Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2013) 37: 793 – 805
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doi:10.3906/mat-1110-29

A characterization of Auslander category

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Received: 16.10.2011 • **Accepted:** 10.08.2012 • **Published Online:** 26.08.2013 • **Printed:** 23.09.2013

Abstract: In this paper, we discuss the Bass class and the Auslander class with respect to a semidualizing module over an associative ring. Let ${}_SC_R$ be a semidualizing module we proved that the Bass class $\mathcal{B}_C(R)$ is a right orthogonal subcategory of some right R-module; and that the Auslander class $\mathcal{A}_C(S)$ is a left orthogonal subcategory of the character module of some left S-module. As an application, we introduce the notion of the minimal semidualizing module, and get a one to one correspondence between the isomorphism classes of minimal semidualizing R-modules and maximal classes among coresolving preenvelope classes of $\operatorname{Mod} R$ with the same Ext-projective generators in $\operatorname{gen}^* R$.

Key words: Semidualizing module, Auslander class, Bass class

1. Introduction

Semidualizing modules provide a common generalization of a dualizing module and a free module of rank one over a commutative noetherian local ring. Foxby [8] first defined them (PG-modules of rank one), while many people furthered their study in other names (see for example [2, 13]). In [10], Henric Holm and Diana White extended the definition of semidualizing modules to a non-commutative non-noetherian ring, which coincided with the notion of a Wakamatsu tilting module introduced by T. Wakamatsu in [14].

A semidualizing module over a commutative noetherian ring gives rise to two full subcategories of the category of R-modules, namely the so-called Auslander class $\mathcal{A}_C(R)$ and Bass class $\mathcal{B}_C(R)$ defined by Avramov and Foxby [5, 8]. Semidualizing modules and their Auslander/Bass classes have caught, attention of several authors (see for instance [4, 6, 8]). In [10], Henric Holm and Diana White also extended the definition of Auslander classes and Bass classes to arbitrary associative rings. In this paper, we discuss the Auslander class and the Bass class with respect to a semidualizing module over an associative ring.

This paper is organized as follows. In Section 2, we give some terminology and some preliminary results which are often used in this paper. In Section 3, we give a characterization of the Auslander class and the Bass class with respect to a semidualizing module. And our main results are as follows:

Theorem 1.1 *Let ${}_SC_R$ be a semidualizing module. Then*

(1) $\mathcal{B}_C(R) = N^{\perp}$, for some right R-module N.

(2) $\mathcal{A}_C(S) = {}^{\perp}M^{+}$, for some left S-module M.

Where $N^{\perp}$ is a right orthogonal subcategory of N and M^{+} is the character module of M.

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2000 *AMS Mathematics Subject Classification:* 16D20, 16E30.

We call a semidualizing module C a *minimal semidualizing module* if there is no proper direct summand of C which is also a semidualizing module. As an application of Theorem 1.1, we have the following theorem.

Theorem 1.2 *Let C be an R-module with $S = \mathrm{End}_R C$. Then*

(1) $C \to \mathcal{B}_C(R)$ gives a one to one correspondence between the isomorphism classes of minimal semidualizing R-modules and maximal classes among those coresolving preenvelope classes of $\mathrm{Mod}\, R$ *with the same Ext-projective generators in* $\mathrm{gen}^* R$.

(2) $C \to \mathcal{A}_C(S)$ gives a one to one correspondence between the isomorphism classes of minimal semidualizing S-modules and maximal classes among those resolving precover classes of $\mathrm{Mod}\, S$ *with the same Ext-injective cogenerators in* $\mathrm{gen}^* S$.

2. Preliminaries

Throughout this paper, all rings are associative with identities and all modules are unitary. M_R $({}_RM)$ denotes a right (left) R-module. We denote by $\mathrm{Mod}\, R$ the category of right R-modules. For an R-module M, we denote by M^+ the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M. M^I $(M^{(I)})$ is the direct product (sum) of copies of a module M indexed by a set I. As usual, $\mathrm{Add}_R M$ $(\mathrm{add}_R M)$ denotes the full subcategory of $\mathrm{Mod}\, R$ whose objects are the direct summands of (finite) direct sums of copies of M. Similarly, $\mathrm{Prod}_R M$ stands for the full subcategory of $\mathrm{Mod}\, R$ whose objects are the direct summands of direct products of copies of M. We denote by $\mathrm{Gen}\, M$ the full subcategory of $\mathrm{Mod}\, R$ consisting of those modules X such that there is an epimorphism $M_0 \to X$ with $M_0 \in \mathrm{Add}_R M$. Dually we define $\mathrm{Cogen}\, M$.

In this paper, all subcategories are closed under finite direct sums, finite direct summands, and isomorphisms. Following [7], a full subcategory $\mathcal{C}$ of $\mathrm{Mod}\, R$ is called a *resolving subcategory* if it is closed under extensions and kernels of epimorphisms and if it contains all the projective modules. Dually, a full subcategory $\mathcal{C}$ of $\mathrm{Mod}\, R$ is called a *coresolving subcategory* if it is closed under extensions and cokernels of monomorphisms and if it contains all the injective modules.

Let $\mathcal{C}$ be a full subcategory of $\mathrm{Mod}\, R$. We denote by $\mathcal{C}^{\perp}$ (resp., ${}^{\perp}\mathcal{C}$) the subcategory of R-modules N such that $\mathrm{Ext}_R^{i\geq 1}(X, N) = 0$ (resp., $\mathrm{Ext}_R^{i\geq 1}(N, X) = 0$) for any $X \in \mathcal{C}$. Recall that $\mathcal{C}$ is a *self-orthogonal subcategory* of $\mathrm{Mod}\, R$, if $\mathcal{C} \subset {}^{\perp}\mathcal{C}$. We say that an R-module $C \in \mathcal{C}$ is *Ext-projective* in $\mathcal{C}$, if $C \in {}^{\perp}\mathcal{C}$. Moreover, C is an *Ext-projective generator* for $\mathcal{C}$, if it is an Ext-projective module, and for any module $M \in \mathcal{C}$, there exists an exact sequence: $0 \to M' \to C' \to M \to 0$ with $C' \in \mathrm{Add}_R C$ and $M' \in \mathcal{C}$. Dually, we define an Ext-injective module and an Ext-injective cogenerator for $\mathcal{C}$.

Given a full subcategory $\mathcal{C}$ of $\mathrm{Mod}\, R$, we denote by $\mathrm{gen}^* \mathcal{C}$ (resp., $\mathrm{Gen}^* \mathcal{C}$) the subcategory of all modules N such that there exists a long exact sequence: $\cdots \xrightarrow{f_2} M^1 \xrightarrow{f_1} M^0 \xrightarrow{f_0} N \to 0$ with each $M^i \in \mathcal{C}$ (resp., $M^i \in \mathrm{Add}_R \mathcal{C}$) and each $\mathrm{Ext}_R^1(\mathcal{C}, \mathrm{Ker}\, f_i) = 0$. Dually, we define $\mathrm{cogen}^* \mathcal{C}$ (resp., $\mathrm{Cogen}^* \mathcal{C}$) the subcategory of all modules N such that there exists a long exact sequence: $0 \to N \xrightarrow{g_0} M_0 \xrightarrow{g_1} M_1 \xrightarrow{g_2} \cdots$, where $M_i \in \mathcal{C}$ (resp., $M_i \in \mathrm{Prod}_R \mathcal{C}$) and $\mathrm{Ext}_R^1(\mathrm{Coker}\, g_i, \mathcal{C}) = 0$ for all $i \geq 0$. If the category $\mathcal{C}$ is of the form $\mathrm{add}_R M$ for some R-module M, often simply replace the category with the module M in the corresponding notations. For example, we use $\mathrm{gen}^* M$ instead of $\mathrm{gen}^* \mathcal{C}$.

Let R and S be two rings. Following [10], an (S, R)-bimodule C is *a semidualizing module*, if (1) $C_R \in \mathrm{gen}^* R$; (2) ${}_SC \in \mathrm{gen}^* S$; (3) The homothety map ${}_SS_S \to \mathrm{Hom}_R(C, C)$ is an isomorphism; (4) The homothety map ${}_RR_R \to \mathrm{Hom}_S(C, C)$ is an isomorphism; (5) $\mathrm{Ext}_R^{i\geq 1}(C_R, C_R) = \mathrm{Ext}_S^{i\geq 1}({}_SC, {}_SC) = 0$. In [1] an (S,R)-bimodule C is called a *faithfully balanced bimodule*, if it satisfies (3) and (4). On the other hand, in [14] C_R is a *Wakamatsu tilting module*, if it satisfies (1) $C \in \mathrm{gen}^* R$; (2) $R \in \mathrm{cogen}^* C$; (3) C is self-orthogonal. In fact, following ([14], Lemma 3.2), an (S,R)-bimodule C is a semidualizing module if and only if C_R is a Wakamatsu tilting module with $S = \mathrm{End}(C_R)$ if and only if ${}_SC$ is a Wakamatsu tilting module with $\mathrm{End}({}_SC) = R$.

Let ${}_SC_R$ be a semidualizing bimodule. Following [10], the *Auslander class* $\mathcal{A}_C(S)$ with respect to ${}_SC_R$ consists of all S-modules M satisfying

(A1) $\mathrm{Tor}^S_{i\geq 1}(M,C) = 0$,

(A2) $\mathrm{Ext}^{i\geq 1}_R(C, M\otimes_S C) = 0$, and

(A3) The natural evaluation homomorphism $\gamma_M\colon\ M \to \mathrm{Hom}_R(C, M\otimes_S C)$, defined by $\gamma(m)(c) = m\otimes c$ for any $m\in M$ and $c \in C$, is an isomorphism (of S-modules).

The *Bass class* $\mathcal{B}_C(R)$ with respect to ${}_SC_R$ consists of all R-modules N satisfying

(B1) $\mathrm{Ext}^{i\geq 1}_R(C,N) = 0$,

(B2) $\mathrm{Tor}^S_{i\geq 1}(\mathrm{Hom}_R(C,N), C) = 0$, and

(B3) The natural evaluation homomorphism $\nu_N :\ \mathrm{Hom}_R(C,N)\otimes_S C \to N$, defined by $\nu(f\otimes c) = f(c)$ for any $c\in C$ and $f \in \mathrm{Hom}_R(C,N)$, is an isomorphism (of R-modules).

Let us now recall some notions concerning precover classes and preenvelope classes in [7]. Let $\mathcal{C}$ be a full subcategory of $\mathrm{Mod}\,R$. A homomorphism $f\colon\ C \to M$ in $\mathrm{Mod}\,R$ is called a *$\mathcal{C}$-precover* of M if $C \in \mathcal{C}$ and the sequence $\mathrm{Hom}_R(X,C) \xrightarrow{f_*} \mathrm{Hom}_R(X,M) \to 0$ is exact for all $X \in \mathcal{C}$. Dually, we define a $\mathcal{C}$-preenvelope. Recall that $\mathcal{C}$ is a *precover class* (resp., preenvelope class) provided each R-module admits a $\mathcal{C}$-precover (resp., $\mathcal{C}$-preenvelope). A $\mathcal{C}$-precover $f : C \to M$ of M is called *special*, if f is surjective and $\mathrm{Ext}^1_R(N, \mathrm{Ker}\, f) = 0$ for all $N \in \mathcal{C}$. Dually, we define a special $\mathcal{C}$-preenvelope. $\mathcal{C}$ is called a *special precover class* (resp., special preenvelope classes), if each R-module M has a special $\mathcal{C}$-precover (resp., special $\mathcal{C}$-preenvelope). An analogous theory has independently been discovered and studied by M. Auslander and other authors. Following [7, 9], let $\mathcal{C},\mathcal{D} \subseteq \mathrm{Mod}\,R$; the pair $(\mathcal{C},\mathcal{D})$ is called a *cotorsion pair*, if $\mathcal{C} = \{M \in \mathrm{Mod}\,R \mid \mathrm{Ext}^1_R(M,D) = 0 \text{ for all } D\in\mathcal{D}\}$ and $\mathcal{D} = \{N \in \mathrm{Mod}\,R \mid \mathrm{Ext}^1_R(C,N) = 0 \text{ for all } C \in \mathcal{C}\}$. A cotorsion pair $(\mathcal{C},\mathcal{D})$ in $\mathrm{Mod}\,R$ is called *complete* if either $\mathcal{C}$ is a special precover class or $\mathcal{D}$ is a special preenvelope class (see [9], P102, Lemma 2.2.6).

The following observations will be very useful.

Lemma 2.1 *[12, 11] Let M be an R-module. $\mathrm{Add}_R\, M$ is a precover class, and $\mathrm{Prod}_R\, M$ is a preenvelope class.*

Let R and S be two rings and ${}_SC_R$ a faithfully balanced bimodule. For any R-module X, we have a natural map $\nu_X\colon\ \mathrm{Hom}_R(C,X)\otimes_S C \to X$, defined by $\nu(f\otimes c) = f(c)$ for any $c \in C$ and $f \in \mathrm{Hom}_R(C,X)$. Dually, for any S-module Y, we have a natural map $\gamma_Y\colon\ Y \to \mathrm{Hom}_R(C, Y\otimes_S C)$, defined by $\gamma(y)(c) = y\otimes c$, for any $y\in Y$ and $c \in C$. It is easy to see that ν_X (resp., γ_Y) is an isomorphism, provided $X \in \mathrm{add}_R\, C$ (resp., $Y \in \mathrm{add}_S\, C^+$). The following result is maybe known, and we give a proof for safety.

Lemma 2.2 *Let ${}_SC_R$ be a faithfully balanced bimodule.*

(1) If C_R is finitely generated, then for any $X \in \mathrm{Add}_R\, C$, the natural map ν_X is an isomorphism.

(2) If ${}_SC$ is finitely generated, then for any $Y \in \mathrm{Prod}_S\, C^+$, the natural map γ_Y is an isomorphism.

Proof We only prove (1). The proof of (2) is similar. We first claim that $\nu_{C^{(I)}}$ is an isomorphism for some index set I. Since C_R is finitely generated, there is an isomorphism $\mathrm{Hom}_R(C, C^{(I)}) \to \mathrm{Hom}_R(C, C)^{(I)}$ defined by $f \to (p_i f)$, where $p_i \colon C^{(I)} \to C$ is the ith projection for $i \in I$. Thus we have an isomorphism

$$\beta_1 : \mathrm{Hom}_R(C, C^{(I)}) \otimes_S C \to \mathrm{Hom}_R(C, C)^{(I)} \otimes_S C,$$

given by $f \otimes c \to (p_i f) \otimes c$ for $f \in \mathrm{Hom}_R(C, C^{(I)})$.

Note that $- \otimes_S C$ commutes with direct sums, hence we have an isomorphism

$$\beta_2 : \mathrm{Hom}_R(C, C)^{(I)} \otimes_S C \to (\mathrm{Hom}_R(C, C) \otimes_S C)^{(I)},$$

given by $(g_i) \otimes c \to (g_i \otimes c)$ for $c \in C$ and $(g_i) \in \mathrm{Hom}_R(C, C)^{(I)}$.

Since ${}_SC_R$ is faithful and balanced, the homothety map $\sigma \colon S \to \mathrm{Hom}_R(C, C)$, given by $\sigma(s)(c) = sc$ for $s \in S$ and $c \in C$, is an isomorphism. Hence there is an isomorphism

$$\beta_3 : (\mathrm{Hom}_R(C, C) \otimes_S C)^{(I)} \to (S \otimes_S C)^{(I)},$$

given by $(g_i \otimes c_i) \to (\sigma^{-1}(g_i) \otimes c_i)$, where $g_i \in \mathrm{Hom}_R(C, C)$ and $c_i \in C$ for $i \in I$.

And the natural isomorphism $S \otimes_S C \to C$ induces an isomorphism

$$\beta_4 : (S \otimes_S C)^{(I)} \to C^{(I)}$$

given by $(s_i \otimes c_i) \to (s_i c_i)$, where $s_i \in S$ and $c_i \in C$ for $i \in I$.

Let $f \in \mathrm{Hom}_R(C, C^{(I)})$ and $c \in C$. Then $\beta_4\beta_3\beta_2\beta_1(f \otimes c) = \beta_4\beta_3\beta_2((p_i f) \otimes c) = \beta_4\beta_3((p_i f \otimes c)) = \beta_4(\sigma^{-1}(p_i f) \otimes c) = \sigma^{-1}(p_i f)(c) = (p_i f(c)) = f(c)$. It is easy to see that $\nu_{C^{(I)}} = \beta_4\beta_3\beta_2\beta_1$ is an isomorphism.

Let $X \in \mathrm{Add}_R\, C$. There is an R-module Y such that $X \oplus Y = C^{(I)}$ for some index set I. Then there is a split exact sequence $0 \to X \xrightarrow{\lambda} C^{(I)} \xrightarrow{p} Y \to 0$ which induces the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(C, X) \otimes_S C \arrow[r, "\lambda_* \otimes 1"] \arrow[d, "\nu_X"] & \mathrm{Hom}_R(C, C^{(I)}) \otimes_S C \arrow[r, "p_* \otimes 1"] \arrow[d, "\nu_{C^{(I)}}"'] & \mathrm{Hom}_R(C, Y) \otimes_S C \arrow[r] \arrow[d, "\nu_Y"'] & 0 \\
0 \arrow[r] & X \arrow[r, "\lambda"] & C^{(I)} \arrow[r, "p"] & Y \arrow[r] & 0.
\end{tikzcd}$$

The Five Lemma shows that ν_X is a monomorphism. Thus ν_Y is also a monomorphism, and hence ν_X is an isomorphism by the Five Lemma again. □

Lemma 2.3 *(Ext-Tor relations)[9] Let R and S be two rings and A a right R-module, and $n \geq 1$ a nature number.*

(1) Let B be an (S, R)-bimodule and C an injective right S-module. Then

$$\mathrm{Ext}^n_R(A, \mathrm{Hom}_S(B, C)) \cong \mathrm{Hom}_S(\mathrm{Tor}^R_n(A, B), C).$$

(2) Let $A \in \mathrm{gen}^* R$, and let B be an (S, R)-bimodule and C an injective left S-module. Then

$$\mathrm{Tor}^R_n(A, \mathrm{Hom}_S(B, C)) \cong \mathrm{Hom}_S(\mathrm{Ext}^n_R(A, B), C).$$

3. Main results

Taking ${}_{\mathcal{C}}\overline{\mathcal{X}} = \mathcal{C}^{\perp} \cap \operatorname{Gen}^* \mathcal{C}$ and $\overline{\mathcal{Y}}_{\mathcal{C}} = {}^{\perp}\mathcal{C} \cap \operatorname{Cogen}^* \mathcal{C}$, we have the following proposition.

Proposition 3.1 *Let ${}_S C_R$ be a semidualizing module. Then*

(1)$\mathcal{B}_C(R) = {}_{C_R}\overline{\mathcal{X}}$;

(2)$\mathcal{A}_C(S) = \overline{\mathcal{Y}}_{{}_S C^+}$.

Proof (1) We first claim that $\operatorname{Add}_R C \subseteq \mathcal{B}_C(R)$. In fact, it suffices to show that $C^{(I)} \in \mathcal{B}_C(R)$ for any index set I. Because ${}_S C_R$ is a semidualizing module, we have $\operatorname{Ext}^i_R(C, C^{(I)}) \cong \operatorname{Ext}^i_R(C, C)^{(I)} = 0$ for any $i \geq 1$, and $\operatorname{Hom}_R(C, C^{(I)}) \cong \operatorname{Hom}_R(C, C)^{(I)} = S^{(I)}$. Hence, $\operatorname{Tor}^S_{i\geq 1}(\operatorname{Hom}_R(C, C^{(I)}), C^{(I)}) \cong \operatorname{Tor}^S_{i\geq 1}(S^{(I)}, C^{(I)}) = 0$. By Lemma 2.2, the natural map $\nu_{C^{(I)}}\colon \operatorname{Hom}_R(C, C^{(I)} \otimes_S C) \to C^{(I)}$ is an isomorphism. And we obtain our claim.

Given any $M \in {}_{C_R}\overline{\mathcal{X}}$, there is a long exact sequence:

$$\cdots \xrightarrow{f_3} C_2 \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0 \xrightarrow{f_0} M \to 0 \tag{3.1}$$

with each $C_i \in \operatorname{Add}_R C$ and each $\operatorname{Ker} f_i \in C^{\perp}$, which induces a projective resolution of $\operatorname{Hom}_R(C, M)$ in $\operatorname{Mod} S$:

$$\cdots \to \operatorname{Hom}_R(C, C_1) \xrightarrow{f_{1*}} \operatorname{Hom}_R(C, C_0) \xrightarrow{f_{0*}} \operatorname{Hom}_R(C, M) \to 0. \tag{3.2}$$

by applying $\operatorname{Hom}_R(C, -)$ to the sequence (3.1), because $C \in \operatorname{gen}^* R$. Applying the functor $- \otimes_S C$ to the sequence (3.2), we get a complex

$$\cdots \to \operatorname{Hom}_R(C, C_1) \otimes_S C \xrightarrow{f_{1*}\otimes 1_C} \operatorname{Hom}_R(C, C_0) \otimes_S C \xrightarrow{f_{0*}\otimes 1_C} \operatorname{Hom}_R(C, M) \otimes_S C \to 0. \tag{3.3}$$

Note that the functor $- \otimes_S C$ is right exact, and so we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
C_1 \arrow[r, "f_1"] \arrow[d, "\nu_{C_1}"] & C_0 \arrow[r, "f_0"] \arrow[d, "\nu_{C_0}"'] & M \arrow[r] \arrow[d, "\nu_M"'] & 0 \\
\operatorname{Hom}_R(C, C_1) \otimes_S C \arrow[r, "f_{1*}\otimes_S 1_C"] & \operatorname{Hom}_R(C, C_0) \otimes_S C \arrow[r, "f_{0*}\otimes_S 1_C"] & \operatorname{Hom}_R(C, M) \otimes_S C \arrow[r] & 0.
\end{tikzcd}$$

By Lemma 2.2, ν_{C_0}, ν_{C_1} are isomorphisms, and hence ν_M is an isomorphism, by the Five Lemma. Since $\nu_{C_i}\colon \operatorname{Hom}_R(C, C_i) \otimes_S C \to C_i$ is an isomorphism for all $i \geq 0$, we can obtain that the complex (3.3) is isomorphic to the long exact sequence (3.1). This immediately yields $\operatorname{Tor}^S_{i\geq 1}(\operatorname{Hom}_R(C, M), C) = 0$. Therefore, $M \in \mathcal{B}_C(R)$.

Conversely, let $X \in \mathcal{B}_C(R)$, and there is an $\operatorname{Add}_R C$-precover $g_0\colon C_0 \to X$, which induces an epimorphism $g_{0*} \otimes 1\colon \operatorname{Hom}_R(C, C_0) \otimes_S C \to \operatorname{Hom}_R(C, X) \otimes_S C$, by Lemma 2.1. Furthermore we have the following commutative diagram:

$$\begin{tikzcd}
C_0 \arrow[r, "g_0"] \arrow[d, "\nu_{C_0}"'] & X \arrow[d, "\nu_X"'] & \\
\operatorname{Hom}_R(C, C_0) \otimes_S C \arrow[r, "g_{0*}\otimes_S 1_C"] & \operatorname{Hom}_R(C, X) \otimes_S C \arrow[r] & 0.
\end{tikzcd}$$

Since ν_{C_0}, ν_X are isomorphisms, g_0 is an epimorphism.

Taking $K_0 = \operatorname{Ker} g_0$, there exists an exact sequence

$$0 \to K_0 \to C_0 \xrightarrow{g_0} X \to 0. \tag{3.4}$$

Applying the functor $\operatorname{Hom}_R(C,-)$ to the sequence (3.4), we get a long exact sequence:

$$0 \to \operatorname{Hom}_R(C,K_0) \to \operatorname{Hom}_R(C,C_0) \xrightarrow{g_{0*}} \operatorname{Hom}_R(C,X) \to \operatorname{Ext}^1_R(C,K_0) \to \operatorname{Ext}^1_R(C,C_0).$$

Since g_0 is an $\operatorname{Add}_R C$-precover, g_{0*} is an epimorphism, and $\operatorname{Ext}^{i\geq 1}_R(C,C_0) = 0$ because $C_0 \in \mathcal{B}_C(R)$. Hence, $\operatorname{Ext}^1_R(C,K_0) = 0$, and that $X, C_0 \in C^{\perp}$ implies $K_0 \in C^{\perp}$.

We claim that $K_0 \in \mathcal{B}_C(R)$. Applying the functor $\operatorname{Hom}_R(C,-)\otimes_S C$ to the sequence (3.4), we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & K_0 \arrow[r] \arrow[d, "\nu_{K_0}"] & C_0 \arrow[r, "g_0"] \arrow[d, "\nu_{C_0}"'] & X \arrow[r] \arrow[d, "\nu_X"'] & 0 \\
0 \arrow[r] & \operatorname{Hom}_R(C,K_0)\otimes_S C \arrow[r] & \operatorname{Hom}_R(C,C_0)\otimes_S C \arrow[r, "g_{0*}\otimes 1_C"] & \operatorname{Hom}_R(C,X)\otimes_S C \arrow[r] & 0
\end{tikzcd}$$

The Five Lemma shows that ν_{K_0} is an isomorphism. We obtain $\operatorname{Tor}^S_i(\operatorname{Hom}_R(C,K_0),C) \cong \operatorname{Tor}^S_{i+1}$ $(\operatorname{Hom}_R(C,X),C) = 0$, for all $i \geq 1$. Thus we get our claim.

Repeating the same argument on K_0, and so on, we have $X \in {}_C\overline{\mathcal{X}}$.

(2) We first claim that $\operatorname{Prod}_S C^+ \subseteq \mathcal{A}_C(S)$. Indeed, it is enough to show $({}_SC^+)^J \in \mathcal{A}_C(S)$ for any index set J. Since ${}_SC \in \operatorname{gen}^* S$, we have isomorphisms
$\operatorname{Tor}^S_i((C^+)^J,C) \cong (\operatorname{Tor}^S_i(C^+,C))^J$ and $(\operatorname{Tor}^S_i(C^+,C))^J \cong ((\operatorname{Ext}^i_S(C,C)^+)^J = 0$ for any $i \geq 1$, by Lemma 2.3. Also, $\operatorname{Ext}^i_R(C,(C^+)^J \otimes_S C) \cong \operatorname{Ext}^i_R(C,(C^+\otimes_S C)^J) \cong \operatorname{Ext}^i_R(C,(R^+)^J) = 0$, since R^+ is an injective cogenerator of $\operatorname{Mod} R$, and the natural map $\gamma_{(C^+)^J}\colon (C^+)^J \to \operatorname{Hom}_R(C,(C^+)^J\otimes_S C)$ is an isomorphism by Lemma 2.2. Thus, we get our claim.

Given any $X \in \overline{\mathcal{Y}}_{{}_SC^+}$, there is a long exact sequence:

$$0 \to X \xrightarrow{f_0} D_0 \xrightarrow{f_1} D_1 \xrightarrow{f_2} D_2 \to \cdots \tag{3.5}$$

with each $D_i \in \operatorname{Prod}_S C^+$ and each $\operatorname{Coker} f_i \in {}^{\perp}C^+$, which induces an exact sequence

$$\cdots \to \operatorname{Hom}_S(D_2,C^+) \xrightarrow{f_2^*} \operatorname{Hom}_S(D_1,C^+) \xrightarrow{f_1^*} \operatorname{Hom}_S(D_0,C^+) \xrightarrow{f_0^*} \operatorname{Hom}_S(X,C^+) \to 0,$$

by applying the functor $\operatorname{Hom}_S(-,C^+)$ to this sequence (3.5). Since $\operatorname{Hom}_S(D_i,C^+) \cong (D_i\otimes_S C)^+$ for any $i \geq 0$, we have an injective resolution of $(X\otimes_S C)_R$ in $\operatorname{Mod} R$:

$$0 \to X\otimes_S C \xrightarrow{f_0\otimes 1_C} D_0\otimes_S C \xrightarrow{f_1\otimes 1_C} D_1\otimes_S C \to \cdots. \tag{3.6}$$

Applying the functor $\mathrm{Hom}_R(C,-)$ to the left exact sequence $0 \to X \otimes_S C \xrightarrow{f_0 \otimes 1_C} D_0 \otimes_S C \xrightarrow{f_1 \otimes 1_C} D_1 \otimes_S C$, we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f_0"] \arrow[d, "\gamma_X"] & D_0 \arrow[r, "f_1"] \arrow[d, "\gamma_{D_0}"'] & D_1 \arrow[d, "\gamma_{D_1}"'] \\
0 \arrow[r] & \mathrm{Hom}_R(C, X \otimes_S C) \arrow[r, "(f_0 \otimes 1_C)_*"] & \mathrm{Hom}_R(C, D_0 \otimes_S C) \arrow[r, "(f_1 \otimes 1_C)_*"] & \mathrm{Hom}_R(C, D_1 \otimes_S C)
\end{tikzcd}$$

Note that $\gamma_{D_0}, \gamma_{D_1}$ are isomorphisms, so is γ_X, by the Five Lemma.

By applying $\mathrm{Hom}_R(C,-)$ to the sequence (3.6), we have the complex

$$0 \to \mathrm{Hom}_R(C, X \otimes_S C) \to \mathrm{Hom}_R(C, D_0 \otimes_S C) \to \mathrm{Hom}_R(C, D_1 \otimes_S C) \to \cdots,$$

which is isomorphic to the sequence (3.5), is a long exact sequence, since each natural map $\gamma_{D_i} : \mathrm{Hom}_R(C, D_i \otimes_S C) \to D_i$ is an isomorphism by Lemma 2.2. Therefore, $\mathrm{Ext}^i_R(C, X \otimes_S C) = 0$ for any $i \geq 1$.

Conversely, given any $Y \in \mathcal{A}_C(S)$, we first claim $Y \in \mathrm{Cogen}_S C^+$. Let $g_0 \colon Y \to D_0$ be a $\mathrm{Prod}_S C^+$-preenvelope of Y by Lemma 2.1. Taking $H = \mathrm{Ker}\, g_0$, we have an exact sequence:

$$0 \to H \to Y \xrightarrow{g_0} D_0. \tag{3.7}$$

Applying the functor $\mathrm{Hom}_S(-, C^+)$ to the sequence (3.7), we have an exact sequence: $\mathrm{Hom}_S(D_0, C^+) \xrightarrow{g_0^*} \mathrm{Hom}_S(Y, C^+) \to 0$. Since $\mathrm{Hom}_S(-, C^+) \cong (- \otimes_S C)^+$, we have an exact sequence: $(D_0 \otimes_S C)^+ \xrightarrow{(g_0 \otimes 1_C)^+} (Y \otimes_S C)^+ \to 0$. And so the sequence $0 \to Y \otimes_S C \xrightarrow{g_0 \otimes 1_C} D_0 \otimes_S C$ is exact. Applying the functor $\mathrm{Hom}_S(C,-)$ to this sequence, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r] & Y \arrow[r, "g_0"] \arrow[d, "\gamma_Y"'] & D_0 \arrow[d, "\gamma_{D_0}"'] \\
 & 0 \arrow[r] & \mathrm{Hom}_R(C, Y \otimes_S C) \arrow[r, "(g_0 \otimes 1_C)_*"] & \mathrm{Hom}_R(C, D_0 \otimes_S C).
\end{tikzcd}$$

Since γ_Y, γ_{D_0} are isomorphisms, g_0 is a monomorphism, and so $H = 0$. Taking $L_0 = \mathrm{Coker}\, g_0$, we have an exact sequence:

$$0 \to Y \xrightarrow{g_0} D_0 \to L_0 \to 0. \tag{3.8}$$

Applying the functor $\mathrm{Hom}_S(-, C^+)$ to the sequence (3.8), we get an exact sequence:

$$\mathrm{Hom}_S(L_0, C^+) \to \mathrm{Hom}_S(D_0, C^+) \xrightarrow{{g_0}^*} \mathrm{Hom}_S(Y, C^+) \to \mathrm{Ext}^1_S(L_0, C^+) \to \mathrm{Ext}^1_S(D_0, C^+).$$

Since g_0 is a $\mathrm{Prod}_S C^+$-preenvelope of Y, ${g_0}^*$ is epic. By Lemma 2.3, $\mathrm{Ext}^1_S(D_0, C^+) \cong (\mathrm{Tor}^S_1(D_0, C))^+ = 0$, because $D_0 \in \mathcal{A}_C(S)$. Thus $\mathrm{Ext}^1_S(L_0, C^+) = 0$. And that $Y, D_0 \in {}^{\perp}C^+$ implies $L_0 \in {}^{\perp}C^+$. By Lemma 2.3, there is an isomorphism $\mathrm{Ext}^i_S(L_0, C^+) \cong (\mathrm{Tor}^S_i(L_0, C))^+$, and hence we have $\mathrm{Tor}^S_i(L_0, C) = 0$ for any $i \geq 1$. Thus there is an exact sequence $0 \to Y \otimes_S C \to D_0 \otimes_S C \to L_0 \otimes_S C \to 0$. Applying the functor $\mathrm{Hom}_R(C,-)$

to this sequence, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, "\gamma_Y"] & D_0 \arrow[r] \arrow[d, "\gamma_{D_0}"'] & L_0 \arrow[r] \arrow[d, "\gamma_{L_0}"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(C, Y \otimes_S C) \arrow[r] & \mathrm{Hom}_R(C, D_0 \otimes_S C) \arrow[r] & \mathrm{Hom}_R(C, L_0 \otimes_S C) \arrow[r] & 0.
\end{tikzcd}$$

Since γ_Y, γ_{D_0} are isomorphisms, we can obtain that γ_{L_0} is an isomorphism, by the Five Lemma. And by dimension shift we obtain an isomorphism $\mathrm{Ext}_R^i(C, L_0 \otimes_S C) \cong \mathrm{Ext}_R^{i+1}(C, Y \otimes_S C) = 0$, for any $i \geq 1$. Therefore, $L_0 \in \mathcal{A}_C(S)$. Repeating the above process on L_0, and so on, we get our result. □

Putting ${}_C\mathcal{X} = \mathcal{C}^{\perp} \cap \mathrm{gen}^* \mathcal{C}$, (resp., $\mathcal{Y}_{\mathcal{C}} = {}^{\perp}\mathcal{C} \cap \mathrm{cogen}^* \mathcal{C}$) and $\mathcal{B}_C^f(R) = \mathcal{B}_C(R) \cap \mathrm{gen}^* R$ (resp, $\mathcal{A}_C^f(S) = \mathcal{A}_C(S) \cap \mathrm{cogen}^*(S^+)$), we have the following corollary.

Corollary 3.2 *Let ${}_SC_R$ be a semidualizing module. Then*

(1) $\mathcal{B}_C^f(R) = {}_{C_R}\mathcal{X}$

(2) $\mathcal{A}_C^f(R) = \mathcal{Y}_{{}_SC^+}$

Proof We have to show (1), the proof of (2) is similar. Because $C \in \mathrm{gen}^* R$, for any $M \in {}_C\mathcal{X}$, we have $M \in \mathrm{gen}^* R$ from ([14], Lemma 3.4). And by Proposition 3.1, we have $M \in \mathcal{B}_C^f(R)$.

Conversely, let $N \in \mathcal{B}_C^f(R)$. Since N is finitely generated, there is an $\mathrm{add}_R C$-precover $g_0 : C_0 \to N$, which induces an epimorphism $g_{0*} \otimes 1 : \mathrm{Hom}_R(C, C_0) \otimes_S C \to \mathrm{Hom}_R(C, N) \otimes_S C$, by Lemma 2.1. Furthermore we have the following commutative diagram:

$$\begin{tikzcd}
C_0 \arrow[r, "g_0"] \arrow[d, "\nu_{C_0}"'] & N \arrow[d, "\nu_N"'] & \\
\mathrm{Hom}_R(C, C_0) \otimes_S C \arrow[r, "g_{0*} \otimes_S 1_C"] & \mathrm{Hom}_R(C, N) \otimes_S C \arrow[r] & 0.
\end{tikzcd}$$

Since ν_{C_0}, ν_N are isomorphisms, we obtain that g_0 is an epimorphism.

Taking $K_0 = \mathrm{Ker}\, g_0$, we have an exact sequence:

$$0 \to K_0 \to C_0 \xrightarrow{g_0} N \to 0. \tag{3.9}$$

Applying the functor $\mathrm{Hom}_R(C, -)$ to the sequence (3.9), we get a long exact sequence:

$$0 \to \mathrm{Hom}_R(C, K_0) \to \mathrm{Hom}_R(C, C_0) \xrightarrow{g_{0*}} \mathrm{Hom}_R(C, N) \to \mathrm{Ext}_R^1(C, K_0) \to \mathrm{Ext}_R^1(C, C_0).$$

Since g_0 is an $\mathrm{add}_R C$-precover, g_{0*} is an epimorphism. And $\mathrm{Ext}_R^{i \geq 1}(C, C_0) = 0$, because C is self-orthogonal. Hence, $\mathrm{Ext}_R^1(C, K_0) = 0$. And that $N, C_0 \in C^{\perp}$ implies $K_0 \in C^{\perp}$. We claim $K_0 \in \mathcal{B}_C^f(R)$. Since $N \in \mathrm{gen}^* R$, we have an exact sequence $0 \to L \to P_0 \to N \to 0$, where P_0 is a finitely generated projective R-module and

$L \in \mathrm{gen}^* R$. Consider the following pullback diagram:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \\ & L \arrow[r, equal] \arrow[d] & L \arrow[d] & \\ 0 \arrow[r] & K_0 \arrow[r] \arrow[d, equal] & Q \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K_0 \arrow[r] & C_0 \arrow[r, "g_0"] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\ & & 0 & 0. & \end{tikzcd}$$

We have $Q \cong K_0 \oplus P$ and an exact sequence: $0 \to L \to P_0 \oplus K_0 \to C_0 \to 0$. Note that $L, C_0 \in \mathrm{gen}^* R$, and so is K_0, by ([14], Lemma 2.2(2)). Applying the functor $\mathrm{Hom}_R(C,-) \otimes_S C$ to the sequence (3.9), we have the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & K_0 \arrow[r] \arrow[d, "\nu_{K_0}"] & C_0 \arrow[r, "g_0"] \arrow[d, "\nu_{C_0}"'] & N \arrow[r] \arrow[d, "\nu_N"'] & 0 \\ 0 \arrow[r] & \mathrm{Hom}_R(C,K_0) \otimes_S C \arrow[r] & \mathrm{Hom}_R(C,C_0) \otimes_S C \arrow[r, "g_{0*} \otimes 1_C"] & \mathrm{Hom}_R(C,N) \otimes_S C \arrow[r] & 0 \end{tikzcd}$$

The Five Lemma shows that ν_{K_0} is an isomorphism. And we obtain $\mathrm{Tor}_i^S(\mathrm{Hom}_R(C,K_0), C) \cong \mathrm{Tor}_{i+1}^S(\mathrm{Hom}_R(C,N), C) = 0$, for all $i \geq 1$. Thus we get our claim.

Repeating the same argument on K_0, and so on, we have $N \in {}_C\mathcal{X}$. □

Lemma 3.3 *Let $\mathcal{C}$ be a self-orthogonal full subcategory of* $\mathrm{Mod}\, R$.

(1) If $\mathcal{C}$ is a preenvelope class with $Q \in \mathrm{gen}^* \mathcal{C}$, for some injective cogenerator Q, then there exists a long exact sequence,

$$\cdots \xrightarrow{f_3} C_2 \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0 \xrightarrow{f_0} Q \to 0,$$

such that $\overline{\mathcal{Y}}_{\mathcal{C}} = {}^{\perp}(\prod_{i\in\mathbb{N}}(\mathrm{Ker}\, f_i)) \cap {}^{\perp}\mathcal{C}$.

(2) If $\mathcal{C}$ is a precover class with $R \in \mathrm{cogen}^* \mathcal{C}$, then there exists a long exact sequence:

$$0 \to R \xrightarrow{g_0} C_0 \xrightarrow{g_1} C_1 \xrightarrow{g_2} C_2 \xrightarrow{g_3} \cdots,$$

such that ${}_C\overline{\mathcal{X}} = (\coprod_{i\in\mathbb{N}}(\mathrm{Coker}\, g_i))^{\perp} \cap \mathcal{C}^{\perp}$.

Proof We only prove (1), the proof of (2) is similar. Taking $L_i = \mathrm{Ker}\, f_i$ for any $i \geq 0$, we have to verify that, given any $X \in \overline{\mathcal{Y}}_{\mathcal{C}}$, $X \in {}^{\perp}L_i$. Let us consider the exact sequences

$$(*) \quad 0 \to X \xrightarrow{\alpha} C^0 \to X_1 \to 0 \qquad (**) \quad 0 \to L_0 \to C_0 \xrightarrow{f_0} Q \to 0,$$

where $C^0, C_0 \in \mathcal{C}$ and $X_1 \in \overline{\mathcal{Y}}_{\mathcal{C}}$. Applying $\mathrm{Hom}_R(X,-)$ to $(**)$, we obtain an exact sequence: $0 \to \mathrm{Hom}_R(X,L_0) \to \mathrm{Hom}_R(X,C_0) \xrightarrow{f_{0*}} \mathrm{Hom}_R(X,Q) \to \mathrm{Ext}_R^1(X,L_0) \to 0$. To prove that $\mathrm{Ext}_R^1(X,L_0) = 0$, it suffices to show that f_{0*} is an epimorphism. Note that Q is injective, any morphism $f \in \mathrm{Hom}_R(X,Q)$ extends to a morphism $f' \in \mathrm{Hom}_R(C^0,Q)$. Finally, applying the functor $\mathrm{Hom}_R(C^0,-)$ to $(**)$, we have that f' lifts a morphism $f'' \in \mathrm{Hom}_R(C^0,C_0)$. Thus, $\alpha f'' f_0$ extends to f. And hence $\mathrm{Ext}^1_R(X,L_0)=0$. Moreover, by applying the functor $\mathrm{Hom}(-,L_{i+1})$ to $(**)$ and the functor $\mathrm{Hom}(X_1,-)$ to the exact sequence $0 \to L_{i+1} \to C_i \to L_i \to 0$, we obtain that $\mathrm{Ext}^1_R(X,L_{i+1}) = \mathrm{Ext}^2_R(X_1,L_{i+1}) \cong \mathrm{Ext}^1_R(X_1,L_i)$ for any $i \geq 0$. By induction we conclude that $\mathrm{Ext}^i_R(X_1,L_i)=0$ for any $i \geq 1$, and by dimension shift we get $\mathrm{Ext}^j_R(X,L_i) \cong \mathrm{Ext}^{j+1}_R(X_1,L_i)=0$ for any $i,j \geq 1$.

Conversely, given $Y \in {}^{\perp}(\prod_{i\in\mathbb{N}} L_i) \cap {}^{\perp}\mathcal{C}$, we want to show that $Y \in \mathrm{Cogen}\,\mathcal{C}$. Since Q is an injective cogenerator, there is a monomorphism $i: Y \to Q^I$ for some index set I. Consider the following pullback diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_0^I \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & K_0^I \arrow[r] & C_0^I \arrow[r, "f_0^I"] \arrow[d] & Q^I \arrow[r] \arrow[d] & 0 \\
 & & L \arrow[r, equal] \arrow[d] & L \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

Since $\mathrm{Ext}^1_R(Y,K_0^I) \cong \mathrm{Ext}^1_R(Y,K_0)^I = 0$, by ([7], P74 exercise 4), the first row is splits. And so Y is cogenerated by $\mathcal{C}$. Since $\mathcal{C}$ is a preenvelope class, there is a $\mathcal{C}$-preenvelope of Y:

$$0 \to Y \xrightarrow{h_0} C^0 \to Y_0 \to 0. \tag{3.10}$$

For any $C \in \mathcal{C}$, applying the functor $\mathrm{Hom}_R(-,C)$ to the sequence (3.10), there exists an exact sequence:

$$\mathrm{Hom}_R(Y_0,C) \to \mathrm{Hom}_R(C^0,C) \xrightarrow{{h_0}^*} \mathrm{Hom}_R(Y,C) \to \mathrm{Ext}^1_R(Y_0,C) \to \mathrm{Ext}^1_R(C^0,C)$$

Because h_0 is a $\mathcal{C}$-preenvelope of Y, ${h_0}^*$ is epic. And since $\mathcal{C}$ is self-orthogonal, we have $\mathrm{Ext}^1_R(C^0,C)=0$. Hence $\mathrm{Ext}^1_R(Y_0,\mathcal{C})=0$. Furthermore, we have $Y_0 \in {}^{\perp}\mathcal{C}$, because $Y \in {}^{\perp}\mathcal{C}$. Moreover, by applying the functor $\mathrm{Hom}_R(-,L_i)$ to $(\dagger)$, and applying the functor $\mathrm{Hom}_R(Y_0,-)$ to the exact sequence $0 \to L_{i+1} \to C_i \to L_i \to 0$, we get isomorphisms $\mathrm{Ext}^j_R(Y_0,L_i) \cong \mathrm{Ext}^{j+1}_R(Y_0,L_{i+1}) \cong \mathrm{Ext}^j_R(Y,L_i)=0$, for all $j \geq 1$ and $i \geq 0$. Thus we have $Y_0 \in {}^{\perp}(\prod_{i\in\mathbb{N}} L_i) \cap {}^{\perp}\mathcal{C}$. Repeating the same argument for Y_0, and so on, we have $Y \in \mathrm{Cogen}^*\,\mathcal{C}$. Thus we complete the proof of (1). □

Let ${}_SC_R$ be a semidualizing bimodule. Following ([14], Lemma 3.2), we have a long exact sequence:

$$0 \to R \xrightarrow{f_0} C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_1} C_2 \to \cdots$$

with $C_i \in \mathrm{add}_R C$ and $\mathrm{Ext}^1_R(\mathrm{Ker}\, f_i, C)=0$ for all $i \geq 0$. Let $K_i = \mathrm{Ker}\, f_i$ and $N = C \coprod (\coprod_{i\in\mathbb{N}} K_i)$.

On the other hand, applying ([14], Lemma 3.2) again, we have $S \in \mathrm{cogen}^*\, {}_SC$, and so there is an exact sequence:

$$0 \to S \to C_0' \xrightarrow{g_0} C_1' \xrightarrow{g_1} C_2' \xrightarrow{g_2} C_3' \xrightarrow{g_3} \to \cdots \tag{3.11}$$

with $C_i' \in \operatorname{add}_S C$ and $\operatorname{Ext}_S^1(\operatorname{Coker} g_i, \mathcal{C}) = 0$ for all $i \geq 0$. Putting $L_i = \operatorname{Coker} g_i$ for all $i \geq 0$ and $M = (\coprod_{i\in\mathbb{N}} L_i) \coprod C$, we have the following results.

Theorem 3.4 *Let ${}_SC_R$ be a semidualizing module. Then*

(1) $\mathcal{B}_C(R) = N^{\perp}$;

(2) $\mathcal{A}_C(S) = {}^{\perp}M^{+}$.

Proof (1) Taking $\mathcal{C} = \operatorname{Add}_R C$, $\mathcal{C}$ is a precover class, by Lemma 2.1. Since $C \in \operatorname{gen}^* R$, $\mathcal{C}$ is a self-orthogonal subcategory of $\operatorname{Mod} R$. Thus we obtain our result immediately by Proposition 3.1(1) and Lemma 3.3(2).

(2) By the long exact sequence (3.11), we have a long exact sequence:

$$\cdots \to (C_2')^+ \xrightarrow{g_2^+} (C_1')^+ \xrightarrow{g_1^+} (C_0')^+ \xrightarrow{f_0^+} S^+ \to 0$$

such that $\operatorname{Ker} g_i^+ = L_i^+$. Taking $\mathcal{C} = \operatorname{Prod}_S C^+$, we have that $\mathcal{C}$ is a preenvelope class, by Lemma 2.1. And we claim that $\mathcal{C}$ is self-orthogonal. Indeed, for any index sets I, J, we have $(C^+)^I \cong (C^{(I)})^+$ and $(C^+)^J \cong (C^{(J)})^+$, by ([7], Proposition 1.2.7). Since ${}_SC \in \operatorname{gen}^* S$, we have isomorphisms $\operatorname{Ext}_S^i((C^+)^I, (C^+)^J) \cong \operatorname{Ext}_S^i((C^{(I)})^+, (C^{(J)})^+) \cong \operatorname{Tor}_i^S((C^{(J)})^+, C^{(I)})^+ \cong (\operatorname{Tor}_i^S((C^{(J)})^+, C)^+)^I \cong (((\operatorname{Ext}_S^I(C, C^{(J)}))^+)^+)^I \cong (((\operatorname{Ext}_S^i(C, C)^{(J)})^+)^+)^I = 0$, by ([7], Theorem 3.2.1) and ([7], Theorem 3.2.15). And we get our claim.

By Proposition 3.1(2) and Lemma 3.3(1), we have $\mathcal{A}_C(S) = \overline{\mathcal{Y}}_{SC^+} = {}^{\perp}(\prod_{i\in N}(L_i)^+ \prod C^+)$. On the other hand, by ([7], P74 exercises 4), we have $(\prod_{i\in N}(L_i)^+) \prod C^+ \cong (\coprod_{i\in\mathbb{N}} L_i) \coprod C)^+ = M^+$. Hence, $\mathcal{A}_C(S) = {}^{\perp}(M^+)$. □

Corollary 3.5 *Let ${}_SC_R$ be a semidualizing module, we have*

(1) $\mathcal{B}_C(R)$ is a coresolving preenvelope class with an Ext-projecive generator C;

(2) $\mathcal{A}_C(S)$ is a resolving precover class with an Ext-injective cogenerator C.

Proof We have only to show (1). The proof of (2) is similar. By Proposition 3.1, $\mathcal{B}_C(R) = C^{\perp} \cap \operatorname{Gen}^* C$. Clearly, C is an Ext-projective generator of $\mathcal{B}_C(R)$. By Theorem 3.4, we have that $\mathcal{B}_C(R) = N^{\perp}$ is a coresolving subcategory. Therefore, $({}^{\perp}\mathcal{B}_C(R), \mathcal{B}_C(R))$ is a complete cotorsion pair by ([9], Theorem 3.2.1). Therefore we get that $\mathcal{B}_C(R)$ is a preenvelope class. □

Proposition 3.6 *Let $\mathcal{C}$ be a coresolving preenvelope class with an Ext-projective generator $C \in \operatorname{gen}^* R$, then C is a semidualizing module.*

Proof Since C is an Ext-projective R-module, we have that C is self-orthogonal. Let $g_0 \colon R \to T_0$ be a $\mathcal{C}$-preenvelope of R, and $i \colon R \to E$ be the injective envelope of R. Since $\mathcal{C}$ is a coresolving subcategory, there is a morphism $g \colon T_0 \to E$ such that $i = gg_0$. And so g_0 is a monomorphism. Since C is an Ext-projective generator, there is an exact sequence: $0 \to Y_0 \to C_0' \xrightarrow{\alpha} T_0 \to 0$ with $C_0' \in \operatorname{Add}_R C$ and $Y_0 \in \mathcal{C}$. There is a morphism $f_0 \in \operatorname{Hom}_R(R, C_0')$ such that $g_0 = \alpha f_0$. Note that g_0 is a $\mathcal{C}$-preenvelope and a monomorphism, so is f_0. Since R is finitely generated, there exists an R-module $C_0 \in \operatorname{add}_R C$, such that $\operatorname{Im} f_0 \subseteq C_0$. And hence we have an exact sequence $0 \to R \xrightarrow{f_0} C_0 \to K_0 \to 0$. For any $X \in \mathcal{C}$, applying the functor $\operatorname{Hom}_R(-, X)$ to this sequence, we get an exact sequence:

$$0 \to \mathrm{Hom}_R(K_0, X) \to \mathrm{Hom}_R(C_0, X) \xrightarrow{{f_0}^*} \mathrm{Hom}_R(R, X) \to \mathrm{Ext}^1_R(K_0, X) \to \mathrm{Ext}^1_R(C_0, X).$$

${f_0}^*$ is an epimorphism since f_0 is a $\mathcal{C}$-preenvelope of R. And $\mathrm{Ext}^1_R(C_0, X) = 0$, because $\mathcal{C}$ is Ext-projective. And so $\mathrm{Ext}^1_R(K_0, X) = 0$. Hence $\mathrm{Ext}^1_R(K_0, \mathcal{C}) = 0$. And that $R, C_0 \in {}^{\perp}\mathcal{C}$ implies $K_0 \in {}^{\perp}\mathcal{C}$. Since $C_0 \in \mathrm{gen}^* R$, it is easy to show that $K_0 \in \mathrm{gen}^* R$. Continuing this process, we have $R \in \mathrm{cogen}^* C$. □

We call a semidualizing module C *minimal*, if every proper direct summand of C is not a semidualizing module. Clearly, every basic Wakamatsu tilting module over an Artin algebra is a minimal semidualizing module.

Theorem 3.7 *Let C be an R-module with $S = \mathrm{End}\, C_R$; we have:*

(1) $C \to \mathcal{B}_C(R)$ gives a one to one correspondence between the isomorphism classes of minimal semidualizing modules and maximal classes among coresolving preenvelope classes of $\mathrm{Mod}\, R$ *with the same Ext-projective generators in* $\mathrm{gen}^* R$.

(2) $C \to \mathcal{A}_C(S)$ gives a one to one correspondence between the isomorphism classes of minimal semidualizing modules and maximal classes among resolving precover classes of $\mathrm{Mod}\, S$ *with the same Ext-injective cogenerators in* $\mathrm{gen}^* S$.

Proof We only show (1). The proof of (2) is similar. We define a map $\phi\colon\ C \to \mathcal{B}_C(R)$. By Corollary 3.5, ϕ is a map between the isomorphism classes of minimal semidualizing modules and coresolving preenvelope classes of $\mathrm{Mod}\, R$ with Ext-projective generators in $\mathrm{gen}^* R$. On the other hand, for any coresolving preenvelope class $\mathcal{C}$ with an Ext-projective generator in $\mathrm{gen}^* R$, we define $\psi\colon\ \mathcal{C} \to C$, where C is an Ext-projective generator, such that there is no proper direct summand T of C which is also an Ext-projective generator of $\mathcal{C}$. By Proposition 3.6, ψ is well-defined. Furthermore, it follows that $\psi\phi(C) = C$ for any minimal semidualizing module C.

Let $\mathcal{C}$ be a coresolving subcategory with an Ext-projective generator in $\mathrm{gen}^* R$. Then $\mathcal{C} \subseteq \phi\psi(\mathcal{C})$, by Proposition 3.1. Thus, for any minimal semidualizing module C, $\mathcal{B}_C(R)$ is a maximal class among those coresolving subcategories with the same Ext-projective generator C. Conversely, if $\mathcal{C}$ is a maximal class among those coresolving preenvelope classes of $\mathrm{Mod}\, R$ with the same Ext-projective generator in $\mathrm{gen}^* R$, then $\mathcal{C} = \phi\psi(\mathcal{C})$. And we complete our theorem. □

The following corollary follows directly from Theorem 3.7.

Corollary 3.8 *Let R be a noetherian ring and C an R-module with $S = \mathrm{End}\, C_R$. If S is also a noetherian ring, then:*

(1) $C \to \mathcal{B}_C(R)$ gives a one to one correspondnce between the isomorphism classes of minimal semidualizing modules and maximal classes among those coresolving preenvelope classes of $\mathrm{Mod}\, R$ *with the same finitely generated Ext-projective generators.*

(2) $C \to \mathcal{A}_C(S)$ gives a one to one correspondence between the isomorphism classes of minimal semidualizing modules and maximal classes among those resolving precover classes of $\mathrm{Mod}\, S$ *with the same finitely generated Ext-injective cogenerators.*

Acknowledgements

This research was partially supported by the Specialized Research Fund for the Doctoral Program of Higher Education (Grant No. 20100091110034), NSFC (Grant Nos. 1117142, 11201220, 11126092, 11126169) and NSF of Jiangsu Province of China (Grant No. BK2010047). The author thanks the referee for the useful suggestions.

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