Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2013) 37: 899 – 907
© TÜBİTAK
doi:10.3906/mat-1207-13

Labelings of type $(1, 1, 1)$ for toroidal fullerenes

Martin BAČA,[1,2,*] **Muhammad NUMAN,**[2] **Ayesha SHABBIR**[2]
[1]Department of Applied Mathematics and Informatics, Technical University, Košice, Slovakia
[2]Abdus Salam School of Mathematical Sciences, GC University, 68-B, New Muslim Town, Lahore, Pakistan

Received: 10.07.2012 • **Accepted:** 01.11.2012 • **Published Online:** 23.09.2013 • **Printed:** 21.10.2013

Abstract: In this paper we deal with the problem of labeling the vertices, edges, and faces of a toroidal fullerene $\mathbb{H}_m^n$ with mn hexagons by the consecutive integers from 1 up to $|V(\mathbb{H}_m^n)| + |E(\mathbb{H}_m^n)| + |F(\mathbb{H}_m^n)|$ in such a way that the set of face-weights of 6-sided faces forms an arithmetic progression with common difference d, where by face-weight we mean the sum of the label of that face and the labels of vertices and edges surrounding that face.

The paper examines the existence of such labelings for several differences d.

Key words: Toroidal fullerene, toroidal polyhex, super d-antimagic labeling

1. Introduction

Let $\mathbb{G}$ be a family of cubic graphs embedded on the surface of a sphere or a torus such that each face is a hexagon. Let V, E, and F be the vertex set, the edge set, and the face set of a graph $G \in \mathbb{G}$.

A labeling of type $(1, 1, 1)$ assigns labels from the set $\{1, 2, \dots, |V(G)| + |E(G)| + |F(G)|\}$ into the vertices, edges, and faces of a graph G in such a way that each vertex, edge, and face receives exactly one label and each number is used exactly once as a label.

The *weight* of a 6-sided face under a labeling of type $(1, 1, 1)$ is the sum of labels carried by that face and the edges and vertices on its boundary.

A labeling of type $(1, 1, 1)$ of a graph $G \in \mathbb{G}$ is called *d-antimagic* if weights of 6-sided faces form an arithmetic sequence starting from a and having common difference d, where $a > 0$ and $d \geq 0$ are 2 given integers.

The concept of the d-antimagic labeling of the plane graphs was defined in [9]. The d-antimagic labelings of type $(1, 1, 1)$ for the generalized Petersen graph $P(n, 2)$, the hexagonal planar maps, and the grids can be found in [6], [7], and [8].

In particular for $d = 0$, Lih [12] called such labeling *magic* and described magic (0-antimagic) labelings for the wheels, the friendship graphs, and the prisms. However, the subject of magic labeling can be traced back to the 13th century when similar notions were investigated in very classical Chinese mathematics (see [12]). The magic (0-antimagic) labelings for grid graphs and honeycombs are given in [2] and [3], respectively.

A d-antimagic labeling is called *super* if the smallest possible labels appear on the vertices. The super d-antimagic labelings of type $(1, 1, 1)$ for antiprisms are described in [4], and for disjoint union of prisms, they

*Correspondence: martin.baca@tuke.sk
2000 *AMS Mathematics Subject Classification:* 05C78.

are given in [1]. The existence of super d-antimagic labeling of type $(1,1,1)$ for the plane graphs containing a special Hamilton path is examined in [5] and for disconnected plane graphs is investigated in [10].

In this paper we study d-antimagic labelings for toroidal fullerenes. Classical fullerene is an all-carbon molecule in which the atoms are arranged on a pseudospherical framework made up entirely of pentagons and hexagons. Its molecular graph is a finite trivalent graph embedded on the surface of a sphere with only hexagonal and (exactly 12) pentagonal faces. Deza et al. [11] considered fullerene's extension to other closed surfaces and showed that only 4 surfaces are possible, namely sphere, torus, Klein bottle, and projective plane. Unlike spherical fullerenes, toroidal and Klein bottle fullerenes have been regarded as tessellations of entire hexagons on their surfaces since they must contain no pentagons; see [11] and [13].

A *toroidal fullerene* (toroidal polyhex) is a cubic bipartite graph embedded on the torus such that each face is a hexagon. Note that the torus is a closed surface that can carry the graph of a toroidal polyhex such that all its vertices have degree 3 and all faces of the embedding are hexagons; see Figure 1.

More precisely, let L be a regular hexagonal lattice and P_m^n be an $m \times n$ quadrilateral section (with m hexagons on the top and bottom sides and n hexagons on the lateral sides, n is even) cut from the regular hexagonal lattice L. First identify 2 lateral sides of P_m^n to form a cylinder, and finally identify the top and bottom sides of P_m^n at their corresponding points; see Figure 2. From this we get a toroidal polyhex $\mathbb{H}_m^n$ with mn hexagons, $2mn$ vertices, and $3mn$ edges.

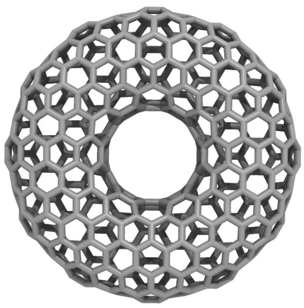

Figure 1. Toroidal polyhex.

There have been a few works on the enumeration of perfect matchings of toroidal polyhexes by applying various techniques, such as the transfer matrix [14, 15]. In [16, 17, 18], a k-resonance of toroidal polyhexes was studied.

2. Necessary conditions

In this section we shall find bounds for a feasible value of d for the super d-antimagic labeling of type $(1,1,1)$ of the toroidal polyhex $\mathbb{H}_m^n$.

Theorem 1 *For every toroidal polyhex $\mathbb{H}_m^n$, n even and $m,n \geq 2$, there is no super d-antimagic labeling of type $(1,1,1)$ with $d \geq 41$.*

Proof Suppose that $\mathbb{H}_m^n$ with $2mn$ vertices, $3mn$ edges, and mn faces has a super d-antimagic labeling of type $(1,1,1)$, $g : V(\mathbb{H}_m^n) \cup E(\mathbb{H}_m^n) \cup F(\mathbb{H}_m^n) \to \{1,2,\dots,6mn\}$, and $\{a, a+d, a+2d, \dots, a+(mn-1)d\}$ is the set of face-weights.

The minimum possible face-weight of a 6-sided face is the sum of the 6 smallest possible vertex labels, namely $1,2,\dots,6$, and the 7 smallest possible labels from the set $\{2mn+1, 2mn+2, \dots, 6mn\}$ for 6 edges and 1 face. Thus,

$$a \geq \sum_{i=1}^{6} i + \sum_{j=1}^{7} (2mn + j) = 14mn + 49.$$

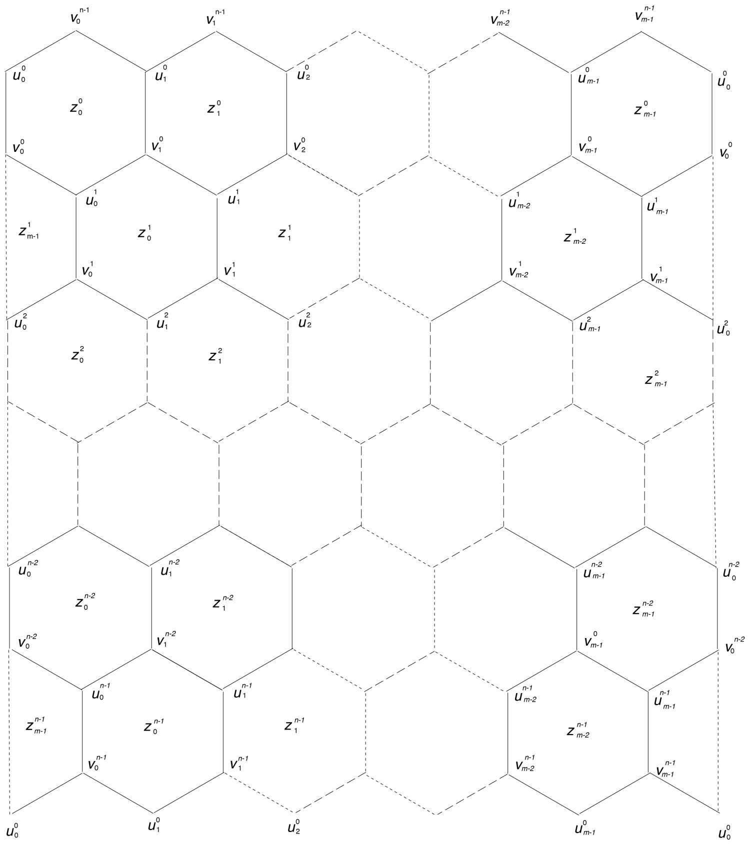

Figure 2. Quadrilateral section P_m^n cuts from the regular hexagonal lattice.

To calculate all the face-weights each vertex label is used 3 times, each edge label is used twice, and each face label is used once. In this case the vertices receive the smallest possible labels $1, 2, \dots, 2mn$. The edges could conceivably receive the $3mn$ smallest labels from the set $\{2mn+1, 2mn+2, \dots, 6mn\}$ and faces receive the remaining labels from the set or, at the other extreme, the faces receive mn smaller labels from the set and the edges can receive $3mn$ largest labels or anything in between. Thus, we get:

$$\frac{1}{2}(65m^2n^2 + 13mn) \leq 3\sum_{v \in V} g(v) + 2\sum_{e \in E} g(e) + \sum_{f \in F} g(f)$$

$$\leq \frac{1}{2}(71m^2n^2 + 13mn). \tag{1}$$

The sum of all the face-weights is

$$a + (a+d) + \dots + (a + (mn-1)d) = mna + \frac{(mn-1)mnd}{2}. \tag{2}$$

Since the minimum possible face-weight is at least $14mn + 49$, from Eqs. (1) and (2) we get the following inequality:

$$(mn-1)d \leq 71mn + 13 - 2a$$

and

$$d \leq 43 - \frac{42}{mn-1} < 43.$$

On the other hand, the maximum possible face-weight is the sum of the 6 largest possible vertex labels, namely $2mn-5, 2mn-4, \dots, 2mn$, and 7 largest possible labels from the set $\{2mn+1, 2mn+2, \dots, 6mn\}$ for 6 edges and 1 face. It is no more than

$$\sum_{i=1}^{6}(2mn-i+1) + \sum_{j=1}^{7}(6mn-j+1) = 54mn - 36.$$

Then we get an upper bound on the parameter d as

$$a + (mn-1)d \leq 54mn - 36$$

and

$$d \leq 40 - \frac{45}{mn-1} < 40.$$

Thus, we have arrived at the desired result. □

3. Results on super d-antimagic labelings of toroidal polyhex

Since $\mathbb{H}_m^n$ is 2-colorable cubic graph, there exist a 1-factor (perfect matching) and a 2-factor (a collection of n cycles on $2m$ vertices each). Denote the edges in 1-factor by the symbols $u_i^j v_i^j$, for $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$. Let $u_0^j v_0^{j-1} u_1^j v_1^{j-1} u_2^j v_2^{j-1} \dots v_{m-2}^{j-1} u_{m-1}^j v_{m-1}^{j-1} u_0^j$ and $v_0^j u_0^{j+1} v_1^j u_1^{j+1} v_2^j u_2^{j+1} \dots u_{m-2}^{j+1} v_{m-1}^j u_{m-1}^{j+1} v_0^j$ be the $2m$-cycles in 2-factor for j even, $0 \leq j \leq n-2$. Let z_i^j, $0 \leq i \leq m-1$, $0 \leq j \leq n-1$ be the 6-sided face of $\mathbb{H}_m^n$; see Figure 2.

For $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$, we construct the vertex labeling $g : V(\mathbb{H}_m^n) \to \{1, 2, \dots, 2mn\}$ in the following way:

$g(u_i^j) = mj + i + 1$ and

$g(v_i^j) = (n+j)m + i + 1$.

Now, we label by consecutive values $3mn+1, 3mn+2, \dots, 5mn$ the edges of $2m$-cycles as follows:

$h(u_i^j v_i^{j-1}) = (5n-j)m - i$ if j is even,

$h(u_i^j v_{i+1}^{j-1}) = (5n-j)m - i$ if j is odd,

$h(v_i^j u_{i+1}^{j+1}) = (4n-j)m - i$ if j is odd,

$h(v_i^j u_i^{j+1}) = (4n-j)m - i$ if j is even,

where $0 \leq i \leq m-1$, $0 \leq j \leq n-1$ and i is taken modulo m, j is taken modulo n.

With the vertex labeling g and the edge labeling h in hand, we prove the following lemmas.

Lemma 1 *Let* $w(z_i^j) = g(u_i^j) + h(u_i^j v_{i+1}^{j-1}) + g(v_{i+1}^{j-1}) + h(v_{i+1}^{j-1} u_{i+1}^j) + g(u_{i+1}^j) + g(v_i^j) + h(v_i^j u_{i+1}^{j+1}) + g(u_{i+1}^{j+1}) + h(u_{i+1}^{j+1} v_{i+1}^j) + g(v_{i+1}^j)$ *be a partial weight of the face* z_i^j *for* j *odd,* $1 \le j \le n-1$*, and* $0 \le i \le m-1$*. For* n *even,* $n \ge 2$ *and* $m \ge 2$*, the partial weight* $w(z_i^j) = (21n+2j)m + 2i + 8$*,* $0 \le i \le m-2$*, and* $w(z_{m-1}^j) = (21n+2j)m + 6$*.*

Proof By direct computation for the partial weight of the face z_i^j, j odd, we obtain:

$w(z_i^j) = mj+i+1+(5n-j)m-i+(n+j-1)m+(i+1)+1+(4n-j+1)m-(i+1)+mj+(i+1)+1+(n+j)m+i+1+(4n-j)m-i+(j+1)m+(i+1)+1+(5n-j-1)m-(i+1)+(n+j)m+(i+1)+1 = (21n+2j)m+2i+8$ for $0 \le i \le m-2$ and

$w(z_{m-1}^j) = mj+(m-1)+1+(5n-j)m-(m-1)+(n+j-1)m+1+(4n-j+1)m+mj+1+(n+j)m+(m-1)+1+(4n-j)m-(m-1)+m(j+1)+1+(5n-j-1)m+m(n+j)+1 = (21n+2j)m+6$. □

Lemma 2 *Let* $w(z_i^j) = g(u_i^j) + h(u_i^j v_i^{j-1}) + g(v_i^{j-1}) + h(v_i^{j-1} u_{i+1}^j) + g(u_{i+1}^j) + g(v_i^j) + h(v_i^j u_i^{j+1}) + g(u_i^{j+1}) + h(u_i^{j+1} v_{i+1}^j) + g(v_{i+1}^j)$ *be a partial weight of the face* z_i^j *for* j *even,* $0 \le j \le n-2$*, and* $0 \le i \le m-1$*. For* n *even,* $n \ge 2$ *and* $m \ge 2$*, the partial weight* $w(z_i^j) = (21n+2j)m + 2i + 8$*,* $0 \le i \le m-2$*, and* $w(z_{m-1}^j) = (21n+2j)m + 6$*.*

Proof It is easily seen that for the partial weight of the face z_i^j, j even, we have:

$w(z_i^j) = mj+i+1+(5n-j)m-i+(n+j-1)m+i+1+(4n-j+1)m-i+mj+(i+1)+1+(n+j)m+i+1+(4n-j)m-i+m(j+1)+i+1+(5n-j-1)m-i+(n+j)m+(i+1)+1 = (21n+2j)m+2i+8$ for $0 \le i \le m-2$ and

$w(z_{m-1}^j) = mj+(m-1)+1+(5n-j)m-(m-1)+(n+j-1)m+(m-1)+1+(4n-j+1)m-(m-1)+mj+1+(n+j)m+(m-1)+1+(4n-j)m-(m-1)+m(j+1)+(m-1)+1+(5n-j-1)m-(m-1)+(n+j)m+1 = (21n+2j)m+6$. □

By Lemmas 1 and 2, we get:

Theorem 2 *For* n *even,* $n \ge 2$*, and* $m \ge 2$*, the toroidal polyhex* $\mathbb{H}_m^n$ *has a super* 1*-antimagic labeling and a super* 3*-antimagic labeling of type* $(1,1,1)$*.*

Proof Label the vertices of $\mathbb{H}_m^n$ by the labeling g and the edges of $2m$-cycles by the labeling h. From the previous lemmas it follows that the partial weights of faces z_i^j, $0 \le j \le n-1$, $0 \le i \le m-1$, constitute an arithmetic sequence of difference 2, namely the sequence $21mn+6, 21mn+8, \dots, 23mn+2, 23mn+4$.

Let us distinguish 2 cases.

Case 1. $d = 1$. If we label the edges in 1-factor with values in the set $\{2mn+1, 2mn+2, \dots, 3mn\}$ such that

$h(u_i^j v_i^j) = (3n-j)m - i$ for $0 \le j \le n-1$, $0 \le i \le m-1$, then

$w'(z_i^j) = w(z_i^j) + h(u_i^j v_i^j) + h(u_{i+1}^j v_{i+1}^j) = (21n+2j)m+2i+8+(3n-j)m-i+(3n-j)m-(i+1) = 27mn+7$ for $0 \le j \le n-1$, $0 \le i \le m-2$ and

$w'(z_{m-1}^j) = w(z_{m-1}^j) + h(u_{m-1}^j v_{m-1}^j) + h(u_0^j v_0^j) = (21n+2j)m+6+(3n-j)m-(m-1)+(3n-j)m = (27n-1)m+7$ for $0 \le j \le n-1$.

If we complete the face labeling with values in the set $\{5mn+1, 5mn+2, \dots, 6mn\}$ such that the every face z_i^j receives a value

$f(z_i^j) = (5n+j)m + i + 1$, for $0 \le j \le n-1$, $0 \le i \le m-1$,

then for face-weights we have

$wt(z_i^j) = w'(z_i^j) + f(z_i^j) = 27mn + 7 + (5n+j)m + i + 1 = (32n+j)m + i + 8$, for $0 \le j \le n-1$, $0 \le i \le m-2$ and $wt(z_{m-1}^j) = w'(z_{m-1}^j) + f(z_{m-1}^j) = (27n-1)m + 7 + (5n+j)m + (m-1) + 1 = (32n+j)m + 7$, for $0 \le j \le n-1$.

It can be seen that face-weights successively attain values from $32mn+7$ up to $33mn+6$. This implies that the resulting labeling of type $(1,1,1)$ is a super 1-antimagic.

Case 2. $d = 3$. Complete the edge labeling h by labels of edges in 1-factor such that

$h(u_i^j v_i^j) = (2n+j)m + i + 1$ for $0 \le j \le n-1$, $0 \le i \le m-1$.

Now, define a face labeling in the following way:

$f(z_i^j) = (6n-j)m - i$, for $0 \le j \le n-1$, $0 \le i \le m-1$.

Observe that labels of edges in 1-factor are from the set $\{2mn+1, 2mn+2, \dots, 3mn\}$ and labels of faces are from the set $\{5mn+1, 5mn+2, \dots, 6mn\}$. For face-weights we get:

$wt(z_i^j) = w(z_i^j) + h(u_i^j v_i^j) + h(u_{i+1}^j v_{i+1}^j) + f(z_i^j) = (21n+2j)m + 2i + 8 + (2n+j)m + i + 1 + (2n+j)m + (i+1) + 1 + (6n-j)m - i = (31n+3j)m + 3i + 11$, for $0 \le j \le n-1$, $0 \le i \le m-2$ and

$wt(z_{m-1}^j) = w(z_{m-1}^j) + h(u_{m-1}^j v_{m-1}^j) + h(u_0^j v_0^j) + f(z_{m-1}^j) = (21n+2j)m + 6 + (2n+j)m + (m-1) + 1 + (2n+j)m + 1 + (6n-j)m - (m-1) = (31n+3j)m + 8$, for $0 \le j \le n-1$.

We can see that face-weights constitute a set $\{31mn+8, 31mn+11, \dots, 34mn+5\}$, i.e. constitute the arithmetic sequence of difference 3. Thus, the resulting labeling is a super 3-antimagic labeling of type $(1,1,1)$. □

Next, we are going to show a super 5-antimagic labeling of type $(1,1,1)$ for the toroidal polyhex $\mathbb{H}_m^n$.

Let $u_i^j v_{i+1}^{j-1}$ and $v_i^j u_{i+1}^{j+1}$ be edges in a 1-factor of $\mathbb{H}_m^n$ for j odd, $1 \le j \le n-1$, $0 \le i \le m-1$. Let $u_i^0 v_i^0 u_i^1 v_i^1 u_i^2 v_i^2 \dots u_i^{n-2} v_i^{n-2} u_i^{n-1} v_i^{n-1}$ be $2n$-cycles in a 2-factor of $\mathbb{H}_m^n$ for $0 \le i \le m-1$.

For $0 \le i \le m-1$ and $0 \le j \le n-1$, define the bijection $\alpha : V(\mathbb{H}_m^n) \to \{1, 2, \dots, 2mn\}$ as follows:

$$\alpha(u_i^j) = 2ni + 2j + 2,$$

$$\alpha(v_i^j) = \begin{cases} 2ni + 2j + 3, & \text{if } 0 \le j \le n-2 \\ 2ni + 1, & \text{if } j = n-1. \end{cases}$$

The edges of $2n$-cycles, for $0 \le i \le m-1$, we label by the consecutive integers $2mn+1, 2mn+2, \dots, 4mn$ such that

$\beta(u_i^j v_i^j) = 2n(2m-i) - 2j - 1$ if $0 \le j \le n-1$,

$\beta(v_i^j u_i^{j+1}) = 2n(2m-i) - 2j - 2$ if $0 \le j \le n-2$,

$\beta(v_i^{n-1} u_i^0) = 2n(2m-i)$, where j is taken modulo n.

Lemma 3 *Let* $w(z_i^j) = \alpha(u_i^j) + \beta(u_i^j v_i^j) + \alpha(v_i^j) + \beta(v_i^j u_i^{j+1}) + \alpha(u_i^{j+1}) + \alpha(v_{i+1}^j) + \beta(v_{i+1}^j u_{i+1}^j) + \alpha(u_{i+1}^j) +$

$\alpha(v_i^{j-1})+\beta(v_i^{j-1}u_i^j)$ *be a partial weight of the face* z_i^j *of* $\mathbb{H}_m^n$ *for* j *even,* $0 \le j \le n-2$*, and* $0 \le i \le m-1$*. For* n *even,* $n \ge 2$ *and* $m \ge 2$*, the partial weight* $w(z_i^j) = 2n(8m+2i+1)+4j+11$*,* $0 \le i \le m-2$*,* $0 \le j \le n-2$*, and* $w(z_{m-1}^j) = 2n(9m-1)+4j+11$*,* $0 \le j \le n-2$*.*

Proof Suppose that j is even. For $0 \le j \le n-2$ and $0 \le i \le m-2$ the partial weights of z_i^j are

$w(z_i^j) = 2ni+2j+2+2n(2m-i)-2j-1+2ni+2j+3+2n(2m-i)-2j-2+2ni+2(j+1)+2+2n(i+1)+2j+3+2n(2m-i-1)-2j-1+2n(i+1)+2j+2+2ni+2(j-1)+3+2n(2m-i)-2(j-1)-2 = 2n(8m+2i+1)+4j+11$.

For $0 \le j \le n-2$ we have $w(z_{m-1}^j) = 2n(m-1)+2j+2+2n(2m-m+1)-2j-1+2n(m-1)+2j+3+2n(2m-m+1)-2j-2+2n(m-1)+2(j+1)+2+2j+3+4mn-2j-1+2j+2+2n(m-1)+2(j-1)+3+2n(2m-m+1)-2(j-1)-2 = 2n(9m-1)+4j+11$. □

Lemma 4 *Let* $w(z_i^j) = \alpha(u_i^j)+\beta(u_i^jv_i^j)+\alpha(v_i^j)+\alpha(u_{i+1}^{j+1})+\beta(u_{i+1}^{j+1}v_{i+1}^j)+\alpha(v_{i+1}^j)+\beta(u_{i+1}^jv_{i+1}^j)+\alpha(u_{i+1}^j)+\beta(u_{i+1}^jv_{i+1}^{j-1})+\alpha(v_{i+1}^{j-1})$ *be a partial weight of the face* z_i^j *for* j *odd,* $1 \le j \le n-1$*, and* $0 \le i \le m-1$*. For* n *even,* $n \ge 2$ *and* $m \ge 2$*, the partial weight* $w(z_i^j) = 2n(8m+2i+1)+4j+11$*,* $0 \le i \le m-2$*,* $1 \le j \le n-3$*,* $w(z_{m-1}^j) = 2n(9m-1)+4j+11$*,* $1 \le j \le n-3$*,* $w(z_i^{n-1}) = 2n(8m+2i+1)+7$*,* $0 \le i \le m-2$*, and* $w(z_{m-1}^{n-1}) = 2n(9m-1)+7$*.*

Proof Assume that j is odd. For the partial weights of z_i^j we have

$w(z_i^j) = 2ni+2j+2+2n(2m-i)-2j-1+2ni+2j+3+2n(i+1)+2(j+1)+2+2n(2m-i-1)-2j-2+2n(i+1)+2j+3+2n(2m-i-1)-2j-1+2n(i+1)+2j+2+2n(2m-i-1)-2(j-1)-2+2n(i+1)+2(j-1)+3 = 2n(8m+2i+1)+4j+11$ for $1 \le j \le n-3$, $0 \le i \le m-2$,

$w(z_{m-1}^j) = 2n(m-1)+2j+2+2n(2m-m+1)-2j-1+2n(m-1)+2j+3+2(j+1)+2+4mn-2j-2+2j+3+4mn-2j-1+2j+2+4mn-2(j-1)-2+2(j-1)+3 = 2n(9m-1)+4j+11$ for $1 \le j \le n-3$,

$w(z_i^{n-1}) = 2ni+2(n-1)+2+2n(2m-i)-2(n-1)-1+2ni+1+2n(i+1)+2+4mn-2n(i+1)+2n(i+1)+1+2n(2m-i-1)-2(n-1)-1+2n(i+1)+2(n-1)+2+2n(2m-i-1)-2(n-2)-2+2n(i+1)+2(n-2)+3 = 2n(8m+2i+1)+7$ for $0 \le i \le m-2$ and

$w(z_{m-1}^{n-1}) = 2n(m-1)+2(n-1)+2+2n(2m-m+1)-2(n-1)-1+2n(m-1)+1+2+4mn+1+4mn-2(n-1)-1+2(n-1)+2+4mn-2(n-2)-2+2(n-2)+3 = 2n(9m-1)+7$. □

Note that from Lemmas 3 and 4 it follows that partial weights of the faces z_i^j constitute 2 arithmetic progressions with difference 4, namely $2n(8m+1)+7, 2n(8m+1)+11, \dots, 2n(10m-1)+3$ and $2n(9m-1)+7, 2n(9m-1)+11, \dots, 2n(9m+1)+3$.

Theorem 3 *For* n *even,* $n \ge 2$*, and* $m \ge 2$*, the toroidal polyhex* $\mathbb{H}_m^n$ *has a super* 5*-antimagic labeling of type* $(1,1,1)$*.*

Proof Label the vertices of $\mathbb{H}_m^n$ by the labeling α and the edges of $2n$-cycles by the labeling β. Complete the edge labeling β by labels of edges in the 1-factor such that

$\beta(u_{i+1}^jv_i^{j-1}) = n(4m+i)+j+1$ if $1 \le i \le m-2$ and $1 \le j \le n-3$,

$\beta(u_{i+1}^0v_i^{n-1}) = n(4m+i)+1$ if $0 \le i \le m-2$,

$\beta(u_i^{j+1}v_{i+1}^j) = n(4m+i)+j+2$ if $0 \le i \le m-2$ and $1 \le j \le n-1$,

$\beta(u_0^0 v_{m-1}^{n-1}) = n(6m-1)+1$,

$\beta(u_0^j v_{m-1}^{j-1}) = n(6m-1)+j+1$ if $0 \le j \le n-2$, and

$\beta(u_{m-1}^{j+1}v_0^j) = n(6m-1)+j+2$ if $0 \le j \le n-2$.

The face values we arranged in such a way that

$\gamma(z_i^j) = n(6m-i-1)-j$ for $0 \le j \le n-1$, $0 \le i \le m-1$.

There is no problem in seeing that the completed edge labels and face labels use consecutive integers from the set $\{4mn+1, 4mn+2, \dots, 6mn\}$. If j is even, then for face-weights we get

$wt(z_i^j) = w(z_i^j) + \beta(u_{i+1}^j v_i^{j-1}) + \beta(v_{i+1}^j u_i^{j+1}) + \gamma(z_i^j) = 2n(8m+2i+1)+4j+11+n(4m+i)+j+1+n(4m+i)+j+2+n(6m-i-1)-j = n(30m+5i+1)+5j+14$ for $0 \le j \le n-2$, $0 \le i \le m-2$, and

$wt(z_{m-1}^j) = w(z_{m-1}^j) + \beta(u_0^j v_{m-1}^{j-1}) + \beta(v_0^j u_{m-1}^{j+1}) + \gamma(z_{m-1}^j) = 2n(9m-1)+4j+11+n(6m-1)+j+1+n(6m-1)+j+2+n(6m-(m-1)-1)-j = n(35m-4)+5j+14$ for $0 \le j \le n-2$.

If j is odd, then for face-weights we have

$wt(z_i^j) = w(z_i^j) + \beta(u_i^j v_{i+1}^{j-1}) + \beta(v_i^j u_{i+1}^{j+1}) + \gamma(z_i^j) = 2n(8m+2i+1)+4j+11+n(4m+i)+(j-1)+2+n(4m+i)+(j+1)+1+n(6m-i-1)-j = n(30m+5i+1)+5j+14$ for $1 \le j \le n-3$ and $0 \le i \le m-2$,

$wt(z_{m-1}^j) = w(z_{m-1}^j) + \beta(u_{m-1}^j v_0^{j-1}) + \beta(v_{m-1}^j u_0^{j+1}) + \gamma(z_{m-1}^j) = 2n(9m-1)+4j+11+n(6m-1)+(j-1)+2+n(6m-1)+(j+1)+1+n(6m-(m-1)-1)-j = n(35m-4)+5j+14$ for $1 \le j \le n-3$,

$wt(z_i^{n-1}) = w(z_i^{n-1}) + \beta(u_i^{n-1}v_{i+1}^{n-2}) + \beta(v_i^{n-1}u_{i+1}^0) + \gamma(z_i^{n-1}) = 2n(8m+2i+1)+7+n(4m+i)+(n-2)+2+n(4m+i)+1+n(6m-i-1)-(n-1) = n(30m+5i+1)+9$ for $0 \le i \le m-2$,

$wt(z_{m-1}^{n-1}) = w(z_{m-1}^{n-1}) + \beta(u_{m-1}^{n-1}v_0^{n-2}) + \beta(v_{m-1}^{n-1}u_0^0) + \gamma(z_{m-1}^{n-1}) = 2n(9m-1)+7+n(6m-1)+(n-2)+2+n(6m-1)+1+n(6m-(m-1)-1)-(n-1) = n(35m-4)+9$.

It is not difficult to see that the face-weights constitute the arithmetic progression with the first term $n(30m+1)+9$ and common difference 5, which implies that the resulting labeling is a super 5-antimagic labeling of type $(1,1,1)$. □

4. Concluding remarks

In the foregoing section we studied the existence of super d-antimagic labelings of type $(1,1,1)$ for the toroidal polyhex $\mathbb{H}_m^n$. We labeled the edges of a 1-factor by consecutive integers and then in successive steps we labeled the edges of $2m$-cycles (respectively $2n$-cycles) in a 2-factor by consecutive integers. This technique allows us to construct super d-antimagic labelings of type $(1,1,1)$ for $d \in \{1,3,5\}$. We did not solve the existence of such labelings for $d \in \{0,2,4\}$. We think that in general such labelings exist, but the corresponding labelings have to be constructed by other methods than those described in this article. Therefore, we suggest the following open problem.

Open Problem 1 *Find other possible values of the parameter d and corresponding super d-antimagic labelings of type $(1,1,1)$ for the toroidal polyhex $\mathbb{H}_m^n$.*

Acknowledgment

The research for this article was supported by Slovak VEGA Grant 1/0130/12 and Higher Education Commission of Pakistan Grant HEC(FD)/2007/555.

References

[1] Ali, G., Bača, M., Bashir, F., Semaničová-Feňovčíková, A.: On face antimagic labelings of disjoint union of prisms. Utilitas Math. 85, 97–112 (2011).

[2] Bača, M.: On magic labelings of grid graphs. Ars Combin. 33, 295–299 (1992).

[3] Bača, M.: On magic labelings of honeycomb. Discrete Math. 105, 305–311 (1992).

[4] Bača, M., Bashir, F., Semaničová, A.: Face antimagic labelings of antiprisms. Utilitas Math. 84, 209–224 (2011).

[5] Bača, M., Brankovic, L., Semaničová-Feňovčíková, A.: Labelings of plane graphs containing Hamilton path. Acta Math. Sin. 27, 701–714 (2011).

[6] Bača, M., Jendroľ, S., Miller, M., Ryan, J.: Antimagic labelings of generalized Petersen graphs that are plane. Ars Combin. 73, 115–128 (2004).

[7] Bača, M., Baskoro, E.T., Jendroľ, S., Miller, M.: Antimagic labelings of hexagonal planar maps. Utilitas Math. 66, 231–238 (2004).

[8] Bača, M., Lin, Y., Miller, M.: Antimagic labelings of grids. Utilitas Math. 72, 65–75 (2007).

[9] Bača, M., Miller, M.: On d-antimagic labelings of type $(1,1,1)$ for prisms. J. Combin. Math. Combin. Comput. 44, 199–207 (2003).

[10] Bača, M., Miller, M., Phanalasy, O., Semaničová-Feňovčíková, A.: Super d-antimagic labelings of disconnected plane graphs. Acta Math. Sin. 26, 2283–2294 (2010).

[11] Deza, M., Fowler, P.W., Rassat, A., Rogers, K.M.: Fullerenes as tilings of surfaces. J. Chem. Inf. Comput. Sci. 40, 550–558 (2000).

[12] Lih, K.W.: On magic and consecutive labelings of plane graphs. Utilitas Math. 24, 165–197 (1983).

[13] Klein, D.J.: Elemental benzenoids. J. Chem. Inf. Comput. Sci. 34, 453–459 (1994).

[14] Klein, D.J., Zhu, H.: Resonance in elemental benzenoids. Discrete Appl. Math. 67, 157–173 (1996).

[15] Schmalz, T.G., Seitz, W.A., Klein, D.J., Hite, G.E.: Elemental carbon cages. J. Am. Chem. Soc. 110, 1113–1127 (1988).

[16] Ye, D., Qi, Z., Zhang, H.: On k-resonant fullerene graphs. SIAM J. Discrete Math. 23, 1023–1044 (2009).

[17] Ye, D., Zhang, H.: 2-extendability of toroidal polyhexes and Klein-bottle polyhexes. Discrete Appl. Math. 157, 292–299 (2009).

[18] Ye, D., Zhang, H.: Extremal fullerene graphs with the maximum Clar number. Discrete Appl. Math. 157, 3152–3173 (2009).