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Morphism classes producing (weak) Grothendieck topologies, (weak) Lawvere–Tierney topologies, and universal closure operations

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Abstract: In this article, given a category $\mathcal{X}$, with Ω the subobject classifier in $Set^{\mathcal{X}^{op}}$, we set up a one-to-one correspondence between certain (i) classes of $\mathcal{X}$-morphisms, (ii) Ω-subpresheaves, (iii) Ω-automorphisms, and (iv) universal operators.

As a result we give necessary and sufficient conditions on a morphism class so that the associated (i) Ω-subpresheaf is a (weak) Grothendieck topology, (ii) Ω-automorphism is a (weak) Lawvere–Tierney topology, and (iii) universal operation is an (idempotent) universal closure operation.

We also finally give several examples of morphism classes yielding (weak) Grothendieck topologies, (weak) Lawvere–Tierney topologies, and (idempotent) universal closure operations.

Key words: (Preordered) morphism class, Ω-subpresheaf, (weak) Grothendieck topology, Ω-automorphism, (weak) Lawvere–Tierney topology, universal operation, (idempotent) universal closure operation

1. Morphism classes, subpresheaves of Ω

Let $\mathcal{X}$ be a category. The collection, $\mathcal{X}_1/x$, of all the $\mathcal{X}$-morphisms with codomain x is a preordered class by the relation $f \leq g$ if there exists a morphism h such that $f = g \circ h$. The equivalence relation generated by this preorder is $f \sim g$ if $f \leq g$ and $g \leq f$. For a class $\mathcal{M}$ of $\mathcal{X}$-morphisms, we write $f \sim \mathcal{M}$ whenever $f \sim m$ for some $m \in \mathcal{M}$. We say $\mathcal{M}$ is saturated provided that $f \in \mathcal{M}$ whenever $f \sim \mathcal{M}$.

Denoting the domain and codomain of a morphism f by $d_0 f$ and $d_1 f$ respectively, recall that a sieve in $\mathcal{X}$ [5], generated by one morphism f, is called a principal sieve and is denoted by $\langle f \rangle$, so $\langle f \rangle = \{f \circ g \mid d_1 g = d_0 f\}$; and for a sieve S on x and a morphism f with $d_1 f = x$, $S \cdot f = \{g : f \circ g \in S\}$. Also recall that $\mathcal{X}$ is said to be $\mathcal{M}$-wellpowered [2], provided that for each object $x \in \mathcal{X}$, $\{[f] \mid d_1 f = x, f \in \mathcal{M}\}$ is a set, where $[f]$ is the class of all morphisms with codomain x isomorphic to f.

We say $\mathcal{X}$ is weakly $\mathcal{M}$-wellpowered provided that for each $x \in \mathcal{X}$, $\{\langle f \rangle \mid d_1 f = x, f \in \mathcal{M}\}$ is a set. Obviously $\mathcal{X}$ is weakly $\mathcal{M}$-wellpowered if it is $\mathcal{M}$-wellpowered.

For a class $S \subseteq \mathcal{X}_1/x$, and a morphism f with codomain x, the class of all the largest elements w in $\mathcal{X}_1/d_0 f$ satisfying $f \circ w \leq s$ for some $s \in S$ is denoted by $(f \Rightarrow S)$. Obviously for a sieve S on x, see [5], $(f \Rightarrow S)$ is just the class of maximums of $S \cdot f$.

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Definition 1.1 *A class $\mathcal{M}$ of $\mathcal{X}-$ morphisms is said to satisfy the principality property, if for each x, $f \in \mathcal{X}_1/x$ and $m \in \mathcal{M}/x$, $(f \Rightarrow \langle m \rangle) \cap \mathcal{M}/d_0 f \neq \emptyset$.*

The principality property is equivalent to the fact that for each $m \in \mathcal{M}/x$ and f with codomain x, $\langle m \rangle \cdot f$ has a maximum in $\mathcal{M}/d_0 f$, or equivalently $\langle m \rangle \cdot f = \langle n \rangle$ for some n in $\mathcal{M}/d_0 f$.

If $\mathcal{X}_1$ satisfies the principality and is weakly $\mathcal{X}_1-$wellpowered, the map $P : \mathcal{X}^{op} \to Set$ with $P(x) = \{\langle f \rangle | f \in \mathcal{X}_1/x\}$ and for $f : y \to x$, $P(f) : P(x) \to P(y)$ the function taking $\langle g \rangle$ to $\langle g \rangle \cdot f$ is a functor.

Proposition 1.2 *Suppose that $\mathcal{X}_1$ satisfies the principality property and $\mathcal{X}$ is weakly $\mathcal{X}_1-$ wellpowered:*

(a) Every class $\mathcal{M}$ of morphisms of $\mathcal{X}$ that satisfies the principality property yields a subobject $M : \mathcal{X}^{op} \to Set$ of P.

(b) Every subobject M of P yields a class $\mathcal{M}$ of morphisms of $\mathcal{X}$ that satisfies the principality property and is saturated.

(c) Saturated classes $\mathcal{M}$ that satisfy the principality property correspond bijectively to subfunctors of P.

Proof (a) Define M to take each object $x \in \mathcal{X}$ to $M(x) = \{\langle m \rangle \mid d_1 m = x, m \in \mathcal{M}\}$ and each morphism $f : x \to y$ to $M(f) : M(y) \to M(x)$ taking $\langle m \rangle$, with $m \in \mathcal{M}$, to $\langle m \rangle \cdot f$. It follows easily that M is a subobject of P.

(b) Define $\mathcal{M} = \bigcup_{x \in \mathcal{X}_0} \{f \mid \langle f \rangle \in M(x)\}$.

(c) Follows easily from parts (a) and (b). □

Definition 1.3 *Let $\mathcal{M}$ be a class of $\mathcal{X}$-morphisms. $\mathcal{M}$ is said to have:*

(a) enough retractions, if for all objects x in $\mathcal{X}$, $\mathcal{M}/x$ has a retraction.

(b) almost enough retractions, if for all objects x in $\mathcal{X}$, $\mathcal{M}/x = \emptyset$ or $\mathcal{M}/x$ has a retraction.

(c) the identity property if for all objects x in $\mathcal{X}$ and for all sieves S on x whenever $\mathcal{M}_S = \{f \in \mathcal{X}_1/x \mid (f \Rightarrow S) \cap \mathcal{M}/d_0 f \neq \emptyset\}$ has a maximum in $\mathcal{M}/x$, then $\mathcal{M}_S \ni 1_x$.

(d) the maximum property if for all objects x in $\mathcal{X}$ and for all sieves S on x, whenever $S \cap \mathcal{M}/x \neq \emptyset$, then S has a maximum in $\mathcal{M}/x$.

(e) the quasi-meet property if for all objects x in $\mathcal{X}$ and $m_1, m_2 \in \mathcal{M}/x$, there exists $n \in (m_1 \Rightarrow \langle m_2 \rangle)$ such that $m_1 \circ n \sim \mathcal{M}/x$.

Remark 1.4 *In the above Definition:*

(a) Obviously (a) implies (b). Also (d) implies (b), because if $\mathcal{M}/x \neq \emptyset$, then $T_x \cap \mathcal{M}/x \neq \emptyset$ and therefore there is $m \in \mathcal{M}/x$ such that $T_x = \langle m \rangle$. But then m is a retraction and is in $\mathcal{M}/x$. Finally we note that principality and (d) imply (e). To see this, suppose $m_1, m_2 \in \mathcal{M}/x$. By principality, there is $n \in \mathcal{M}/x$ such that $\langle m_2 \rangle \cdot m_1 = \langle n \rangle$. It then follows that $\langle m_1 \rangle \cap \langle m_2 \rangle = \langle m_1 \circ n \rangle$. Setting $S = \langle m_1 \rangle \cup \langle m_2 \rangle$, (d) gives $m_1 \leq m_2$ or $m_2 \leq m_1$. So $m_1 \circ n \sim m_1$ or $m_1 \circ n \sim m_2$, yielding the required conclusion.

(b) *Note that if there exists n in $(m_1 \Rightarrow \langle m_2 \rangle)$ such that $m_1 \circ n \sim \mathcal{M}/x$, then every k in $(m_1 \Rightarrow \langle m_2 \rangle)$ satisfies $m_1 \circ k \sim \mathcal{M}/x$. Also if $n_1 \in (m_1 \Rightarrow \langle m_2 \rangle)$ and $n_2 \in (m_2 \Rightarrow \langle m_1 \rangle)$, then $m_1 \circ n_1 \sim m_2 \circ n_2$.*

Let $\mathcal{X}$ be a small category and Ω be the subobject classifier of $Set^{\mathcal{X}^{op}}$, see [5], we have:

Definition 1.5 *A subobject $A : \mathcal{X}^{op} \to Set$ of Ω in $Set^{\mathcal{X}^{op}}$ is said to be:*

(a) *a filter provided that for each object x in $\mathcal{X}$, $A(x)$ is a filter (i.e. for 2 sieves S_1, S_2 on x, if $S_1 \subseteq S_2$ and $S_1 \in A(x)$ then $S_2 \in A(x)$).*

(b) *closed under binary intersection provided that for each object x in $\mathcal{X}$, $A(x)$ is closed under binary intersection (i.e. for 2 sieves S_1, S_2 on x, if $S_1, S_2 \in A(x)$ then $S_1 \cap S_2 \in A(x)$).*

Given a subobject A of Ω, each sieve S on an object x yields a sieve S_A on x given by $S_A = \{f \mid d_1 f = x, S \cdot f \in A(d_0 f)\}$. Since $S_A \cdot f = (S \cdot f)_A$, $\hat{A}$ defined to take each object x to $\hat{A}(x) = \{S_A \mid S \text{ is a sieve on } x\}$ and each morphism $g : x \to y$ to $\hat{A}(g) : \hat{A}(y) \to \hat{A}(x)$ taking S_A to $S_A \cdot g$ is easily seen to be a subobject of Ω.

Lemma 1.6 *Let S be a sieve on x in $\mathcal{X}$. Whether $\mathcal{M}$ is a collection of morphisms that satisfies the principality property and M is the associated presheaf, or M is a subobject of P and $\mathcal{M}$ is the associated collection of morphisms, we have:*

$$S_M = \{f \in \mathcal{X}_1/x \mid (f \Rightarrow S) \cap \mathcal{M}/d_0(f) \neq \emptyset\}.$$

Proof Follows from straightforward calculations. □

For presheaves $A, B : \mathcal{X}^{op} \longrightarrow Set$, we write $A \leq B$ if for all $x \in \mathcal{X}$, $A(x) \subseteq B(x)$ and we write $A \wedge B$ for pointwise intersection, i.e. $(A \wedge B)(x) = A(x) \cap B(x)$.

With $T : \mathcal{X}^{op} \to Set$ the terminal object defined by $T(x) = \{T_x\}$, where T_x is the maximal sieve on x, we have:

Theorem 1.7 *If $\mathcal{M}$ satisfies the principality property, $\mathcal{X}$ is weakly $\mathcal{M}-$wellpowered and M is the associated presheaf, or if M is a subobject of P and $\mathcal{M}$ is the associated saturated class, then:*

(a) *$T \leq M$ if and only if $\mathcal{M}$ has enough retractions;*

(b) *$\hat{M} \wedge T \leq M$ if and only if $\mathcal{M}$ has almost enough retractions;*

(c) *$\hat{M} \wedge M \leq T$ if and only if $\mathcal{M}$ has the identity property;*

(d) *M is a filter if and only if $\mathcal{M}$ has the maximum property;*

(e) *M is closed under binary intersection if and only if $\mathcal{M}$ has the quasi-meet property.*

Proof

(a) $T \leq M$ if and only if for each x in $\mathcal{X}$, $T_x \in M(x)$ if and only if there is $f \in \mathcal{M}/x$ such that $\langle f \rangle = T_x$ if and only if there is $f \in \mathcal{M}/x$ such that $1_x \in \langle f \rangle$ if and only if there is $f \in \mathcal{M}/x$ such that f is a retraction if and only if $\mathcal{M}$ has enough retractions.

(b) First notice that $\hat{M}(x) \cap T(x) \neq \emptyset$ if and only if $T_x \in \hat{M}(x)$ if and only if there is a sieve S on x such that $T_x = S_M$ if and only if there is S such that $S \in M(x)$ if and only if $\mathcal{M}/x \neq \emptyset$. Now $\hat{M} \wedge T \leq M$

if and only if for each x, $\hat{M}(x) \cap T(x) \subseteq M(x)$ if and only if $\hat{M}(x) \cap T(x) = \emptyset$ or $T_x \in M(x)$ if and only if $\mathcal{M}/x = \emptyset$ or $\mathcal{M}/x$ contains a retraction if and only if $\mathcal{M}$ has almost enough retractions.

(c) First note that $S_M \in \hat{M}(x) \cap M(x)$ if and only if there is $m \in \mathcal{M}/x$ such that $S_M = \langle m \rangle$ if and only if S_M has a maximum in $\mathcal{M}/x$. Now $\hat{M} \wedge M \leq T$ if and only if for each x, $\hat{M}(x) \cap M(x) \subseteq T(x)$ if and only if $S_M \in \hat{M}(x) \cap M(x)$ implies $S_M = T_x$ if and only if S_M has a maximum in $\mathcal{M}/x$ implies $S_M \ni 1_x$, by Lemma 1.6, if and only if $\mathcal{M}$ has the identity property.

(d) M is a filter if and only if for each $x \in \mathcal{X}$, $\langle m \rangle \in M(x)$, and $\langle m \rangle \subseteq S$ implies $S \in M(x)$ if and only if there is $m \in \mathcal{M}$ such that $\langle m \rangle \subseteq S$ implies $S = \langle n \rangle$ for some $n \in \mathcal{M}$ if and only if $S \cap \mathcal{M} \neq \emptyset$ implies S has a maximum in $\mathcal{M}$ if and only if $\mathcal{M}$ has the maximum property.

(e) M is closed under binary intersection if and only if for each $x \in \mathcal{X}$, $\langle m \rangle$, $\langle n \rangle$ in $M(x)$ implies $\langle m \rangle \cap \langle n \rangle \in M(x)$ if and only if $m, n \in \mathcal{M}/x$ implies there exists $f \in (m \Rightarrow \langle n \rangle)$, since M is a functor, such that $m \circ f \sim \mathcal{M}/x$ if and only if $\mathcal{M}$ has the quasi-meet property. □

Corollary 1.8 *Let $\mathcal{M}$ be a class of $\mathcal{X}$-morphisms satisfying the principality property. The induced Ω-subobject M is a (weak) Grothendieck topology ([3, 5]) if and only if $\mathcal{M}$ satisfies ((a) and (d)) (a), (c), and (d) of Definition 1.3.*

Proof Follows from Remark 1.4 and Theorem 1.7. □

2. Subpresheaves of Ω, automorphisms on Ω

With $\mathcal{X}$ a small category, we know subobjects $i : M \rightarrowtail \Omega$ in $Set^{\mathcal{X}^{op}}$ are in one-to-one correspondence with Ω-automorphisms j via the following pullback square, see [5].

$$\begin{tikzcd} M \arrow[r, "!_M"] \arrow[d, tail, "i"'] \arrow[dr, phantom, "p.b."] & 1 \arrow[d, tail, "t"] \\ \Omega \arrow[r, "j"'] & \Omega \end{tikzcd} \qquad (I)$$

Note that for a given M, j is defined by the maps j_x that take each sieve S on x to S_M; and for a given j, M is defined by $M(x) = \{S : j_x(S) = T_x\}$.

With M and j corresponding to each other, we obviously have $j_x(S) = T_x$ if and only if $S \in M(x)$.

For parallel morphisms $\Phi, \Psi : A \to \Omega$ in $Set^{\mathcal{X}^{op}}$ define $\Phi \preceq \Psi$ if $\Phi_x(s) \subseteq \Psi_x(s)$ for all x in $\mathcal{X}$ and $s \in A(x)$. The meet $\Phi \wedge \Psi = \wedge \circ \langle \Phi, \Psi \rangle$, where $\wedge : \Omega \times \Omega \to \Omega$ is the internal meet (see [5]), is obviously the one induced by the partial order $\preceq$.

Theorem 2.1 *With $j : \Omega \longrightarrow \Omega$ and subobject M of Ω corresponding to each other, we have:*

(a) $j \circ t = t$ *if and only if* $T \leq M$;

(b) $j \preceq j \circ j$ *if and only if* $\hat{M} \wedge T \leq M$;

(c) $j \circ j \preceq j$ *if and only if* $\hat{M} \wedge M \leq T$;

(d) $j \circ \wedge \preceq \wedge \circ (j \times j)$ *if and only if* M *is a filter;*

(e) $\wedge \circ (j \times j) \preceq j \circ \wedge$ *if and only if* M *is closed under binary intersection.*

Proof

(a) Suppose that $j \circ t = t$. Let $x \in \mathcal{X}$, so $j_x(T_x) = T_x$ and so by the above statements $T_x \in M(x)$.

Suppose that $T \leq M$. It is enough to show that $j_x(T_x) = T_x$ for all $x \in \mathcal{X}$. Since $T_x \in M(x)$ for all $x \in \mathcal{X}$, the result follows.

(b) Suppose that $j \preceq j \circ j$. For $x \in \mathcal{X}$, if $S_M = T_x$ for a sieve S on x, then $S \in M(x)$. Hence $j_x(S) = T_x$ and since by assumption $j_x(S) \subseteq j_x(j_x(S))$, $j_x(j_x(S)) = T_x$. So $j_x(S) \in M(x)$ and since $j_x(S) = S_M$, the result follows.

Suppose that $\hat{M} \wedge T \leq M$. For $x \in \mathcal{X}$ and $S \in \Omega(x)$, if $f \in j_x(S)$, then $S \cdot f \in M(d_0 f)$, and so $j_{d_0 f}(S \cdot f) = T_{d_0 f}$. Since $j_{d_0 f}(S \cdot f) \in \hat{M}(d_0 f)$, by assumption $j_{d_0 f}(S \cdot f) \in M(d_0 f)$ and since $S_M \cdot f = (S \cdot f)_M$, $j_x(S) \cdot f \in M(d_0 f)$ and this means $f \in j_x(j_x(S))$.

(c) Suppose that $j \circ j \preceq j$. For $x \in \mathcal{X}$ and $S \in \Omega(x)$, if $S_M \in M(x)$ then $S_{(S_M)} = T_x$, since $S_{(S_M)} \subseteq S_M$, $S_M = T_x$.

Suppose that $\hat{M} \wedge M \leq T$. Let $x \in \mathcal{X}$ and $S \in \Omega(x)$. If $f \in j_x(j_x(S))$, then $j_x(S) \cdot f \in M(d_0 f)$ and so $j_{d_0 f}(S \cdot f) \in M(d_0 f)$. Since $j_{d_0 f}(S \cdot f) \in \hat{M}(d_0 f)$, by assumption $j_{d_0 f}(S \cdot f) = T_{d_0 f}$. Therefore $S \cdot f \in M(d_0 f)$, and thus $f \in j_x(S)$.

(d) Suppose that $j \circ \wedge \preceq \wedge \circ (j \times j)$. Let $x \in \mathcal{X}$ and $S_1, S_2 \in \Omega(x)$. If $S_1 \subseteq S_2$ and $S_1 \in M(x)$, then $j_x(S_1 \cap S_2) = j_x(S_1) = T_x$ and so by assumption $j_x(S_2) = T_x$. Hence $S_2 \in M(x)$.

Suppose that M is a filter. Let $x \in \mathcal{X}$ and $S_1, S_2 \in \Omega(x)$. If $f \in j_x(S_1 \cap S_2)$, then $(S_1 \cap S_2) \cdot f \in M(d_0 f)$. So $(S_1 \cdot f) \cap (S_2 \cdot f) \in M(d_0 f)$. Since $(S_1 \cdot f) \cap (S_2 \cdot f) \subseteq S_1 \cdot f$ and M is a filter, $S_1 \cdot f \in M(d_0 f)$. Similarly $S_2 \cdot f \in M(d_0 f)$. Therefore $f \in j_x(S_1) \cap j_x(S_2)$.

(e) Suppose that $\wedge \circ (j \times j) \preceq j \circ \wedge$. Let $x \in \mathcal{X}$ and $S_1, S_2 \in \Omega(x)$. If $S_1, S_2 \in M(x)$, then $j_x(S_1) = j_x(S_2) = T_x$. So by assumption $j_x(S_1 \cap S_2) = T_x$ and therefore $S_1 \cap S_2 \in M(x)$.

Suppose that M is closed under binary intersection. Let $x \in \mathcal{X}$ and $S_1, S_2 \in \Omega(x)$. If $f \in j_x(S_1) \cap j_x(S_2)$, then $S_1 \cdot f$ and $S_2 \cdot f$ are in $M(d_0 f)$. So by assumption $(S_1 \cdot f) \cap (S_2 \cdot f) \in M(d_0 f)$, and therefore $(S_1 \cap S_2) \cdot f \in M(d_0 f)$. Hence $f \in j_x(S_1 \cap S_2)$. $\square$

Corollary 2.2 *Let* $\mathcal{M}$ *be a class of* $\mathcal{X}$*-morphisms that satisfies the principality property. The induced map* $j : \Omega \longrightarrow \Omega$ *is a (weak) Lawvere–Tierney topology ([3, 5]) if and only if* $\mathcal{M}$ *satisfies ((a) and (d)) (a), (c), and (d) of Definition 1.3.*

Proof Follows from Remark 1.4 and Theorems 1.7 and 2.1. $\square$

3. Automorphisms on Ω, universal operations

We call a collection of functions $-_X : \mathcal{S}ub(X) \longrightarrow \mathcal{S}ub(X)$ indexed on objects X an operation. An operation is called universal if for each arrow $f : x \longrightarrow y$ and each $\alpha \in \mathcal{S}ub(X)$, $\overline{f^{-1}(\alpha)} = f^{-1}(\overline{\alpha})$. A universal (or natural) closure operation, see [4, 5], is a universal operation that satisfies extensive and monotone properties.

It is known that, in any topos, the Lawvere–Tierney topologies correspond to universal closure operations, see [4, 5].

This correspondence also holds between arrows $j : \Omega \longrightarrow \Omega$ and universal operations "$-$". Here is how this correspondence works. For a given $j : \Omega \longrightarrow \Omega$, for each X, $- : \mathcal{S}ub(X) \longrightarrow \mathcal{S}ub(X)$ is defined by the following pullbacks:

$$\begin{tikzcd} A \arrow[r, "!_A"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "p.b."] & 1 \arrow[d, "t"] \\ X \arrow[r, "\hat{\alpha}"'] & \Omega \end{tikzcd} \qquad (I) \qquad\qquad \begin{tikzcd} \bar{A} \arrow[r, "!_{\bar{A}}"] \arrow[d, "\bar{\alpha}"'] \arrow[dr, phantom, "p.b."] & 1 \arrow[d, "t"] \\ X \arrow[r, "j\circ\hat{\alpha}"'] & \Omega \end{tikzcd} \qquad (II)$$

where $\hat{\alpha}_c(x) = \{f : a \to c \in \mathcal{X}_1 \ \mid \ X(f)(x) \in \alpha_a(A(a))\}$ and we have $\hat{\bar{\alpha}} = j \circ \hat{\alpha}$.

Conversely given a universal operation "$-$", the Ω-automorphism j is obtained by the following pullback.

$$\begin{tikzcd} \bar{1} \arrow[r, "!_{\bar{1}}"] \arrow[d, "\bar{t}"'] \arrow[dr, phantom, "p.b."] & 1 \arrow[d, "t"] \\ \Omega \arrow[r, "j"'] & \Omega \end{tikzcd} \qquad (III)$$

Now if j induces the universal operation "$-$" and this operation induces the map j', then $j' = \hat{\bar{t}} = j \circ \hat{t}$ and since $\hat{t} = 1$, $j' = j$. Conversely, if the universal operation "$-$" induces the map j and this map induces the universal operation "$\tilde{}$", then for any subobject α of any X we have, $\tilde{\alpha} = (j \circ \hat{\alpha})^{-1}(t) = \hat{\alpha}^{-1}(j^{-1}(t)) = \hat{\alpha}^{-1}(\bar{t}) = \overline{\hat{\alpha}^{-1}(t)} = \bar{\alpha}$.

Lemma 3.1 *Let X be an object in $Set^{\mathcal{X}^{op}}$. We have:*

(a) For 2 subobjects $\alpha : A \rightarrowtail X$ and $\beta : B \rightarrowtail X$ of X,

$$\alpha \leq \beta \text{ if and only if } \alpha \wedge \beta = \alpha.$$

(b) For 2 parallel morphisms $\Phi, \Psi : X \to \Omega$ in $Set^{\mathcal{X}^{op}}$,

$$\Phi \preceq \Psi \text{ if and only if } \Phi \wedge \Psi = \Phi.$$

Proof

(a) Suppose that $\alpha \leq \beta$. So there exists γ such that $\beta \circ \gamma = \alpha$. Since β is a mono, the following is a pullback square.

$$\begin{tikzcd} A \arrow[r, "\gamma"] \arrow[d, "1"'] \arrow[dr, phantom, "p.b."] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"'] & X \end{tikzcd}$$

Hence $\alpha \wedge \beta = \alpha$.

Suppose that $\alpha \wedge \beta = \alpha$. We have $\alpha = \alpha \wedge \beta = \beta \circ \beta^{-1}(\alpha)$. So $\alpha \leq \beta$.

(b) Suppose that $\Phi \preceq \Psi$ for 2 parallel morphisms $\Phi, \Psi : X \to \Omega$. Therefore $(\Phi \wedge \Psi)_x(s) = \Phi_x(s) \cap \Psi_x(s)$ and since $\Phi_x(s) \subseteq \Psi_x(s)$, $\Phi_x(s) \cap \Psi_x(s) = \Phi_x(s)$. Thus $\Phi \wedge \Psi = \Phi$.

Suppose that $\Phi \wedge \Psi = \Phi$. For each object $x \in \mathcal{X}$ and sieve $S \in \Omega(x)$, $(\Phi \wedge \Psi)_x(S) = \Phi_x(S)$, i.e. $\Phi_x(S) \cap \Psi_x(S) = \Phi_x(S)$. So $\Phi_x(S) \subseteq \Psi_x(S)$ and therefore $\Phi \preceq \Psi$. $\square$

Lemma 3.2 *Let the morphism* $j:\Omega \longrightarrow \Omega$ *and the universal operation "$-$" correspond to each other. For each object* X *in* $Set^{\mathcal{X}^{op}}$ *and* $\alpha,\beta \in Sub(X)$*, we have:*

(a) $\widehat{\alpha\wedge\beta} = \hat{\alpha}\wedge\hat{\beta}$*;*

(b) $\widehat{\overline{\alpha\wedge\beta}} = j\circ(\hat{\alpha}\wedge\hat{\beta})$*;*

(c) $\widehat{\bar{\alpha}\wedge\bar{\beta}} = (j\circ\hat{\alpha})\wedge(j\circ\hat{\beta})$*.*

Proof

(a) Using the pullbacks:

$$\begin{tikzcd} A \arrow[r, "!_A"] \arrow[d, hook, "\alpha"'] \arrow[dr, phantom, "p.b."] & 1 \arrow[d, "t"] \\ X \arrow[r, "\hat{\alpha}"'] & \Omega \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "!_A"] \arrow[d, hook, "\beta"'] \arrow[dr, phantom, "p.b."] & 1 \arrow[d, "t"] \\ X \arrow[r, "\hat{\beta}"'] & \Omega \end{tikzcd} \qquad \begin{tikzcd} A\cap B \arrow[r, tail, "\pi_2"] \arrow[d, hook, "\pi_1"'] \arrow[dr, phantom, "p.b."] & B \arrow[d, hook, "\beta"] \\ A \arrow[r, tail, "\alpha"'] & X \end{tikzcd}$$

and that $\alpha\wedge\beta = \alpha\circ\pi_1 = \beta\circ\pi_2$, one can show the following is a pullback square.

$$\begin{tikzcd} A\cap B \arrow[r, "!"] \arrow[d, hook, "\alpha\circ\pi_1"'] \arrow[dr, phantom, "///"] & 1 \arrow[d, "\langle t,t\rangle"] \\ X \arrow[r, "\langle\hat{\alpha},\hat{\beta}\rangle"'] & \Omega\times\Omega \end{tikzcd}$$

The result then follows from the fact that the classifying map of $\langle t,t\rangle$ is the internal meet $\wedge:\Omega\times\Omega\to\Omega$.

(b) Since the classifying map of $\overline{\alpha\wedge\beta}$ is $j\circ(\widehat{\alpha\wedge\beta})$, the result follows by part (a).

(c) Follows from part (a) and the equality $\hat{\bar{\alpha}} = j\circ\hat{\alpha}$.

□

Lemma 3.3 *Let "$-$" be a universal operation.*

(a) Let $X \in Set^{\mathcal{X}^{op}}$ *and* $\alpha,\beta\in\mathcal{S}ub(X)$*. Then* $\alpha\leq\beta$ *if and only if* $\hat{\alpha}\preceq\hat{\beta}$*;*

(b) Let $X\in Set^{\mathcal{X}^{op}}$*. For all* $\alpha,\beta\in\mathcal{S}ub(X)$*,* $(\alpha\leq\beta\Rightarrow\bar{\alpha}\leq\bar{\beta})$ *if and only if for all* $\alpha,\beta\in\mathcal{S}ub(X)$*,* $\overline{\alpha\wedge\beta}\leq\bar{\alpha}\wedge\bar{\beta}$*.*

Proof

(a) Using the previous Lemma, $\alpha\leq\beta$ if and only if $\alpha\wedge\beta=\alpha$ if and only if $\widehat{\alpha\wedge\beta}=\hat{\alpha}$ if and only if $\hat{\alpha}\wedge\hat{\beta}=\hat{\alpha}$ if and only if $\hat{\alpha}\preceq\hat{\beta}$.

(b) Suppose that for all α,β, $\alpha\leq\beta\Rightarrow\bar{\alpha}\leq\bar{\beta}$. Given α,β, since $\alpha\wedge\beta\leq\alpha$ and $\alpha\wedge\beta\leq\beta$, by assumption $\overline{\alpha\wedge\beta}\leq\bar{\alpha}$ and $\overline{\alpha\wedge\beta}\leq\bar{\beta}$. Hence $\overline{\alpha\wedge\beta}\leq\bar{\alpha}\wedge\bar{\beta}$.

Conversely, suppose for all α,β, $\overline{\alpha\wedge\beta}\leq\bar{\alpha}\wedge\bar{\beta}$. Given α,β, such that $\alpha\leq\beta$, we have $\alpha\wedge\beta=\alpha$ and so $\overline{\alpha\wedge\beta}=\bar{\alpha}$. Hence $\bar{\alpha}\leq\bar{\alpha}\wedge\bar{\beta}$. Since $\bar{\alpha}\wedge\bar{\beta}\leq\bar{\beta}$, $\bar{\alpha}\leq\bar{\beta}$. □

Theorem 3.4 *Let the morphism* $j:\Omega \longrightarrow \Omega$ *and the universal operation "$-$" correspond to each other. We have:*

(a) For all X *in* $Set^{\mathcal{X}^{op}}$ *and* $\alpha \in \mathcal{S}ub(X)$, $\alpha \leq \bar{\alpha}$ *if and only if* $t \leq \bar{t}$ *if and only if* $j \circ t = t$;

(b) For all X *in* $Set^{\mathcal{X}^{op}}$ *and* $\alpha \in \mathcal{S}ub(X)$, $\bar{\alpha} \leq \bar{\bar{\alpha}}$ *if and only if* $\bar{t} \leq \bar{\bar{t}}$ *if and only if* $j \preceq j \circ j$;

(c) For all X *in* $Set^{\mathcal{X}^{op}}$ *and* $\alpha \in \mathcal{S}ub(X)$, $\bar{\bar{\alpha}} \leq \bar{\alpha}$ *if and only if* $\bar{\bar{t}} \leq \bar{t}$ *if and only if* $j \circ j \preceq j$;

(d) For all X *in* $Set^{\mathcal{X}^{op}}$ *and* $\alpha, \beta \in \mathcal{S}ub(X)$, $\overline{\alpha \wedge \beta} \leq \bar{\alpha} \wedge \bar{\beta}$ *if and only if* $\overline{\langle t, t\rangle} \leq \overline{\langle 1, t\circ !_\Omega\rangle} \wedge \overline{\langle t\circ !_\Omega, 1\rangle}$ *if and only if* $j \circ \wedge \preceq \wedge \circ (j \times j)$;

(e) For all X *in* $Set^{\mathcal{X}^{op}}$ *and* $\alpha, \beta \in \mathcal{S}ub(X)$ $\bar{\alpha} \wedge \bar{\beta} \leq \overline{\alpha \wedge \beta}$ *if and only if* $\overline{\langle 1, t\circ !_\Omega\rangle} \wedge \overline{\langle t\circ !_\Omega, 1\rangle} \leq \overline{\langle t, t\rangle}$ *if and only if* $\wedge \circ (j \times j) \preceq j \circ \wedge$.

Proof

(a) Suppose that for all X in $Set^{\mathcal{X}^{op}}$ and $\alpha \in \mathcal{S}ub(X)$, $\alpha \leq \bar{\alpha}$. Choose $\alpha = t$ to get the result.

Suppose that $t \leq \bar{t}$. So there is $\gamma : 1 \to \bar{1}$ such that $\bar{t} \circ \gamma = t$. Since $j \circ \bar{t} = t\circ !_{\bar{1}}$, $j \circ \bar{t} \circ \gamma = t\circ !_{\bar{1}} \circ \gamma$ and so $j \circ t = t \circ 1 = t$.

Suppose that $j \circ t = t$. Let $\alpha : A \to X$ be a subobject of X for some object X. If $\hat{\alpha}$ is the classifying map of α, then $\hat{\alpha} \circ \alpha = t\circ !_A$. So $j \circ \hat{\alpha} \circ \alpha = j \circ t\circ !_A = t\circ !_A$, i.e. the following square commutes.

$$\begin{tikzcd} A \arrow[r, "!_A"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "///"] & 1 \arrow[d, "t"] \\ X \arrow[r, "j\circ\hat{\alpha}"'] & \Omega \end{tikzcd}$$

Since $j \circ \hat{\alpha}$ is the classifying map of $\bar{\alpha}$, there exists a unique $\gamma : A \to \bar{A}$ such that $\bar{\alpha} \circ \gamma = \alpha$. Thus $\alpha \leq \bar{\alpha}$.

(b) To get the first implication, choose $\alpha = t$.

Suppose that $\bar{t} \leq \bar{\bar{t}}$. Since $\hat{\bar{t}} = j$ and $\hat{\bar{\bar{t}}} = j \circ j$, the result follows from Lemma 3.3-(a).

Suppose that $j \preceq j \circ j$. Let X be an object in $Set^{\mathcal{X}^{op}}$ and α be in $\mathcal{S}ub(X)$. For each object x in $\mathcal{X}$ and $S \in \Omega(x)$, $\hat{\alpha}_x(S) \in \Omega(x)$ and so $j_x(\hat{\alpha}_x(S)) \subseteq (j_x \circ j_x)(\hat{\alpha}_x(S))$. Thus $(j \circ \hat{\alpha})_x(S) \subseteq (j \circ j \circ \hat{\alpha})_x(S)$ and therefore $j \circ \hat{\alpha} \preceq j \circ j \circ \hat{\alpha}$. Hence $\hat{\bar{\alpha}} \preceq \hat{\bar{\bar{\alpha}}}$ and so by Lemma 3.3-(a), we have $\bar{\alpha} \leq \bar{\bar{\alpha}}$.

(c) The proof follows by arguments similar to part (b).

(d) Suppose that for all X in $Set^{\mathcal{X}^{op}}$ and $\alpha, \beta \in \mathcal{S}ub(X)$, $\overline{\alpha \wedge \beta} \leq \bar{\alpha} \wedge \bar{\beta}$. We know $\langle t, t\rangle = \langle t\circ !_\Omega, 1\rangle \wedge \langle 1, t\circ !_\Omega\rangle$. The result then follows by assumption.

Suppose that $\overline{\langle t, t\rangle} \leq \overline{\langle 1, t\circ !_\Omega\rangle} \wedge \overline{\langle t\circ !_\Omega, 1\rangle}$. Let $\alpha = \langle t\circ !_\Omega, 1\rangle$ and $\beta = \langle 1, t\circ !_\Omega\rangle$. One can easily verify that $\hat{\alpha} = \pi_1 : \Omega \times \Omega \longrightarrow \Omega$ and $\hat{\beta} = \pi_2 : \Omega \times \Omega \longrightarrow \Omega$ are the projections, and $\alpha \wedge \beta = \langle t, t\rangle$. By assumption, $\overline{\alpha \wedge \beta} \leq \bar{\alpha} \wedge \bar{\beta}$. By Lemma 3.3-(a), $\widehat{\overline{\alpha \wedge \beta}} \preceq \widehat{\bar{\alpha} \wedge \bar{\beta}}$. By Lemma 3.2-(b) and (c), $j \circ \wedge \circ \langle \hat{\alpha}, \hat{\beta}\rangle \preceq \wedge \circ (j \times j) \circ \langle \hat{\alpha}, \hat{\beta}\rangle$. Since $\langle \hat{\alpha}, \hat{\beta}\rangle$ is the identity, the result follows.

Suppose that $j \circ \wedge \preceq \wedge \circ (j \times j)$. Let X be an object in $Set^{\mathcal{X}^{op}}$ and $\alpha, \beta \in \mathcal{S}ub(X)$. We have $j \circ \wedge \circ \langle \hat{\alpha}, \hat{\beta}\rangle \preceq \wedge \circ (j \times j) \circ \langle \hat{\alpha}, \hat{\beta}\rangle$. By Lemma 3.2-(b) and (c), $\widehat{\overline{\alpha \wedge \beta}} \preceq \widehat{\bar{\alpha} \wedge \bar{\beta}}$ and by Lemma 3.3-(a),

$\overline{\alpha \wedge \beta} \preceq \bar{\alpha} \wedge \bar{\beta}$.

(e) The proof follows by arguments similar to part (d).

□

Corollary 3.5 *Let $\mathcal{M}$ be a class of $\mathcal{X}$-morphisms that satisfies the principality property. The induced universal operation "$-$" is a universal closure operation if and only if $\mathcal{M}$ satisfies (a) and (d) of Definition 1.3. In addition, "$-$" is idempotent if and only if $\mathcal{M}$ satisfies (c) as well.*

Proof Follows from Remark 1.4 and Theorems 1.7, 2.1, and 3.4. □

4. Examples

Throughout this section, the collection of all the identity morphisms, all the retractions, all the monomorphisms, all the epimorphisms, and all the isomorphisms, in a category $\mathcal{X}$ is denoted by $Ids(\mathcal{X})$, $Ret(\mathcal{X})$, $Mono(\mathcal{X})$, $Epi(\mathcal{X})$, and $Iso(\mathcal{X})$, respectively.

We also assume $\mathcal{X}$ is a small category and so for any collection $\mathcal{M} \subseteq \mathcal{X}_1$, $\mathcal{X}$ is $\mathcal{M}$-wellpowered.

Example 4.1 *Let $\mathcal{X}$ be a category. Consider $\mathcal{M} \subseteq Ret(\mathcal{X})$ (in particular $\mathcal{M} = Iso(\mathcal{X})$ or $\mathcal{M} = Ids(\mathcal{X})$). It is easy to verify that the principality property holds if and only if for all morphisms $f \in \mathcal{X}_1$, $\mathcal{M}/d_1 f \neq \emptyset$ implies that $\mathcal{M}/d_0 f \neq \emptyset$; $\mathcal{M}$ has enough retractions if and only if for all objects $x \in \mathcal{X}$, $\mathcal{M}/x \neq \emptyset$; that $\mathcal{M}$ does have almost enough retractions; and that the identity property, the maximum property, and the quasi-meet property all hold.*

Assuming the principality, the induced presheaf $M : \mathcal{X}^{op} \to Set$, where for each x, $M(x) = \{T_x\}$ or $M(x) = \emptyset$, satisfies all the conditions of Theorem 1.7 but (a). If for all x, $\mathcal{M}/x \neq \emptyset$, then (a) holds, too, and M is the smallest Grothendieck topology.

With $\mathcal{N}/x = \{f : d_1 f = x, \mathcal{M}/d_0 f \neq \emptyset\}$, the induced automorphism $j : \Omega \longrightarrow \Omega$ is obtained by $j_x(S) = S \cap \mathcal{N}/x$ and by Theorem 2.1, we have $j = j \circ j$ and $j \circ \wedge = \wedge \circ (j \times j)$. Since $\mathcal{M}/x \neq \emptyset$ is equivalent to $\mathcal{N}/x = \mathcal{X}_1/x$, in case for all x, $\mathcal{M}/x \neq \emptyset$, j reduces to the identity Lawvere–Tierney topology.

The induced universal operation "$-$" on $Set^{\mathcal{X}^{op}}$ sends a subobject $\alpha : A \longrightarrow X$ of X to $\bar{\alpha} : \bar{A} \longrightarrow X$, which is determined for each x, by the inclusion $\bar{\alpha}_x : \bar{A}(x) \hookrightarrow X(x)$, where $\bar{A}(x) = \{u \in X(x) : \hat{\alpha}_x(u) \cap \mathcal{N}/x = T_x\} = \begin{cases} \alpha_x(A(x)) & \text{if } \mathcal{M}/x \neq \emptyset \\ \emptyset & \text{if } \mathcal{M}/x = \emptyset \end{cases}$. This universal operation satisfies all the properties listed in Theorem 3.4 but (a). In case for all x, $\mathcal{M}/x \neq \emptyset$, then (a) holds too and "$-$" is isomorphic to the identity universal closure operation.

As special cases, let $\mathcal{X}$ be any small full subcategory of groupoids and $\mathcal{M}$ be any collection of morphisms, or let $\mathcal{X}$ be the category of finite ordinals and $\mathcal{M} \subseteq Epi(\mathcal{X}) = Ret(\mathcal{X})$, or let $\mathcal{X}$ be the category Mat, see [1], and $\mathcal{M} = Ret(\mathcal{X}) = \{A : A \text{ is an } m \times n \text{ matrix with rank } n\}$.

Example 4.2 *Let $\mathcal{X}$ be a category and $\mathcal{M} \subseteq Epi(\mathcal{X})$ be pullback stable. The principality property holds by pullback stability and the fact that for every $m \in \mathcal{M}/x$ and $f \in \mathcal{X}_1/x$, the pullback, $f^{-1}(m)$, of m along f is in $(f \Rightarrow \langle m \rangle)$. If furthermore $\mathcal{M}$ has enough retractions and is weakly closed under composition (i.e. for 2* *composable morphisms* f *and* g, $f \circ g \sim \mathcal{M}$*), which both hold for* $\mathcal{M} = Epi(\mathcal{X})$*, then the identity property as well as the quasi-meet property hold.*

Thus, under the above hypothesis, the induced functor M*, map* j*, and universal operation "$-$" satisfy (a), (b), (c), and (e) of Theorems 1.7, 2.1, and 3.4, respectively.*

As a special case let $\mathcal{X}$ *be the full subcategory of* Top *consisting of finite ordinal topological spaces and* $\mathcal{M} = Epi(\mathcal{X})$.

Example 4.3 *Let* $\mathcal{X}$ *be a category and* $\mathcal{M} \subseteq Mono(\mathcal{X})$ *be pullback stable, which is the case for* $\mathcal{M} = Mono(\mathcal{X})$*. The principality property holds by pullback stability and the fact that for every* $m \in \mathcal{M}/x$ *and* $f \in \mathcal{X}_1/x$*,* $f^{-1}(m)$ *is in* $(f \Rightarrow \langle m \rangle)$*. If furthermore* $\mathcal{M}$ *has enough retractions and is weakly closed under composition (i.e. for 2 composable morphisms* f *and* g*,* $f \circ g \sim \mathcal{M}$*), which holds for* $\mathcal{M} = Mono(\mathcal{X})$*, then the identity property and the quasi-meet property hold.*

Under the above hypothesis, the induced M*,* j*, and "$-$" satisfy (a), (b), (c), and (e) of Theorems 1.7, 2.1, and 3.4, respectively.*

As special cases let $\mathcal{X}$ *be the full subcategory of* Top *consisting of finite ordinal topological spaces and* $\mathcal{M} = Mono(\mathcal{X})$*, or let* $\mathcal{X}$ *be the category* Mat*, see [1], and* $\mathcal{M} = Mono(\mathcal{X}) = \{A : A \text{ is an } m \times n \text{ matrix with rank } m\}$.

Example 4.4 *Let* $(X, \leq)$ *be a preordered set and* $\mathcal{X} = C(X, \leq)$ *be the category it induces, see [1]. We know in case* $x \leq y$*,* $Hom(x, y)$ *has a unique morphism, which we denote by* (x, y)*. It is not hard to see that* $\langle (a, x) \rangle \cdot (b, x) = \{(c, b) : c \leq b \text{ and } c \leq a\}$ *and that* $((b, x) \Rightarrow \langle (a, x) \rangle) \neq \emptyset$ *if and only if a meet* $a \wedge b$ *exists, in which case* $(a \wedge b, b) \in ((b, x) \Rightarrow \langle (a, x) \rangle)$ *or equivalently* $\langle (a, x) \rangle \cdot (b, x) = \langle (a \wedge b, b) \rangle$.

Let $\mathcal{M}$ *be a class of morphisms of* $\mathcal{X}$*. One can verify that* $\mathcal{M}$ *satisfies the principality property if and only if for each* $(a, x) \in \mathcal{M}/x$ *and* $(b, x) \in \mathcal{X}_1/x$*, a meet* $a \wedge b$ *exists and* $(a \wedge b, b) \in \mathcal{M}/b$*;* $\mathcal{M}$ *has enough retractions (almost enough retractions) if and only if for each* x*,* $\mathcal{M}/x \ni 1_x$ *(*$\mathcal{M}/x = \emptyset$ *or* $\mathcal{M}/x \ni 1_x$*);* $\mathcal{M}$ *has the identity property if and only if for all* x *and for all sieves* S *on* x*, if the set* $\mathcal{M}_S = \{(a, x) \mid \exists y_a \leq a \ \ni: (y_a, a) \in \mathcal{M}, (y_a, x) \in S \text{ and } \forall y \leq a \ \ ((y, x) \in S \Rightarrow y \leq y_a)\}$ *has a maximum in* $\mathcal{M}$*, then it contains* 1_x*;* $\mathcal{M}$ *has the maximum property if and only if for all* x*,* $(\mathcal{M}/x, \leq^{op})$ *is weakly well-ordered (i.e. every nonempty subset of* $\mathcal{M}/x$ *has a maximum) and also for all* x *and* $(a, x) \in \mathcal{X}_1/x$*, either there is* (b, x) *in* $\mathcal{M}/x$ *such that* $b \cong a$ *or for all* $(b, x) \in \mathcal{M}/x$*,* $b \geq a$*; and finally* $\mathcal{M}$ *has the quasi-meet property if and only if* $\mathcal{M}$ *has local binary meet (i.e. for all objects* x*,* $\mathcal{M}/x$ *has binary meet).*

In case $(X, \leq)$ *is a partially ordered set, every maximum or meet that exists is unique and if* $(X, \leq)$ *is a lattice then every binary meet exists and is unique.*

As special cases consider the following examples.

(a) Let $(X, \leq)$ *be any partially ordered set such that every nonempty subset of* X *has a maximum (*$\leq^{op}$ *is then indeed a total order and* $(X, \leq^{op})$ *is well-ordered). Obviously every sieve on an object* $x \in \mathcal{X}$ *is principal and* $\langle (b, x) \rangle \cdot (a, x) = \langle (a \wedge b, a) \rangle$*. For* $\mathcal{M} \subseteq \mathcal{X}_1$*,* $\mathcal{M}_{\langle (b,x) \rangle} = \{(a, x) \mid (a \wedge b, a) \in \mathcal{M}\}$*. Now suppose for all* x*,* $\mathcal{M}/x \neq \emptyset$ *and for* $a \leq b \leq x$*,* $(a, x) \in \mathcal{M}$ *if and only if* $(a, b) \in \mathcal{M}$ *and* $(b, x) \in \mathcal{M}$*. One can then verify that* $\mathcal{M}$ *satisfies the principality as well as all the properties listed in Definition 1.3; and that* $\langle (a, x) \rangle_M = \langle (\hat{a}, x) \rangle$*, where* $\hat{a} \geq a$ *is the largest element of* X *with* $(a, \hat{a}) \in \mathcal{M}$.

So by Theorems 1.7, 2.1, and 3.4, the induced presheaf M*, where* $M(x) = \{\langle (a, x) \rangle : (a, x) \in \mathcal{M}\}$*, is a Grothendieck topology; the induced* j*, where* $j_x(\langle (a, x) \rangle) = \langle (\hat{a}, x) \rangle$ *is a Lawvere–Tierney topology; and* *the induced universal operation "$-$", which takes* $\alpha : A \longrightarrow X$ *to* $\bar{\alpha} : \bar{A} \longrightarrow X$, *where* $\bar{\alpha}_x : \bar{A}(x) \hookrightarrow X(x)$ *is the inclusion with* $\bar{A}(x) = \{u : (\dot{x}, x) \in \mathcal{M}$ *where* $\dot{x} \leq x$ *is the largest with* $X(\dot{x}, x)(u) \in \alpha_{\dot{x}}(A_{\dot{x}})\}$, *is an idempotent universal closure operation.*

As a special case one can take $X = \{\cdots, -3, -2, -1\}$ *in the usual order and* $\mathcal{M}/n = \{(k, n) | -5 \leq k \leq n\}$ *for* $-5 \leq n \leq -1$, $\mathcal{M}/n = \{(n, n)\}$ *otherwise.*

(b) With $\mathbb{N}$ *the set of natural numbers and* $\leq$ *the usual order, let* $\mathcal{X} = C(\mathbb{N}, \leq^{op})$. *In this category all the sieves are principal. For each* n, *set* $\mathcal{M}/n = \{1_n, f_n\}$, *where* $f_n \in Hom_{\mathcal{X}}(n+1, n)$. *It is not hard to verify that* $\mathcal{M}$ *has the principality property and satisfies all the conditions of Definition 1.3 except the identity property. The latter property does not hold, since if* $S = \langle f_n \circ f_{n+1} \rangle$ *is a sieve on* n, *then* $S_M = \langle f_n \rangle$ *has the maximum* $f_n \in \mathcal{M}/n$ *but* S *does not have a maximum in* $\mathcal{M}/n$.

The induced functor M, *where* $M(n) = \{T_n, \langle f_n \rangle\}$, *the induced map* j *that sends each member of* $M(n)$ *to the total sieve and for* $m, n \in \mathbb{N}$, $j_n(\langle f_n \circ f_{n+1} \circ \ldots \circ f_{n+m} \rangle) = \langle f_n \circ f_{n+1} \circ \ldots \circ f_{n+m-1} \rangle$ *and the induced universal operation "$-$", satisfy (a), (b), (d), and (e) of Theorems 1.7, 2.1, and 3.4, respectively. So* M *is a weak Grothendieck topology and* j *is a weak Lawvere–Tierney topology, see [3].*

(c) Let $\mathcal{X} = C(X, \leq)$, *where* $X = \{\frac{n-1}{n} \mid n \in \mathbb{N}\} \cup \{1\}$ *is the semilattice in the usual order. All the sieves are principal except* $S = \{(\frac{n-1}{n}, 1) \mid n \geq 1\} = T_1 - \{(1,1)\}$, *and we have the following.*

$$\langle (x,t) \rangle \cdot (y,t) = \begin{cases} \langle (x,y) \rangle & \text{if } x < y \\ T_y & \text{if } x \geq y \end{cases} \qquad \text{and} \qquad S \cdot (y,1) = \begin{cases} T_y & \text{if } y < 1 \\ S & \text{if } y = 1 \end{cases}$$

(1) Set $\mathcal{M}/t = \{(x,t) \mid x \in X, \frac{4}{5} \leq x \leq t\}$ *for* $\frac{4}{5} < t \in X$ *and* $\mathcal{M}/t = \{1_t\}$ *for* $t \leq \frac{4}{5}$. *We have the following.*

$$\langle (x,t) \rangle_M = \begin{cases} \langle (x,t) \rangle & \text{if } x < \frac{4}{5} \\ T_t & \text{if } x \geq \frac{4}{5} \end{cases} \qquad \text{and} \qquad S_M = S$$

(2) Set $\mathcal{M}/t = \{(x,t) \mid x \in X, 0 \leq x \leq t\}$ *for* $\frac{4}{5} \geq t \in X$ *and* $\mathcal{M}/t = \{1_t\}$ *for* $t > \frac{4}{5}$. *Denoting by* $x \vee (\frac{4}{5})$ *the maximum of* x *and* $\frac{4}{5}$, *we have:*

$$\langle (x,t) \rangle_M = \begin{cases} \langle (x \vee (\frac{4}{5}), t) \rangle & \text{if } t > \frac{4}{5} \\ T_t & \text{if } t \leq \frac{4}{5} \end{cases} \qquad \text{and} \qquad S_M = S.$$

It is then easy to see that in both cases $\mathcal{M}$ *has the principality property and satisfies all the properties listed in Definition 1.3. So by Theorems 1.7, 2.1, and 3.4,* M *is a Grothendieck topology,* j *is a Lawvere–Tierney topology, and "$-$" is an idempotent universal closure operation.*

(d) Let $X = \{x_0, x_1, x_2, x_3, x_4, x_5\}$ *with* $x_1 \leq x_0$, $x_2 \leq x_1$, $x_3 \leq x_2$, $x_4 \leq x_2$, *and* $x_5 \leq x_3$. *Then the category* $\mathcal{X}$ *is generated by the morphisms* $f_0 : x_1 \longrightarrow x_0$, $f_1 : x_2 \longrightarrow x_1$, $f_2 : x_3 \longrightarrow x_2$, $f_3 : x_4 \longrightarrow x_2$, *and* $f_4 : x_5 \longrightarrow x_3$.

Now let $\mathcal{M}/x_0 = \{1_{x_0}, f_0, f_0 \circ f_1\}$, $\mathcal{M}/x_1 = \{1_{x_1}, f_1\}$, $\mathcal{M}/x_2 = \{1_{x_2}\}$, $\mathcal{M}/x_3 = \{1_{x_3}, f_4\}$, $\mathcal{M}/x_4 = \{1_{x_4}\}$, *and* $\mathcal{M}/x_5 = \{1_{x_5}\}$.

It is easy to see that $\mathcal{M}$ *satisfies the principality property as well as all the properties listed in Definition 1.3. Hence the induced functor* M, *where* $M(x_0) = \{T_{x_0}, \langle f_0 \rangle, \langle f_0 \circ f_1 \rangle\}$, $M(x_1) = \{T_{x_1}, \langle f_1 \rangle\}$, $M(x_2) = \{T_{x_2}\}$, $M(x_3) = \{T_{x_3}, \langle f_4 \rangle\}$, $M(x_4) = \{T_{x_4}\}$, *and* $M(x_5) = \{T_{x_5}\}$, *is a Grothendieck topology,* j *is a Lawvere–Tierney topology, and "$-$" is an idempotent universal closure operation.*

Example 4.5 *Let $\mathcal{X}$ be the category generated by the morphisms $w \xrightarrow{k} z$, $z \xrightarrow{g} y$, $z \xrightarrow{h} y$, $y \xrightarrow{f} x$, and $w \xrightarrow{n} x$, with $f \circ g = f \circ h$ and $g \circ k = h \circ k$. Set $\mathcal{M} = \{1_x, 1_y, 1_z, 1_w, f, k\}$. It is easy to see that $\mathcal{M}$ has the principality property and satisfies all the conditions of Definition 1.3; thus, M is a Grothendieck topology, j is a Lawvere–Tierney topology, and "$-$" is an idempotent universal closure operation.*

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