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Surgery in codimension 3 and the Browder–Livesay invariants

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Abstract: The inertia subgroup $I_n(\pi)$ of a surgery obstruction group $L_n(\pi)$ is generated by elements that act trivially on the set of homotopy triangulations $\mathcal{S}(X)$ for some closed topological manifold X^{n-1} with $\pi_1(X)=\pi$. This group is a subgroup of the group $C_n(\pi)$, which consists of the elements that can be realized by normal maps of closed manifolds. These 2 groups coincide by a recent result of Hambleton, at least for $n\geq 6$ and in all known cases. In this paper we introduce a subgroup $J_n(\pi)\subset I_n(\pi)$, which is generated by elements of the group $L_n(\pi)$, which act trivially on the set $\mathcal{S}^{\partial}(X,\partial X)$ of homotopy triangulations relative to the boundary of any compact manifold with boundary $(X,\partial X)$. Every Browder–Livesay filtration of the manifold X provides a collection of higher-order Browder–Livesay invariants for any element $x\in L_n(\pi)$. In the present paper we describe all possible invariants that can give a Browder–Livesay filtration for computing the subgroup $J_n(\pi)$. These are invariants of elements $x\in L_n(\pi)$, which are nonzero if $x\notin J_n(\pi)$. More precisely, we prove that a Browder–Livesay filtration of a given manifold can give the following invariants of elements $x\in L_n(\pi)$, which are nonzero if $x\notin J_n(\pi)$: the Browder-Livesay invariants in codimensions 0, 1, 2 and a class of obstructions of the restriction of a normal map to a submanifold in codimension 3.

Key words: Surgery assembly map, closed manifolds surgery problem, assembly map, inertia subgroup, splitting problem, Browder–Livesay invariants, Browder–Livesay groups, normal maps, iterated Browder–Livesay invariants, manifold with filtration, Browder–Quinn surgery obstruction groups, elements of the second type of a Wall group

1. Introduction

Throughout the paper we consider finitely presented groups π equipped with orientation homomorphisms $w\colon \pi\to\{\pm 1\}$. Let $L_n(\pi)$ be the Wall surgery obstruction groups $L_n^s(\pi,w)$. As usual, $B\pi=K(\pi,1)$ denotes the classifying space of π. For a manifold X we suppose that the orientation map $w\colon \pi=\pi_1(X)\to\{\pm1\}$ coincides with the Stiefel–Whitney character. We shall work in the category of topological manifolds.

Any element $x\in L_{n+1}(\pi)$, $n\geq 5$, can be represented by a normal map of a compact manifold with boundary. The results of [19] provide the following representation. Choose a closed n-manifold X^n with $\pi_1(X^n)=\pi$. Then there is a normal map

$$(F,B)\colon (W^{n+1};\partial_0W,\partial_1W)\to(X\times I;X\times\{0\},X\times\{1\}) \tag{1.1}$$

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with the obstruction $\sigma(F,B)=x$, where $\partial_0 W = X$, $F|_{\partial_0 W} = \mathrm{Id}$, $\partial_1 W = M^n$, and

$$f = F|_M : M \to X$$

is a simple homotopy equivalence. Here, $B:\nu_W \to \nu_{X\times I}$ is a bundle map covering F, the source ν_W is the stable normal bundle of W in Euclidean space, and $\nu_{X\times I}$ is a bundle over $X\times I$. In what follows, we shall not mention the maps of stable bundles if this does not lead to confusion. We call this the *Wall realization* of x on X.

Let X be a closed n-dimensional topological manifold. An orientation preserving simple homotopy equivalence $f\ M^n \to X^n$ of n-manifolds is called a *homotopical triangulation* of the manifold X. Two homotopical triangulations $f_i : M_i \to X$ $(i=1,2)$ are said to be *equivalent* if there exists a homeomorphism $h: M_1 \to M_2$ fitting in the following homotopy commutative diagram (see [16], [17], and [19]).

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & X. \\ \downarrow_h & \nearrow_{f_2} & \\ M_2 & & \end{array} \tag{1.2}$$

The set of equivalence classes $\mathcal{S}^{TOP}(X)$ is *the topological structure set* of the closed manifold X. If n is bigger than 4 it fits into the surgery exact sequence (see [16], [17], and [19])

$$\cdots \to [\Sigma(X_+), G/TOP] \longrightarrow L_{n+1}(\pi) \xrightarrow{\lambda} \mathcal{S}^{TOP}(X) \longrightarrow [X, G/TOP] \xrightarrow{\sigma} L_n(\pi). \tag{1.3}$$

The map λ is defined by the Wall realization on X. For a homotopy triangulation $g\colon M \to X$, the Wall realization in (1.3) is defined in a similar way (see [14] and [19]) as in representation (1.1) that gives the action of $x \in L_{n+1}(\pi)$ on the trivial triangulation $\mathrm{Id}\colon X \to X$. By definition, $[\lambda(x)]\,(\mathrm{Id})$ is the homotopy triangulation

$$F|_{\partial_1 W}\colon \partial_1 W \to X \times \{1\}$$

of the manifold X [19, §10]. For $n \geq 5$, let an element $x \in L_{n+1}(\pi)$ act trivially on a homotopy triangulation of some manifold X^n with $\pi_1(X)=\pi$. We denote by $I_{n+1}(\pi)$ the subgroup generated by such elements.

Now, let $(X,\partial X)$ be a compact n-dimensional manifold with boundary. A homotopy triangulation relative to the boundary of $(X,\partial X)$ is given by a pair of maps

$$(f,\partial f) : (M,\partial M) \to (X,\partial X),$$

for which the map f is a simple homotopy equivalence and the restriction ∂f is a homeomorphism. Two such homotopy triangulations

$$(f_i,\partial f_i) : (M_i,\partial M_i) \to (X,\partial X),\ i=0,1$$

are *concordant* if there exists a simple homotopy equivalence of the 4-ad

$$(F; g, f_0, f_1) : (W; V, M_0, M_1) \to (X\times I; \partial X \times I, X\times\{0\}, X\times\{1\})$$

with

$$\partial W = M_0 \cup M_1 \cup V, \quad \partial V = \partial M_0 \cup \partial M_1, \quad V = \partial M_0 \times I,$$

and

$$\partial f_0 = \partial f_1, \quad g = \partial f_0 \times I : V \to \partial X \times I.$$

The set of concordance classes is denoted by $\mathcal{S}^{\partial}(X, \partial X)$.

In fact, Wall [19] (see also [14]) constructed an action of an element $x \in L_{n+1}(\pi)$, $n \geq 5$, on any homotopy triangulation relative to the boundary

$$(f, \partial f)\colon (M, \partial M) \to (X, \partial X) \tag{1.4}$$

of compact manifolds.

Let $\pi_1(X^n) = \pi_1(\partial X) = \pi$. Then the results of Wall provide a normal map of 4-ads

$$(F, B)\colon (W^{n+1}; \partial_0 W, \partial_1 W, V) \to (M \times I; M \times \{0\}, M \times \{1\}, \partial M \times I) \tag{1.5}$$

with $\sigma(F, B) = x$. In (1.5) we have $\partial_0 W = M \times \{0\}$, $V = \partial M \times I$, $F|_{\partial_0 W \cup V} = \mathrm{Id}$, the boundary of $\partial_1 W$ is ∂M, and the map

$$F|_{\partial_1 W}\colon \partial_1 W \to M \times \{1\} \tag{1.6}$$

is a simple homotopy equivalence. The result of the action of an element $x \in L_{n+1}(\pi)$ on f is the class $[f \circ F|_{\partial_1 W} : \partial_1 W \to X]$.

Denote by $J_{n+1}(\pi)$ the subgroup of $L_{n+1}(\pi)$, which is generated by elements that act trivially on the set $\mathcal{S}^{\partial}(X, \partial X)$ for any compact manifold $(X, \partial X)$ with $\pi_1(X^n) = \pi_1(\partial X) = \pi$. It follows immediately from the definition that $J_n(\pi) \subset I_n(\pi)$. We cannot assume that $J_n(\pi)$ is equal to $I_n(\pi)$.

There is a map

$$A : H_n(B\pi; \mathbf{L}_\bullet) \to L_n(\pi) \tag{1.7}$$

called the *assembly map*, where $B\pi$ is the classifying space for π. These maps fit into the algebraic surgery exact sequence of Ranicki [16]

$$\cdots \to L_{n+1}(\pi) \to \mathcal{S}_{n+1}(B\pi) \to H_n(B\pi; \mathbf{L}_\bullet) \xrightarrow{A} L_n(\pi) \to \cdots \tag{1.8}$$

which holds for n bigger than or equal to zero and where $\mathcal{S}_{n+1}(B\pi)$ is the algebraic structure set of $B\pi$ and $\mathbf{L}_\bullet$ is the Ω-spectrum that is a 1-connected cover of the simply connected surgery Ω-spectrum $\mathbf{L}_\bullet(1)$ with $\pi_n(\mathbf{L}_\bullet(1)) = L_n(1)$, $n > 0$, and $\mathbf{L}_{\bullet 0} \simeq G/TOP$ [16].

For a closed n-dimensional topological manifold X with $n \geq 5$ the surgery exact sequence (1.3) is isomorphic to the left part (from the group $L_n(\pi)$) of the algebraic surgery exact sequence (1.8) of the space X with

$$H_n(X; \mathbf{L}_\bullet) \cong [X, G/TOP] \ \text{ and } \ \mathcal{S}_{n+1}(X) = \mathcal{S}^{TOP}(X)$$

(see [16], [17], and [19]).

Let $C_n(\pi)$ consist of the surgery obstructions $\sigma(f, b) \in L_n(\pi)$, where (f, b) is a normal map of closed manifolds with a given orientation map w.

It was proven in [5] that $I_n(\pi) \subset C_n(\pi)$. Additionally, it follows from [7] that images of these groups coincide in the projective Novikov–Wall groups L^p_*. It was proven in a recent paper by Hambleton [8] that $I_n(\pi) = C_n(\pi)$ for $n \geq 6$.

The iterated Browder–Livesay invariants provide a collection of the higher-order invariants in the closed manifold surgery problem for a group π with a subgroup of index 2 (see [2], [3], [5], [7], [12], and [13]). These are invariants of elements $x \in L_n(\pi)$, which are nonzero if $x \notin C_n(\pi)$. A natural way to describe iterated

Browder–Livesay invariants for the group $L_n(\pi)$ is the Browder–Livesay filtration of the manifold X with $\pi_1(X) = \pi$ (see [2], [6], and [15]) (the definition of a Browder–Livesay filtration and iterated invariants is given in Section 2 below).

In the present paper we describe the application of this approach to computing the subgroup $J_n(\pi)$. More precisely, we prove that a Browder–Livesay filtration of a given manifold can give the following invariants of elements $x \in L_n(\pi_1(X))$, which are nonzero if $x \notin J_n(\pi)$: the Browder–Livesay invariants in codimensions 0, 1, 2 and a class of obstructions of a restriction of a normal map to a submanifold in codimension 3.

In Section 2 we give necessary preliminary material and state our main result (Theorem 2.5), and in Section 3 we formulate and prove all key theorems.

2. Browder–Livesay filtration and iterated invariants

A pair of closed manifolds (X^n, Y^{n-1}) is called a *Browder–Livesay pair* if Y is a one-sided submanifold in X and the homomorphism $\pi_1(Y) \to \pi_1(X)$ induced by the inclusion map is an isomorphism (see [1], [4], [5], [7], [13], and [19, §11]). Let $\rho = \pi_1(X \setminus Y)$ and let $i\colon \rho \to \pi$ be the natural map induced by the inclusion.

Let U be a tubular neighborhood of Y in X with boundary ∂U. We obtain a pushout square

$$F = \begin{pmatrix} \pi_1(\partial U) & \to & \pi_1(X \setminus Y) \\ \downarrow & & \downarrow \\ \pi_1(Y) & \to & \pi_1(X) \end{pmatrix} = \begin{pmatrix} \rho & \to & \rho \\ \downarrow & & \downarrow \\ \pi^{\mp} & \to & \pi^{\pm} \end{pmatrix} \tag{2.1}$$

of fundamental groups with orientation in which horizontal maps are isomorphisms and vertical maps are inclusions of index 2. Thus $i\colon \rho \to \pi$ is an inclusion of index 2 subgroup. In (2.1) the upper horizontal map and the vertical maps agree with the orientations. The bottom horizontal maps preserve the orientation on the image of the vertical maps and reverse orientations outside these images. We shall denote this fact by superscripts "+" or "−". We shall omit these superscripts if the orientation follows from the context.

By definition (see [17, §7] and [19, §11]), a simple homotopy equivalence $f : M \to X$ *splits along the submanifold* Y if it is homotopic to a map $g : M \to X$, which is transversal to Y with $N = g^{-1}(Y)$, and whose restrictions

$$g|_N : N \to Y \quad \text{and} \quad g|_{(M \setminus N)} : M \setminus N \to X \setminus Y \tag{2.2}$$

are simple homotopy equivalences. The splitting obstruction groups

$$LN_{n-1}(\pi_1(X \setminus Y) \to \pi_1(X)) = LN_{n-1}(\rho \to \pi)$$

for a Browder–Livesay manifold pair (X, Y) have been widely investigated (see [1], [3], [5], [7], [17, §7], and [19]) and are called the *Browder–Livesay groups.* The algebraic definition of the LN_*-groups was given in [18]. These groups depend functorially on the oriented inclusion $\rho \to \pi$ and the dimension $n-1 \bmod 4$.

These groups fit in the following braid of exact sequences (see [1], [7], [13], [17], and [18])

$$\begin{array}{ccccccccccc}
\to & & L_n(\rho) & & \xrightarrow{i_*} & & L_n(\pi) & & \xrightarrow{\partial} & LN_{n-2}(\rho \to \pi) & \to \\
& \nearrow & & \searrow & & s \nearrow & & \searrow & \nearrow & & \searrow \\
& & & & LP_{n-1}(F) & & \Gamma \downarrow & & L_n(i_*) & & \\
& \searrow & & \nearrow & & q \searrow & & \nearrow & \searrow & & \nearrow \\
\to & LN_{n-1}(\rho \to \pi) & & & \xrightarrow{c} & & L_{n-1}(\pi^-) & & \xrightarrow{i^!_-} & L_{n-1}(\rho) & \to
\end{array} \tag{2.3}$$

where $LP_{n-1}(F) \cong L_n(i^!_-)$ are the surgery obstruction groups for the manifold pair (X, Y) (see [1], [17], and [19]), and $L_n(i_*)$ are the relative surgery obstruction groups for the inclusion i (see [1], [11], [17], and [19]). The upper and bottom rows of Diagram (2.3) are chain complexes, and Γ is an isomorphism of the corresponding homology groups. Note that the maps s and q are the natural forgetful maps, and the map c denotes passing from surgery problem inside the manifold X to an abstract surgery problem [19]. The map $i^!_-$ is the surgery transfer map, and the map ∂ is the composition

$$L_n(\pi) \xrightarrow{\lambda} \mathcal{S}^{TOP}(X) \to LN_{n-2}(\rho \to \pi)$$

of the action of an element x on the trivial triangulation of a closed manifold X^{n-1} and taking an obstruction to splitting along the submanifold $Y^{n-2} \subset X$ on the top boundary of the bordism as in (1.1). For an element $x \in L_n(\pi)$, which represents a homology class

$$[x] \in \operatorname{Ker} \partial / \operatorname{Im} i_*$$

we have a class

$$\Gamma([x]) = \{qs^{-1}(x) | x \in \operatorname{Ker} \partial\} \in \operatorname{Ker} i^!_- / \operatorname{Im} c$$

that is represented by an element $q(y)$, where $y \in LP_{n-1}(F)$ and $s(y) = x$.

Let $\mathcal{X}$ be a filtration

$$X_k \subset X_{k-1} \subset \cdots \subset X_2 \subset X_1 \subset X_0 = X \tag{2.4}$$

of a closed manifold X^n by means of locally flat closed submanifolds such that every pair of submanifolds is a manifold pair in the sense of Ranicki [17]. A filtration in (2.4), for which every pair of submanifolds (X_i, X_{i+1}), $0 \leq i \leq k-1$, is a Browder–Livesay pair, is called a *Browder–Livesay filtration* (see [2], [6], and [15]). In what follows, we shall consider only Browder–Livesay filtrations and we shall assume that $\dim X_k = n - k \geq 5$. The filtration $\mathcal{X}$ in (2.4) is a *stratified manifold* in the sense of Browder–Quinn (see [4], [15], and [20]).

Let F_i, $0 \leq i \leq k-1$, be a square of fundamental groups in the splitting problem for the manifold pair (X_i, X_{i+1}) of the filtration in (2.4), $G_i = \pi_1(X_i)$, and $\rho_i = \pi_1(X_i \setminus X_{i+1})$. Then $LN_*(\rho_i \to G_i)$ are the splitting obstruction groups for the manifold pair (X_i, X_{i+1}).

Every inclusion $\rho_i \to G_i$ of index 2 gives a commutative braid of exact sequences that is similar to (2.3). Putting together central squares from these diagrams (see [9] and [12]), we obtain the following commutative diagram:

$$
\begin{array}{cccc}
\longrightarrow & L_n(G_0) & \xrightarrow{\partial} & LN_{n-2}(\rho_0 \to G_0) \\
 & s \nearrow \quad \searrow & & \\
LP_{n-1}(F_0) & \Gamma \downarrow & L_n(\rho_0 \to G_0) & \\
 & q \searrow \quad \nearrow & & \\
\longrightarrow & L_{n-1}(G_1) & \xrightarrow{\partial_1} & LN_{n-3}(\rho_1 \to G_1) \\
 & s \nearrow \quad \searrow & & \\
LP_{n-2}(F_1) & \Gamma \downarrow & L_{n-1}(\rho_1 \to G_1) & \\
 & q \searrow \quad \nearrow & & \\
\longrightarrow & L_{n-2}(G_2) & \xrightarrow{\partial_2} & LN_n(\rho_2 \to G_2) \\
 & s \nearrow \quad \searrow & & \\
LP_{n-3}(F_2) & \Gamma \downarrow & L_{n-2}(\rho_2 \to G_2) & \\
 & q \searrow \quad \nearrow & & \\
 & L_{n-3}(G_3) & & \\
 & \vdots & & \\
\longrightarrow & L_{n-k+1}(G_{k-1}) & \xrightarrow{\partial_{k-1}} & LN_{n-k-1}(\rho_{k-1} \to G_{k-1}) \\
 & s \nearrow \quad \searrow & & \\
LP_{n-k}(F_{k-1}) & \Gamma \downarrow & L_{n-k+1}(\rho_{k-1} \to G_{k-1}) & \\
 & q \searrow \quad \nearrow & & \\
 & L_{n-k}(G_k) & &
\end{array} \tag{2.5}
$$

In this diagram we denote by s and q the similar maps from different diagrams. However, in what follows, it will be clear from the context which map is under consideration. Note that the groups and the maps in Diagram (2.5) are defined by the subscripts taken $\bmod 4$ since the L_*-groups and Diagram (2.3) are 4-periodic.

Now we can give an inductive definition of the sets

$$\Gamma^j(x) \subset L_{n-j}(G_j) \text{ for } 0 \leq j \leq k$$

and iterated Browder–Livesay j-invariants $(1 \leq j \leq k)$ with respect to the filtration (2.4) (see [6], [12], [13], and [15]).

Definition 2.1 *Let $x \in L_n(G_0)$. By definition,*

$$\Gamma^0(x) = \{x\} \subset L_n(G_0).$$

The set $\Gamma^0(x)$ is said to be trivial if $x \in \mathrm{Image}\{L_n(\rho_0) \to L_n(G_0)\}$. Let a set

$$\Gamma^j(x) \subset L_{n-j}(G_j), \quad 0 \leq j \leq k-1$$

be defined. For $j \geq 1$, it is called trivial if $0 \in \Gamma^j(x)$.

If $\Gamma^j(x)$, $0 \leq j \leq k-1$, is defined and nontrivial, then the $(j+1)$-th Browder–Livesay invariant with respect to the filtration (2.4) is the set

$$\partial_j(\Gamma^j(x)) \subset LN_{n-j-2}(\rho_{j-1} \to G_{j-1}).$$

The $(j+1)$-th invariant is nontrivial if $0 \notin \partial_j(\Gamma^j(x))$.

If the $(j+1)$*-th,* $1 \leq j \leq k-1$*, the Browder–Livesay invariant is defined and trivial, then the set* $\Gamma^{j+1}(x)$ *is defined as*

$$\Gamma^{j+1}(x) \stackrel{def}{=} \Gamma(\Gamma^j(x)) \stackrel{def}{=} \{qs^{-1}(z) | z \in \Gamma^j(x), \partial_j(z) = 0\} \subset L_{n-j-1}(G_{j+1}).$$

Theorem 2.2 ([12],[15]). *Let* $x \in L_n(G_0)$ *be an element with a nontrivial* j*-th Browder–Livesay invariant for some* $j \geq 1$ *relative to a Browder–Livesay filtration* $\mathcal{X}$ *of the manifold* X *(this is an element of the first type in the sense of [12], [13]). Then the element* x *cannot be realized by a normal map of closed manifolds.*

Let us consider an infinite diagram $\mathcal{D}_\infty$ of groups with orientations

$$\begin{tikzcd}[column sep=small] \cdots \arrow[dr] & & \rho_2 \arrow[dl] \arrow[dr] & & \rho_1 \arrow[dl] \arrow[dr] & & \rho_0 \arrow[dl] \arrow[dr] & \\ & G_3 \arrow[rr, "\cong"] & & G_2 \arrow[rr, "\cong"] & & G_1 \arrow[rr, "\cong"] & & G_0, \end{tikzcd} \tag{2.6}$$

which is commutative as the diagram of groups. The maps $\rho_i \to G_i$ and $\rho_i \to G_{i+1}$ in (2.6) are index 2 inclusions of groups with orientations, and the horizontal maps preserve the orientations on the images of the groups ρ_i and reverse the orientations outside these images. Each commutative triangle from (2.6) defines an algebraic version of diagram (2.3) for the inclusion $\rho_i \to G_i$ [18].

Putting together central squares of these diagrams we obtain an infinite in bottom direction diagram, which is similar to diagram (2.5) (see [2], [6], [9], and [15]). Thus we can define the *iterated Browder–Livesay invariants of an element* $x \in L_n(G_0)$ *relative to the diagram* $\mathcal{D}_\infty$ *in (2.6)* similar to the case of the filtration $\mathcal{X}$. A result similar to Theorem 2.2 is true for Browder–Livesay invariants of an element x relative to the diagram in (2.6) [15]. Note that the finite subdiagram (from G_k until G_0) of diagram (2.6) provides a commutative diagram (2.5). Denote this diagram by $\mathcal{D}_k$.

Definition 2.3 ([2],[6],[13]). *Let* $x \in L_n(G_0)$*. An element* x *is the element of the second type with respect to* $\mathcal{D}_\infty$ *if all sets* $\Gamma^j(x)$*,* $j \geq 0$*, are defined and nontrivial and all Browder–Livesay invariants with respect to* $\mathcal{D}_\infty$ *are defined and trivial.*

Theorem 2.4 ([12],[13]). *Let* $x \in L_n(G_0)$ *be an element of the second type with respect to* $\mathcal{D}_\infty$*. Then* x *cannot be realized by a normal map of closed manifolds.*

Now we state the main result of the present paper:

Theorem 2.5 *Let* $x \in L_n(G_0)$*,* $n \geq 5$*, be an element for which the set* $\Gamma^3(x)$ *is defined and nontrivial with respect to the subdiagram* $\mathcal{D}_3$ *of diagram (2.6). Then the element* x *does not belong to the subgroup* $J_n(\pi)$*.*

Definition 2.6 ([12]). *If* $x \in L_n(G_0)$*,* $n \geq 5$*, is not an element of the first or the second type with respect to any* $\mathcal{D}_\infty$ *then we shall say that such an element is of the third type.*

Remark 2.7 By [12] and [13], the group $L_n(\pi)$ is a union of disjoint subsets $T_n^1(\pi), T_n^2(\pi)$, and $T_n^3(\pi)$ of the elements of the first, the second, and the third type, respectively. The elements of the first and the second type cannot be realized by normal maps of closed manifolds, [12], [13], and hence

$$C_n(\pi) \cap (T_n^1(\pi) \cup T_n^2(\pi)) = \emptyset.$$

To define the j-th Browder–Livesay invariant of x for $j \geq 4$ relative to the diagram $\mathcal{D}_\infty$ we must have a nontrivial set $\Gamma^3(x)$ with respect to subdiagram $\mathcal{D}_3$ of D_∞. Thus, the nontriviality of the j-th Browder–Livesay invariant for $j \geq 4$ automatically implies nontriviality of the invariant $\Gamma^3(x)$. Also, if the element x has the second type, then by Definition 2.3, the invariant $\Gamma^3(x)$ is nontrivial. Define a subset $J'_n(\pi)$ consisting of the elements $x \in L_n(\pi)$ for which there exists a diagram $\mathcal{D}_3$ such that $\Gamma_3(x)$ is defined and nontrivial. Then

$$T^2_n(\pi) \subset J'_n(\pi) \;\text{ and }\; J'(\pi) \cap J_n(\pi) = \emptyset.$$

The problem of mutually arranging the subsets $T^{1\leq i\leq 3}_n(\pi)$, $J_n(\pi)$, and $J'_n(\pi)$ for various groups π is open. This problem is closely related to the computation of differentials in the surgery exact sequence [9].

3. Proof of Theorem 2.5

Lemma 3.1 *Let $X^n = X^n_0$ be a manifold with the fundamental group G_0 and $\mathcal{D}_k$ a finite subdiagram of diagram (2.6) such that $n - k \geq 5$. Then there exists a Browder–Livesay filtration $\mathcal{X}$ of $X = X_0$ as in (2.4), which corresponds to the diagram $\mathcal{D}_k$.*

Proof Consider a map

$$\phi : X^n \to RP^N$$

to a real projective space of high dimension, which induces an epimorphism of fundamental groups $G_0 \to \mathbb{Z}/2$ that has the kernel ρ_0. Using the standard arguments (see [7], [9], [13], and [19, §11]), we can suppose that the map ϕ is transversal to $RP^{N-1} \subset RP^N$ with $Y^{n-1} = \phi^{-1}(RP^{N-1})$ such that the induced map $\pi_1(Y) \to \pi_1(X)$ is an isomorphism. The pair (X, Y) is the Browder–Livesay pair that gives the filtration with $\mathcal{D}_1$. Iterating this process we obtain the desired result. □

Proof of Theorem 2.5 Let the element $x \in L_n(G_0)$ act trivially on a manifold X^{n-1}. Taking the product with $\mathbb{R}P^4$, we can suppose that dimension $n - 1 \geq 8$ (see [19]). Consider a Browder–Livesay filtration

$$X_3 \subset X_2 \subset X_1 \subset X_0 = X^{n-1},$$

which gives the diagram $\mathcal{D}_3$ by Lemma 1. Let U be a tubular neighborhood of X_3 in X. Note that $\pi_1(X \setminus X_3) = \pi_1(X) = G_0$. In accordance with [19], we can construct the action of the element x on the manifold X "outside the tubular neighborhood U". The proof is identical with the proof of Theorem 5.8 (resp. Theorem 6.5) in [19] because $\pi_1(X \setminus U) \cong \pi_1(X)$.

This means that we can represent $X \times I$ as

$$X \times I = (X \setminus U) \times I \bigcup_{\partial U \times I} \overline{U} \times I$$

such that the normal map

$$F\colon (W^n; \partial_0 W, \partial_1 W) \to (X \times I; X \times \{0\}, X \times \{1\}) \tag{3.1}$$

has the following properties.

(i) The manifold W^n is a union

$$W^n = V^n \bigcup_{\partial U \times I} \overline{U} \times I$$

where

$$\partial V = \partial_0 V \bigcup_{\partial U \times \{0\}} \partial U \times I \bigcup_{\partial U \times \{1\}} \partial_1 V,$$

$$\partial_0 V = (X \setminus U) \times \{0\},$$

the boundary of $\partial_0 V$ is equal to

$$\partial U \times \{0\},$$

the boundary of $\partial_1 V$ is equal to

$$\partial U \times \{1\},$$

$$\partial_0 W = X \times \{0\} = (X \setminus U) \times \{0\} \bigcup_{\partial U \times \{0\}} \overline{U} \times \{0\},$$

$$\partial_1 W = \partial_1 V \bigcup_{\partial U \times \{1\}} \overline{U} \times \{1\}.$$

(ii) The restriction

$$F|_{X \times \{0\} \bigcup \overline{U} \times I} \colon X \times \{0\} \bigcup_{\overline{U} \times \{0\}} \overline{U} \times I \to X \times \{0\} \bigcup_{\overline{U} \times \{0\}} \overline{U} \times I$$

is the identity map, the restriction

$$F|_{\partial_1 V} \colon \partial_1 V \to (X \setminus U) \times \{1\}$$

is a homeomorphism that is the identity map on the boundary $\partial U \times \{1\}$, and the restriction

$$F|_{\partial_1 W} \colon \partial_1 W \to X \times \{1\} \tag{3.2}$$

is a homeomorphism.

By changing

$$F|_V \colon V \to (X \setminus U) \times I$$

relative to the boundary ∂V in the class of normal bordisms, we can suppose that it is transversal to

$$(X_1 \setminus (X_1 \cap U)) \times I \subset (X \setminus U) \times I$$

with a transversal preimage $(V_1, \partial V_1)$, where

$$\partial V_1 = \partial_0 V_1 \bigcup_{(X_1 \cap \partial U) \times \{0\}} (X_1 \cap \partial U) \times I \bigcup_{(X_1 \cap \partial U) \times \{1\}} \partial_1 V_1 =$$

$$= [X_1 \setminus (X_1 \cap U)] \times \{0\} \bigcup_{(X_1 \cap \partial U) \times \{0\}} (X_1 \cap \partial U) \times I \bigcup_{(X_1 \cap \partial U) \times \{1\}} \partial_1 V_1,$$

the restriction of $F|_V$ to

$$\partial_0 V_1 \bigcup_{(X_1 \cap \partial U) \times \{0\}} (X_1 \cap \partial U) \times I$$

is the identity map, and the restriction of $F|_V$ to $\partial_1 V_1$ is a homeomorphism

$$F|_{\partial_1 V_1}\colon \partial_1 V_1 \to [X_1 \setminus (X_1 \cap U)] \times I.$$

It follows that the map F in (3.1) is transversal to $X_1 \times I$ with a transversal preimage $(W_1, \partial W_1)$, where $\partial W_1 = \partial_0 W_1 \cup \partial_1 W_1$. Let $U_1 = U \cap X_1$ be a tubular neighborhood of $X_3 \subset X_1$. The restriction $F_1 = F|_{W_1}$ is a normal map

$$F_1\colon (W_1^{n-1}; \partial_0 W_1, \partial_1 W_1) \to (X_1 \times I; X_1 \times \{0\}, X_1 \times \{1\}) \tag{3.3}$$

with the following properties:

(i) the manifold W_1^{n-1} is a union

$$W_1 = V_1 \bigcup_{\partial U_1 \times I} \overline{U_1} \times I$$

where

$$\partial V_1 = \partial_0 V_1 \bigcup_{\partial U_1 \times \{0\}} \partial U_1 \times I \bigcup_{\partial U_1 \times \{1\}} \partial_1 V_1,$$

$$\partial_0 V_1 = (X_1 \setminus U_1) \times \{0\},$$

the boundary of $\partial_0 V_1$ is equal to

$$\partial U_1 \times \{0\},$$

the boundary of $\partial_1 V_1$ is equal to

$$\partial U_1 \times \{1\},$$

$$\partial_0 W_1 = X_1 \times \{0\} = (X_1 \setminus U_1) \times \{0\} \bigcup_{\partial U_1 \times \{0\}} \overline{U_1} \times \{0\} = \partial_0 V_1 \bigcup_{\partial U_1 \times \{0\}} \overline{U_1} \times \{0\},$$

$$\partial_1 W_1 = \partial_1 V_1 \bigcup_{\partial U_1 \times \{1\}} \overline{U_1} \times \{1\};$$

(ii) the restriction

$$F_1|_{X_1 \times \{0\} \bigcup \overline{U_1} \times I}\colon X_1 \times \{0\} \bigcup_{\overline{U_1} \times \{0\}} \overline{U_1} \times I \to X_1 \times \{0\} \bigcup_{\overline{U_1} \times \{0\}} \overline{U_1} \times I$$

is the identity map, the restriction

$$F_1|_{\partial_1 V_1}\colon \partial_1 V_1 \to (X_1 \setminus U_1) \times \{1\}$$

is a homeomorphism that is the identity map on the boundary $\partial U_1 \times \{1\}$, and hence the restriction

$$F_1|_{\partial_1 W_1}\colon \partial_1 W_1 \to X_1 \times \{1\} \tag{3.4}$$

is a homeomorphism.

Now consider the diagram (2.5), for which $x \in L_n(G_0)$. Since $\sigma(F) = x$ and the map in (3.2) is a homeomorphism, we conclude from the geometrical definition of the map

$$\partial_0 : L_n(G_0) \to LN_{n-2}(\rho_0 \to G_0)$$

(see [7], [13], and [19]) that $\partial_0(x) = 0$.

It follows from geometrical definition of the map Γ in diagram (2.5) ([12], [13] and [19]) that the normal map F_1 in (3.3) has a surgery obstruction $\sigma(F_1) = x_1 \in L_{n-1}(G_1)$, which lies in the class $\Gamma(x)$. As above, since the restriction $F_1|_{\partial_1 W_1}$ in (3.4) is a homeomorphism, we obtain $\partial_1(x_1) = 0 \in L_{n-3}(\rho_1 \to G_1)$.

For passing from the map F_1 to a map F_2, which is restriction of F_1 to the transversal preimage of $X_2 \times I$, we can use the same line of arguments as for the passing from the map F to the map F_1. Let $U_2 = U \cap X_2$ be a tubular neighborhood of $X_3 \subset X_2$. Thus we obtain a normal map

$$F_2\colon (W_2^{n-2}; \partial_0 W_2, \partial_1 W_2) \to (X_2 \times I; X_2 \times \{0\}, X_2 \times \{1\}) \tag{3.5}$$

with a decomposition

$$W_2 = V_2 \bigcup_{\partial U_2 \times I} \overline{U_2} \times I,$$

which is similar to the decomposition above. In particular,

$$\partial V_2 = \partial_0 V_2 \bigcup_{\partial U_2 \times \{0\}} \partial U_2 \times I \bigcup_{\partial U_2 \times \{1\}} \partial_1 V_2,$$

$$\partial_0 V_2 = (X_2 \setminus U_2) \times \{0\},$$

the boundary of $\partial_0 V_2$ is equal to

$$\partial U_2 \times \{0\},$$

the boundary of $\partial_1 V_2$ is equal to

$$\partial U_2 \times \{1\},$$

$$\partial_0 W_2 = X_2 \times \{0\}, \quad \partial_1 W_2 = \partial_1 V_2 \bigcup_{\partial U_2 \times \{1\}} \overline{U_2} \times \{1\}.$$

The map F_2 is the identity on

$$\partial_0 W_2 \bigcup_{\overline{U_2} \times \{0\}} \overline{U_2} \times I = X_2 \times \{0\} \bigcup_{\overline{U_2} \times \{0\}} \overline{U_2} \times I,$$

and the restriction of F_2 to $\partial_1 W_2$ is a homeomorphism

$$F_2|_{\partial_1 W_2}\colon \partial_1 W_2 \to X_2 \times \{1\},$$

which is the identity on

$$\overline{U_2} \times \{1\} \subset \partial_1 W_2.$$

As above, the normal map F_2 in (3.5) has a surgery obstruction $\sigma(F_2) = x_2 \in L_{n-2}(G_2)$, which lies in the class $\Gamma^2(x)$ and $\partial_2(x_2) = 0$. By our construction,

$$F_2^{-1}(U_2 \times \{I\}) = U_2 \times \{I\} \subset W_3$$

and $F_2|_{U_2 \times \{I\}}$ is the identity. Since U_2 is a tubular neighborhood of X_3 in X_2 we obtain that a restriction of F_2 to the transversal preimage of $X_3 \times I$

$$F_3 = F_2|_{F_2^{-1}(X_3 \times I)}$$

is the identity. Thus, the surgery obstruction $\sigma(F_3) \in L_{n-3}(G_3)$ is trivial. This obstruction lies in the class $\Gamma^3(x)$, and hence $0 \in \Gamma^3(x)$ and $\Gamma^3(x)$ is trivial. The theorem is thus proved.

There are many examples of nontrivial first and second Browder–Livesay invariants and nontrivial classes Γ^3 (see, for example, [13]). We do not know of any examples with a nontrivial third Browder–Livesay invariant.

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