we call $F(a)$ the F-radical of a.*

(ii). We say that a is F-radical if $F(a) = a$. A prime element p is called F-prime if it is F-radical.

Definition 2.3 *We define F-prime spectrum of L as*

$$Spec_F(L) = \{p \in Spec(L) |\ p = F(p)\}.$$

Definition 2.4 *F-radical elements have the ascending chain condition (acc) if for every sequence $\{a_i\}_{i\in\mathbb{N}}$ of F-radical elements in L the chain $a_0 \leq a_1 \leq a_2 \leq ...$ stabilizes.*

Relations 2.1 *It is natural to ask if F satisfies the following relations for any elements a, b and $\{a_j\}_{j\in J}$ in L:*

(a) $\quad a \leq F(a)$

(b) $\quad F(F(a)) = F(a)$

(c) $\quad F(a \wedge b) = F(a) \wedge F(b) = F(ab)$

(d) $\quad F(\bigvee_{j\in J} a_j) = F(\bigvee_{j\in J} F(a_j))$

(e) $\quad \sqrt{a} \leq F(a)$.

(f) $\quad a \leq b$ *implies* $F(a) \leq F(b)$

(g) $\quad F(\bigvee_{j\in J} a_j) = \bigvee_{j\in J} a_j$ *if* $\{a_j\}_{j\in J}$ *is an ordered family of F-radical elements.*

The following proposition shows the relationships between the items given in Relation 2.1.

Proposition 2.1 *The following hold for $(a) - (f)$ of Relations 2.1.*

1. *If F satisfies (a), (b) and (f) then F satisfies (d).*

2. *If F satisfies (c) then F satisfies (f).*

3. *If F satisfies (a) and (c) then F satisfies (e).*

4. If F satisfies (d) then F satisfies (b).

5. If F satisfies (a) and (d) then F satisfies (f).

In particular the relations (a), (b) and (c) imply (d), (e) and (f).

Proof

1. We have from (a) that $a_j \leq F(a_j)$ for each $j \in J$. It follows that

$$\bigvee_{j\in J} a_j \leq \bigvee_{j\in J} F(a_j).$$

Consequently, we see by (f) that

$$F(\bigvee_{j\in J} a_j) \leq F(\bigvee_{j\in J} F(a_j)).$$

Conversely, since $a_l \leq \bigvee_{j\in J} a_j$ for each $l \in J$, then $F(a_l) \leq F(\bigvee_{j\in J} a_j)$ for each $l \in J$ by (f). Hence $\bigvee_{j\in J} F(a_j) \leq F(\bigvee_{j\in J} a_j)$. This implies, again by (f), that $F(\bigvee_{j\in J} F(a_j)) \leq F(F(\bigvee_{j\in J} a_j))$. So, from (b) we get $F(\bigvee_{j\in J} F(a_j)) \leq F(\bigvee_{j\in J} a_j)$. Hence

$$F(\bigvee_{j\in J} a_j) = F(\bigvee_{j\in J} F(a_j)),$$

that is, (d) holds.

2. Assume (f) is not true. There exist then a, $b \in L$ such that $a \leq b$ but $F(a) \nleq F(b)$. Hence $F(a \wedge b) = F(a) \neq F(a) \wedge F(b)$ which contradicts (c). Thus, F satisfies (f) which follows from (c).

3. From the relation (c) we have $F(t^2) = F(t) \wedge F(t) = F(t)$ for every $t \in L$. By induction on n, we obtain $F(t^n) = F(t)$ for all positive integers n. We know that $\sqrt{b} = \bigvee_{j\in J}\{t_j | t_j^n \leq b\}$. This implies $F(t_j) = F(t_j^n) \leq F(b)$. From relation (a) we have also $t_j \leq F(t_j)$. Hence $t_j \leq F(b)$ and we have proved that $\sqrt{b} \leq F(b)$.

4. If $F(a) \neq F(F(a))$ then $F(\bigvee_{j\in J} a_j) \neq F(\bigvee_{j\in J} F(a_j))$ for $J = 1$ and $a_1 = a$. Thus, F satisfies (b) which follows from (d).

5. If relation (f) does not hold, then there exist a, $b \in L$ such that $a \leq b$ does not imply $F(a) \nleq F(b)$. Then $F(b) < F(a) \vee F(b)$ so we have by (a) that $F(a \vee b) = F(b) \neq F(a) \vee F(b) \leq F(F(a) \vee F(b))$, which contradicts (d). Thus (f) is satisfied under the conditions (a) and (d).

□

Lemma 2.1 *Let p be a prime element in a multiplicative lattice L and let F be an operation on the elements in L satisfying (a) and (f) of Relations 2.1. The following two conditions are equivalent:*

(1) $F(p) = p$

(2) $a \leq p$ *implies* $F(a) \leq p$ *for each element* a *in* L*.*

Proof Assume (1) does not hold, that is by (a) we have that $p < F(p)$ then condition (2) with $a = p$ does not hold either. Thus (2) implies (1).
Conversely, assume that (2) does not hold. Then there is an element a in L such that $a \leq p$ and $F(a) \nleq p$. Then $F(a) \leq F(p)$ and by (a) $p < F(p)$, that is condition (1) does not hold. This shows that (1) implies (2). □

Next, we explain how an operation F is defined as a quasi-radical operation on the elements of a multiplicative lattice. Operations of this kind have been studied by Benhissi, M. Rosenlund and D. Laksov; see [1], [10], [11] and [12].

Definition 2.5 *A quasi-radical operation* F *on the elements in a multiplicative lattice* L *is defined as an operation on the elements in* L *such that for all elements* a *and* b *in* L *the following conditions hold:*

(a) $a \leq F(a)$

(b) $F(F(a)) = F(a)$

(c) $F(a \wedge b) = F(a) \wedge F(b) = F(ab)$.

Remark 2.1 *From Proposition* 2.1 *we see that any quasi-radical operation* F *satisfies (a)–(f) of Relations* 2.1*.*

The following proposition shows that $F(a) = \sqrt{F(a)} = F(\sqrt{a})$ is satisfied for any quasi-radical operation F in a multiplicative lattice.

Proposition 2.2 *A quasi-radical operation* F *on the elements of* L *satisfies* $F(a) = \sqrt{F(a)} = F(\sqrt{a})$ *for any element* $a \in L$*.*

Proof It is clear that $F(a) \leq \sqrt{F(a)}$. Conversely, since

$$\sqrt{F(a)} = \bigvee_{j \in J} \{m_j | m_j^n \leq F(a)\},$$

we have that $F(m_j^n) \leq F(F(a))$ and so, $m_j \leq F(m_j) \leq F(a)$. Hence, $\sqrt{F(a)} \leq F(a)$. Since F is a quasi-radical operation it satisfies (b), (e) and (f) of Relations 2.1. Hence $F(a) \leq F(\sqrt{a}) \leq F(F(a)) = F(a)$. We have now shown that $F(a) = F(\sqrt{a})$ and this finishes our proof. □

Proposition 2.3 *Let* F *be a quasi-radical operation on the elements of L. F satisfies* (g) *of Relation 2.1 if and only if* $F(\bigvee_{j \in J} a_j) = \bigvee_{j \in J} F(a_j)$ *for every ordered family of elements* $\{a_j\}_{j \in J}$ *in* L*.*

Proof Since F is a quasi-radical operation F satisfies (a), (b) and (c) of Relation 2.1. Let $\{a_j\}_{j \in J}$ be an ordered family of elements in L. Then by (f) which follows from (c), we have that $\{F(a_j)\}_{j \in J}$ is an ordered family of F-radical elements in L. Thus by the condition (g), F satisfies $F(\bigvee_{j \in J} F(a_j)) = \bigvee_{j \in J} F(a_j)$ for every ordered family of elements $\{a_j\}_{j \in J}$ in L. Furthermore, F satisfies (d) which follows from (a), (b) and (f) by Proposition 2.1. Hence $F(\bigvee_{j \in J} a_j) = F(\bigvee_{j \in J} F(a_j))$ is satisfied for every ordered family of element in L. This shows that $F(\bigvee_{j \in J} a_j) = \bigvee_{j \in J} F(a_j)$. Conversely, we have $\bigvee_{j \in J} a_j = \bigvee_{j \in J} F(a_j)$ for ordered F-radical elements and $F(\bigvee_{i\in J} a_j) = \bigvee_{j\in J} F(a_j)$, so $\bigvee_{j\in J} a_j = F(\bigvee_{j\in J} a_j)$. Then this shows that (g) is satisfied. $\square$

Theorem 2.1 *Let F be a quasi-radical operation on the elements of L. If L satisfies the ascending chain condition for F-radical elements, then any F-radical element is the infimum of a finite number of F-prime elements.*

Proof Let Ω be the set of F-radical elements which are not the infimum of a finite number of F-prime elements.

Assume that $\Omega \neq \emptyset$. Then Ω admits a maximal element I^*, because the acc for F-radical elements holds. Then I^* is F-radical and cannot be prime. Take $b,\ c \nleq I^*$ such that $bc \leq I^*$, then $I^* < b \vee I^*$ and $I^* < I^* \vee c$. Since I^* is maximal in Ω these two new elements are not in Ω. From (a) we get $I^* < I^* \vee c \leq F(I^* \vee c)$ and $I^* < I^* \vee b \leq F(I^* \vee b)$. Thus the elements $F(I^* \vee b)$ and $F(I^* \vee c)$ are F-radical by (b) but are not in Ω and therefore expressible as an infimum of finite number F-prime elements. By (c) we have

$$\begin{aligned} I^* &\leq F(I^* \vee c) \wedge F(I^* \vee b) = F((I^* \vee c)(I^* \vee b)) \\ &= F({I^*}^2 \vee cI^* \vee bI^* \vee cb) \leq F(I^*) = I^*. \end{aligned}$$

So, $I^* = F(I^* \vee b) \wedge F(I^* \vee c)$ and thus, an infimum for a finite number of F-prime elements, contradicting the assumption that I^* is in Ω. Thus $\Omega = \emptyset$. $\square$

The following definition explains the concept of a radical operation F on the elements in a multiplicative lattice L.

Definition 2.6 *A radical operation F on the elements of L is defined as an operation on the elements of L such that*

$$F(a) = \bigwedge_{a\leq p,\ p\in Q_F} p,\ for\ each\ element\ a\ in\ L \tag{1}$$

for some subset Q_F of Spec(L). If there are no $p \in Q_F$ satisfying $a \leq p$ then $F(a) = I$. We say that F is associated to Q_F.

We will prove that any radical operation F on the elements in a multiplicative lattice is quasi-radical.

Proposition 2.4 *If F is radical operation on elements of L, then F is quasi-radical. In particular (a)–(f) of Relations 2.1 hold.*

Proof Let a be an element of L. The equation (1) holds only for prime elements satisfying $a \leq p$. It is clear that

$$a \leq F(a). \tag{2}$$

Thus the condition (a) of Definition 2.5 holds. Every prime element $p \in Q_F$ with $a \leq p$, contains $F(a)$ so $F(F(a)) \leq F(a)$. By (2) above we have that $F(a) \leq F(F(a))$. Therefore $F(F(a)) = F(a)$ and so F satisfies the condition (b) of Definition 2.5.

By (1) we have

$$F(a \wedge b) = \bigwedge_{a\wedge b\leq p,\ p\in Q_F} p\ ,\ F(ab) = \bigwedge_{ab\leq p,\ p\in Q_F} p\ ,$$

and

$$F(a) \wedge F(b) = (\bigwedge_{a \leq p,\ p \in Q_F} p) \wedge (\bigwedge_{b \leq p,\ p \in Q_F} p).$$

Since for every prime element $p \in L$, $a \leq p$ or $b \leq p$, also since $a \wedge b \leq p$ and $ab \leq p$, we have $F(a \wedge b) \leq F(a) \wedge F(b)$ and $F(ab) \leq F(a) \wedge F(b)$. On the other hand, if a prime element satisfies $a \wedge b \leq p$ or $ab \leq p$ then it satisfies $a \leq p$ or $b \leq p$. Hence $F(a \wedge b) \geq F(a) \wedge F(b)$ and $F(ab) \geq F(a) \wedge F(b)$, it follows that $F(a \wedge b) = F(a) \wedge F(b) = F(ab)$. □

The following propositions explain how a radical operation F is associated to a set Q_F of prime elements in a multiplicative lattice and show that any radical operation is associated to its F-prime spectrum.

Proposition 2.5 *Let F be a radical operation on the elements of L associated to a set $Q_F \subseteq Spec(L)$. Then $Q_F \subseteq Spec_F(L)$ and F coincides with the radical operation associated to the set $Spec_F(L) = \{p \in Spec(L) : F(p) = p\}$*

Proof Since $p = F(p)$ when $p \in Q_F$, we have $Q_F \subseteq Spec_F(L)$ and thus

$$\bigwedge_{\substack{a \leq p,\ p = F(p), \\ \text{p a prime element}}} p \leq F(a), \quad \forall a \in L.$$

Proposition 2.4 shows that F satisfies (a) and (f) of Relation 2.1 and by Lemma 2.1 proves that if p is a prime element such that $a \leq p$ and $p = F(p)$, then $F(a) \leq p$. Consequently, we have

$$F(a) = \bigwedge_{\substack{a \leq p,\ p = F(p), \\ \text{p a prime element}}} p.$$

□

Proposition 2.6 *Let F be a radical operation on the elements of L associated to Q_F where $Q_F \subseteq Spec(L)$. The equality $Q_F = Spec_F(L)$ holds if and only if the following condition is satisfied:*

For each collection of prime elements $\{p_i\}_{i \in I}$ in Q_F such that $p = \bigwedge_{i \in I} p_i$ is a prime element, we have that $p \in Q_F$.

Proof Assume that the condition does not hold. Then there exists a collection of prime elements $\{p_i\}_{i \in I}$ in Q_F such that $p = \bigwedge_{i \in I} p_i$ is a prime element but p is not in Q_F. This implies $Spec_F(L) \neq Q_F$ since $p \in Spec_F(L)$. Hence $Spec_F(L) = Q_F$ implies that the condition holds. To prove the converse inclusion let the condition in the proposition be satisfied and $p \in Spec_F(L)$. Then we have

$$p = F(p) = \bigwedge_{p \leq p', p' \in Q_F} p'.$$

Thus p is the infimum of prime elements in Q_F, and by the condition we have that $p \in Q_F$. Hence $Spec_F(L) \subseteq Q_F$. By Proposition 2.5 we have $Q_F \subseteq Spec_F(L)$ which together with the inclusion shown above proves that $Spec_F(L) = Q_F$. □

Definition 2.7 *A multiplicative lattice* L *is called strongly compact if for any* $a \in L$, $a \leq \bigvee_{j=1}^{n} b_j$ *implies* $a \leq b_l$ *for some* $l \in J$.

Here, we state the theorem which shows that a quasi-radical operation satisfying certain condition must be radical operation.

Theorem 2.2 *Let* F *be a quasi-radical operation and let* L *be a strongly compact multiplicative lattice such that* F *satisfies* (g) *of Relation 2.1. Then* F *is a radical operation.*

Proof Since F is a quasi-radical operation, it satisfies (a), (b), (c) and (f) of Relation 2.1. Let a be an element in L. From Lemma 2.1 it follows that if a prime element $a \leq p$ satisfies $F(p) = p$ then $F(a) \leq p$. Thus if $F(a) = I$ there is no F-prime element greater than a. If $F(a) \neq I$ let $f \nleq F(a)$ be an element in L. Let $\mathcal{F}$ be the set of elements $b \in L$ such that $a \leq b$, $f \nleq b$ and $F(b) = b$. Since from (b) we have that $F(F(a)) = F(a)$ and from (a) that $a \leq F(a)$ we see that $F(a) \in \mathcal{F}$ and thus $\mathcal{F} \neq \emptyset$. Each chain in $\mathcal{F}$ has a maximal element by (g). Thus by Zorn's Lemma there is a maximal element $p \in \mathcal{F}$. Assume that p is not a prime element. Then there exist g, $h \in L$ such that $g \nleq p$, $h \nleq p$ but $gh \leq p$. Thus $g \vee p$ is not in $\mathcal{F}$. So by (a), $F(g \vee p)$ is not in $\mathcal{F}$. By (b), $F(F(g \vee p)) = F(g \vee p)$. Since $a \leq F(g \vee p)$, this implies $f \leq F(g \vee p)$. Similarly $f \leq F(h \vee p)$. Thus $f \leq F(g \vee p) \wedge F(h \vee p) = F((g \vee p)(h \vee p)) = F(gh \vee p) = F(p) = p$ which is a contradiction so p is a prime element. Thus we have shown the existence of F-prime element p such that $a \leq p$ but $f \nleq p$. Since $f \nleq F(a)$ was arbitrary this together with the result of Lemma 2.1, proves $F(a)$ can be realized as the infimum of the F-prime elements p such that $a \leq p$. That is F is a radical operation on the elements of L by Proposition 2.5. □

Acknowledgment

The author wishes to thank the referee for his helpful comments and suggestions.

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