Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2013) 37: 993 – 1000
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doi:10.3906/mat-1111-34

Complemented invariant subspaces of structural matrix algebras

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Received: 25.11.2011 • **Accepted:** 17.12.2012 • **Published Online:** 23.09.2013 • **Printed:** 21.10.2013

Abstract: In this paper, we explore when the lattice of invariant subspaces of a structural matrix algebra can be complemented. We give several equivalent conditions for this lattice to be a Boolean algebra.

Key words: Boolean algebra, structural matrix algebra, invariant subspace, lattice of invariant subspaces

1. Introduction

Let V denote a vector space of finite dimension n over a field F. Let $\mathcal{L}(V)$ denote the set of all subspace of V. Then $\mathcal{L}(V)$ is a modular lattice under the operations $\cap$ and $+$. If $\mathcal{W}$ is a sublattice, it is also modular.

Let $Hom(V)$ denote the algebra of all linear transformations of V onto itself. As usual, $Hom(V)$ can be identified with $M_n(F)$, the algebra of all $n \times n$ matrices over F. We assume that all algebras contain the identity map, I.

Definition 1.1 *Let $\mathcal{V}$ be a sublattice of $\mathcal{L}(V)$ and let $\mathcal{R}$ be a subalgebra of $Hom(V)$. We define*

$$Alg\mathcal{V} = \{\theta \in Hom(V) : W\theta \subset W,\ \textit{for every } W \in \mathcal{V}\}$$

and

$$Lat\mathcal{R} = \{W \in \mathcal{L}(V) : W\theta \subset W,\ \textit{for every } \theta \in \mathcal{R}\}.$$

$Alg\mathcal{V}$ is a subalgebra of $Hom(V)$ and $Lat\mathcal{R}$ is a sublattice of $\mathcal{L}(V)$.

In general, the containments $LatAlg\mathcal{V} \supseteq \mathcal{V}$ and $AlgLat\mathcal{R} \supseteq \mathcal{R}$ are proper. If equality holds, then $\mathcal{V}$ (respectively $\mathcal{R}$) is called reflexive (see [5]).

Let F be a field and let ρ be a reflexive transitive relation on the set $N = \{1, ..., n\}$ for some $n \geq 2$ (more information about ρ will be given in Section 2). The set

$$M_n(F, \rho) = \{A \in M_n(F) : a_{ij} = 0 \text{ whenever } (i, j) \notin \rho\}$$

is a subalgebra of $M_n(F)$ and we call $M_n(F, \rho)$ the algebra of $n \times n$ structural matrices over F (with identity I).

*Correspondence: makkurt@gyte.edu.tr
This work was supported by TÜBİTAK İŞBAP, Project No: 107T897.
2000 *AMS Mathematics Subject Classification:* 15A30, 03G05, 47A15.

Let X be a subset of a set S. If $X \cup \overline{X} = S$ and $X \cap \overline{X} = \emptyset$ for $\overline{X} \subseteq S$, then $\overline{X}$ is called a complement of X in S.

Recall that a lattice L is called a complemented lattice if L has a greatest element and least element, and each element has at least one complement; that is, for $b \in L$, there exists $a \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$.

This work is a follow-up paper of [1], and in this paper we explore when the lattice of invariant subspaces of a structural matrix algebra can be complemented. We recall that a Boolean algebra is a complemented distributive lattice, and we concentrate on the relation between the structural matrix algebra and Boolean algebra. Our aim is to study when the lattice of invariant subspaces of a structural matrix algebra is a Boolean algebra. We give several equivalent conditions for this to hold.

We first prove that a finite sublattice $\mathcal{V}$ of the set $\mathcal{L}(V)$ of all subspaces of a finite dimension vector space V is a Boolean algebra if and only if $Alg(\mathcal{V})$ is a structural matrix algebra $M_n(F,\rho)$, with ρ symmetric. Then, by asking a question with respect to whether we have a necessary and sufficient condition for a subspace to have a complement in a subspace lattice for a structural algebra, we partially answer the question in Propositions 3.6 and 3.7.

Finally, we prove that any basis of a Boolean algebra satisfies the complementation property, and, conversely, if a basis of a lattice of subspaces of a structural algebra satisfies the complementation property, then the lattice is a Boolean algebra.

At the end of the paper, we discuss briefly the structure of the algebra if the finite distributive lattice (i.e. finite reflexive lattice) $\mathcal{L}(V)$ is a Boolean algebra.

2. Preliminaries

Let ρ (or C or $\sim$) be a reflexive transitive relation (i.e. quasi-order relation) on the set $N = \{1, ..., n\}$ for some $n \geq 2$, and then (N, ρ) is called a *quasi-ordered* set (abbreviated as *quoset*). This is also in accordance with [9]. Note that each quoset (N,ρ) gives rise to the partition $J = \{[i] : i \in N\}$ of N whereby $[i] := \{j \mid (i,j) \in \rho$ and $(j,i) \in \rho\}$, and that $(J, \leq)$, $[i] \leq [j]$ defined by $(i,j) \in \rho$, is a *partially ordered* set (abbreviated as *poset*). Thus, the relation $\leq$ is reflexive and antisymmetric (i.e. $[i] \leq [j]$, $[j] \leq [i]$ implies $[i] = [j]$). A subset X of a quoset (N,ρ) is an *order ideal* if for all $j \in X$ it follows from $(i,j) \in \rho$ that also $i \in X$. The family $\mathcal{L}(N,\rho)$ of all order ideals is obviously a sublattice of the powerset lattice $\mathcal{P}(N)$ and thus necessarily distributive. Conversely, for any finite distributive lattice $\mathcal{L}$ let the poset $(N, \leq)$ mimic the order relation among the join irreducible elements of $\mathcal{L}$. Then $\mathcal{L}(N, \leq)$ is isomorphic to $\mathcal{L}$ (Birkhoff's Theorem, [4, p. 61]).

We let $\mathcal{V}$ denote a sublattice of $\mathcal{L}(V)$. Recall that an element $W \in \mathcal{V}$ is (join) *irreducible* if and only if $W = W_1 + W_2$ always implies $W = W_1$ or $W = W_2$ where $W_1 \in \mathcal{V}$ and $W_2 \in \mathcal{V}$. We also say that W *covers* W_0 if and only if $W \supseteq W_0, W \neq W_0$, and there is no $W_1 \in \mathcal{V}$ distinct from W and W_0 such that $W \supseteq W_1 \supseteq W_0$.

The following theorem is implied by [2, Theorem 3.4], but we have a different approach and proof of the result under discussion.

Theorem 2.1 *Let ρ be a quasi-ordered relation on the set $N = \{1, \ldots, n\}$. Let F be a field and $M_n(F,\rho)$ be a structural matrix algebra over F. If $M_n(F,\rho)$ is simple, then $M_n(F,\rho) = M_n(F)$.*

Proof Let $M_n(F,\rho) = M$ be a structural matrix algebra that is simple. We may, without loss of generality, assume the algebra to be in block upper triangular form. We see from consideration of the properly nilpotent elements that the radical, J, is the ideal of all matrices in M that are block strictly upper triangular. Since M is simple, it is semisimple, and so $J = \{0\}$. Then M is block diagonal, that is, a direct sum of full matrix algebras. But since M is simple, it has no proper ideals, and thus consists of a single block. That is, M is the full matrix algebra of order n over F. This completes the proof. □

Corollary 2.2 *Let ρ be a quasi-ordered relation on the set $N = \{1, \ldots, n\}$ and let $M_n(F, \rho)$ be the structural algebra for ρ. The following statements are equivalent:*

(1) *$M_n(F, \rho)$ is simple,*

(2) *$M_n(F, \rho) = M_n(F)$,*

(3) *ρ is the trivial relation, that is, $(i, j) \in \rho$ for all i, j.*

The property isolated in the next definition is central in the study of structural algebras (cf. [7, 8, 9]).

Definition 2.3 A lattice $\mathcal{V} \subset \mathcal{L}(V)$ satisfies the *base property* if and only if there is a basis B of V such that $B \cap W$ is a basis of W for every $W \in \mathcal{V}$.

Remark 2.4 A finite lattice $\mathcal{V} \subset \mathcal{L}(V)$ satisfies the *base property* if and only if it is distributive [1, Theorem 1].

3. Boolean algebras of subspaces

Recall that a Boolean algebra is a complemented distributive lattice. Since we are considering invariant subspace lattices of structural matrix algebras, we want the invariant subspace to have a complement that is also an invariant subspace.

Theorem 3.1 *Let $\mathcal{V}$ be a finite sublattice of $\mathcal{L}(V)$ with $dim(V) = n$. Then the following statements are equivalent:*

(1) *$\mathcal{V}$ is a Boolean algebra;*

(2) *$\mathcal{R} = Alg(\mathcal{V})$ is a structural $n \times n$ matrix algebra $M_n(F, \rho)$ with ρ symmetric;*

(3) *$\mathcal{R} = Alg(\mathcal{V})$ is a semisimple structural matrix algebra $M_n(F, \rho)$.*

Proof As to (1) $\Rightarrow$ (2), when $\mathcal{V}$ is a Boolean algebra it is distributive; by [1, Theorem 3] we have $\mathcal{R} = Alg(\mathcal{V})$. With exactly same idea as in the proof of [1, Theorem 3], the argument is $(e_i^h, e_j^k) \in \rho \Leftrightarrow P_i \subseteq P_j \Leftrightarrow P_i = P_j \Leftrightarrow P_j \subseteq P_i \Leftrightarrow (e_j^k, e_i^h) \in \rho$, and so ρ is symmetric.

As to (2) $\Rightarrow$ (1), by [1, Theorem 3] $\mathcal{V} = Lat(\mathcal{R}) = Lat(M_n(F, \rho))$ is isomorphic to the lattice of all order ideals Y of the quoset (B, ρ) where B is as in the proof of [1, Theorem 3]. Since ρ symmetric, these Y are exactly the 2^s unions of sets $B_i = \{e_i^h \mid 1 \leq h \leq n(i)\}$ $(1 \leq i \leq s)$. So $\mathcal{V}$ is a Boolean algebra.

As to (2) $\Rightarrow$ (3), obviously $M_n(F, \rho) \simeq M_{n(1)}(F) \times \cdots \times M_{n(s)}(F)$, so $M_n(F, \rho)$ is semisimple.

The direction (3) $\Rightarrow$ (2) is slightly less obvious and is proven in [2]. □

We shall give an alternate proof that (3) implies (2). Let $M = M_n(F, \rho)$ be a structural matrix algebra. We are to show that if M is semisimple then ρ is symmetric (i.e. it is an equivalence relation). By relabeling, if necessary, the indices, we may assume that M is in block upper triangular form where the blocks on the diagonal are full matrix algebras and correspond to the equivalence classes of ρ. If A is a matrix in M whose diagonal blocks are zero, then A is properly nilpotent [3, p. 120]. But if M is semisimple, then its radical is zero. However, the radical consists of all properly nilpotent elements [3, Theorem 4.4], whence $A = 0$. Thus, M consists of the block diagonal elements, and so ρ is symmetric.

Theorem 3.2 *Let $\mathcal{R}$ be a subalgebra of $M_n(F)$.*

(1) *If $\mathcal{R}$ is a semisimple structural matrix algebra, then $Lat\mathcal{R}$ is a Boolean algebra.*

(2) *If L is a Boolean algebra of subspaces of F^n, then $AlgL$ is conjugate to a semisimple structural matrix algebra.*

(3) *If $Lat\mathcal{R}$ is a Boolean algebra and if F is algebraically closed, then $\mathcal{R}$ is conjugate to a semisimple structural matrix algebra.*

Proof (1) Let $\mathcal{R}$ be a semisimple structural matrix algebra, say $\mathcal{R} = M_n(F, \rho)$. By Theorem 1 of [1] and Theorem 1.1 of [9], $Lat\mathcal{R}$ is a distributive lattice and satisfies the base property with some basis B. In fact, we may take $B = \{e_1, \ldots, e_n\}$, the standard basis vectors of F^n. Let $W \in Lat\mathcal{R}$ and $B \cap W = \{e_{j_1}, \ldots, e_{j_p}\}$. Let W' be the subspace spanned by the subset of B complementary to $\{e_{j_1}, \ldots, e_{j_p}\}$. We claim $W' \in Lat\mathcal{R}$. If $W' = span\{e_{k_1}, \ldots, e_{k_q}\}$, $p+q = n$, then it suffices to show that for any $(i,j) \in \rho$, $e_{k_l}E_{ij} \in W'$ for $l = 1, \ldots, q$. So let $(i,j) \in \rho$. Then if $k_l \neq i$, we have $e_{k_l}E_{ij} = 0 \in W'$. If $k_l = i$, then $e_{k_l}E_{ij} = e_j$. Suppose $e_j \in W$. Since $\mathcal{R}$ is semisimple, ρ is symmetric [2, Theorem 3.4], such that $E_{ji} \in \mathcal{R}$ as well. Hence, $e_jE_{ji} \in W$ as $W \in Lat\mathcal{R}$. But $W \cap W' = \{0\}$. Thus, $e_j \in W'$ and so $W' \in Lat\mathcal{R}$. Therefore, $Lat\mathcal{R}$ is complemented.

(2) Now suppose L is a Boolean algebra of subspaces of F^n. Then by Theorem 1 of [1], L satisfies the base property for some basis B. If we utilize the construction in the proof of Theorem 1 of [7], we may assume that B consists of the standard unit vectors, that ρ is the defining relation for $AlgL$, and $(i,j) \in \rho$ if and only if $E_{ij} \in AlgL$. Thus, $AlgL$ is a structural matrix algebra. Since L is a Boolean algebra with, say, 2^m elements, then

$$F^n = V_1 \oplus \cdots \oplus V_m$$

where each V_j is a join irreducible element of L. Then

$$AlgL = AlgV_1 \oplus \ldots \oplus AlgV_m,$$

where each $AlgV_j$ is the algebra of all endomorphisms of V_j and is therefore simple. Hence, $AlgL$ is semisimple.

(3) Suppose $\mathcal{R}$ is an algebra such that $Lat\mathcal{R}$ is a Boolean algebra. As in (2), we can write

$$F^n = V_1 \oplus \cdots \oplus V_m$$

where the V_j are join irreducible. Then

$$\mathcal{R} = \mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_m,$$

where $\mathcal{R}_j$ is the restriction of $\mathcal{R}$ to V_j. The join irreducibility of V_j as an element of $Lat\mathcal{R}$ means that $\mathcal{R}_j$ has no invariant subspaces other than V_j and $\{0\}$ since $Lat\mathcal{R}$ is complemented. That is, $\mathcal{R}_j$ is an irreducible algebra. Since F is algebraically closed, by Burnside's theorem $\mathcal{R}_j = Hom(V_j)$. The result follows. $\square$

Example 3.3 *The algebraic closure of F is needed in (3) of Theorem 3.2. For if $F = \mathbb{R}$, the real numbers, and if*

$$R(\theta) = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix},$$

we let $\mathcal{R}$ be the subalgebra of $M_2(\mathbb{R})$ generated by the $R(\theta)$. Then $\mathcal{R}$ is irreducible and is in fact a field, and hence simple. However, $\mathcal{R}$ cannot be isomorphic with a structural ring $M_2(\mathbb{R},\rho)$ of 2×2 matrices since $M_2(\mathbb{R},\rho)$ has zero divisors.

We know that the complement of an element in a Boolean algebra is unique [6, p. 40]. Then we can give the following proposition. Proof is sufficiently obvious by the base property.

Proposition 3.4 *If the lattice of subspaces of a structural algebra is complemented, then the complement is unique.*

Question: If we have a subspace lattice for a structural algebra, do we have necessary and sufficient conditions for a subspace to have a complement?

Remark 3.5 $\mathcal{L}$ is the lattice of invariant subspace for $\mathcal{A} = M_n(F,\rho)$. Let $V \in \mathcal{L}$ and let B be a basis for $\mathcal{L}$. Then $B_1 = B \cap V$ is a basis for V. Put $B_2 = B \backslash B_1$. V has a complement in $\mathcal{L}$ if and only if $spanB_2 \in \mathcal{L}$.

Proposition 3.6 *Let $M_n(F,\rho)$ be a structural matrix algebra with subspace lattice $\mathcal{L}$. Let $V \in \mathcal{L}$ and suppose V is a sum of minimal subspaces, each of which is contained in no other irreducible subspace. Then V has a complement in $\mathcal{L}$.*

Proof Recall first that a minimal nonzero element of $\mathcal{L}$ is irreducible. First suppose V is a minimal subspace that is contained in no other irreducible subspace. Let $W = \sum_j \{V_j : V_j \text{ is irreducible}, V_j \neq V\}$.

We claim W is a complement of V. First $F^n = V + W$ since any element of $\mathcal{L}$ is the sum of irreducible elements. By minimality of V we have $V \cap V_j = \{0\}$ for all j. Thus, $V \cap W = V \cap \left(\sum\limits_j V_j\right) = \left(\sum\limits_j V \cap V_j\right) = \{0\}$. The claim follows.

Now suppose V is the sum of minimal subspaces, none of which is contained in another irreducible subspace, say

$$V = U_1 + \cdots + U_s.$$

By the first part, each U_j has a complement W_j. Put

$$W = W_1 \cap \cdots \cap W_s \in \mathcal{L}$$

a.

$$V \cap W = (U_1 + \cdots + U_s) \cap W \subseteq (U_1 \cap W_1) + \cdots + (U_s \cap W_s) = \{0\}$$

b.

$$V + W = V + (W_1 \cap \cdots \cap W_s) \supseteq (U_1 + W_1) \cap \cdots \cap (U_s + W_s) = F^n$$

The result is established. $\square$

We do not have a necessary condition for subspaces that each satisfy the hypotheses of some V to have a complement. However, if we replace "some V" by "all V", then we have the next result.

Proposition 3.7 *Let $\mathcal{L}$ be the lattice of subspaces of the structural matrix algebra $\mathcal{A}$. $\mathcal{L}$ is a Boolean algebra if and only if there is no chain of nonzero irreducible elements, that is, if and only if V_1, V_2 are irreducible and $0 \neq V_1 \subseteq V_2$ implies $V_1 = V_2$.*

Proof Suppose first that $\mathcal{L}$ is a Boolean algebra and $0 \neq V_1 \subseteq V_2$ where V_1 and V_2 are distinct irreducible subspaces. If B is a basis for $\mathcal{L}$, let $S_i = B \cap V_i,\ i = 1, 2$. Let $B \cap V_1 = \{v_1, v_2, \dots, v_r\},\ B \cap V_2 = \{v_1, v_2, \dots, v_r, t_1, \dots, t_s\}$. Let W be the complement of V_1 in $\mathcal{L}$. If $B = \{v_1, v_2, \dots, v_r, t_1, \dots, t_s, w_1, \dots, w_q\}$, then $B \cap W \subseteq \{t_1, \dots, t_s, w_1, \dots, w_q\}$. But $B \cap W$ is a basis for W. Thus,

$$\dim(span\,(B \cap W) = \dim W - \dim V_1 = s + q.$$

Hence:
$B \cap W = \{t_1, \dots, t_s, w_1, \dots, w_q\}$. Now $V_2 = span\,\{v_1, v_2, \dots, v_r, t_1, \dots, t_s\}$, so by Remark 3.5, $span\,\{w_1, \dots, w_q\} \in \mathcal{L}$. Consequently $V_1 + span\,\{w_1, \dots, w_q\} \in \mathcal{L}$. So again by Remark 3.5, $V = span\,\{t_1, \dots, t_s\} \in \mathcal{L}$, as well. Clearly $V + V_1 = V_2,\ V \cap V_1 = \{0\}$. This contradicts the irreducibility of V_2.

For the converse, note that every element $V \in \mathcal{L}$ is a sum of irreducible subspaces of Proposition 3.6. The converse follows. $\square$

Corollary 3.8 *Let $M_n(F, \rho)$ be a structural matrix algebra with $\mathcal{L} = Lat\,(M_n(F, \rho))$ its lattice. Then $\mathcal{L}$ is a Boolean algebra if and only if $\mathcal{L}$ is an atomic lattice (point lattice).*

Recall that $\mathcal{L}$ is distributive and that an atom (or point) is an element that covers $\{0\}$. The lattice is atomic if and only if every element of $\mathcal{L}$ is the join of the points it contains.

Proof Observe first that the points are the minimal subspaces. A minimal (nonzero) subspace is necessarily join irreducible.

Suppose first that $\mathcal{L}$ is a Boolean algebra and let $W \in \mathcal{L}$. We are to show that W is the join of the points it contains. If W is itself irreducible, then by proposition 3.7, W is a point and there is nothing to prove. Otherwise $W = V_1 + ... + V_k$, where $\{V_j\}_{j=1}^{k}$ is the collection of the irreducible subspaces contained in W. But again by proposition 3.7, each of these is a point, so we are done.

Conversely, suppose $\mathcal{L}$ is a point lattice. From the observation, the points are join irreducible subspaces. Since $\mathcal{L}$ is distributive, then W is the join of points and the converse follows. $\square$

Recall that if B is a basis of F^n, we say that B is a basis of $\mathcal{L}$ if and only if for any $V \in \mathcal{L},\ B \cap V$ is a basis for V. We know that for any lattice $\mathcal{L}$ of invariant subspaces for a structural algebra $\mathcal{A}$, there is a B of $\mathcal{L}$.

Definition 3.9 Let $\mathcal{L}$ be a lattice of subspaces of a structural algebra $\mathcal{A}$ and let B be a basis for $\mathcal{L}$. We say that B satisfies the *complementation property* if for any $S \subseteq B$, the span of S is an element of $\mathcal{L}$; then the complement S' of S in B also spans an element of $\mathcal{L}$.

Remark 3.10 If $S \subseteq B$ is such that $spanS = V \in \mathcal{L}$, then $B \cap V = S$.

Proposition 3.11 (1) *If $\mathcal{L}$ is a Boolean algebra, then any basis B of $\mathcal{L}$ satisfies the complementation property.*

(2) *If some basis B of $\mathcal{L}$ satisfies the complementation property, then $\mathcal{L}$ is a Boolean algebra.*

Remark 3.12 We have seen already that complements, if they exist, in $\mathcal{L}$ are unique.

Proof (proof of proposition) (1) First suppose that $\mathcal{L}$ is a Boolean algebra and that B is a basis for $\mathcal{L}$. Let $V \in \mathcal{L}$ and $S = B \cap V$, say $S = \{x_1, x_2, \ldots, x_m\}$ with the complements $S' = \{x_{m+1}, x_{m+2}, \ldots, x_n\}$. If W is the complement of V in $\mathcal{L}$, then for any $j = 1, 2, \ldots, m$, $x_j \notin B \cap W$, since $x_j \in B \cap V$ and $V \cap W = \{0\}$. Thus, $S' \supseteq B \cap W$. However, $\dim spanS' = n - m = \dim W$ so $spanS' = W$ and B satisfies the complementation property.

(2) Now suppose B is a basis of $\mathcal{L}$ that satisfies the complementation property. If $V \in \mathcal{L}$, let $S = B \cap V$, say $S = \{x_1, x_2, \ldots, x_m\}$. Then $S' = \{x_{m+1}, x_{m+2}, \ldots, x_n\}$ is a complement of S in B. Put $W = spanS'$. By the hypothesis $W \in \mathcal{L}$. Clearly $V + W \supseteq span\,(S \cup S') = F^n$. Also, $V \cap W = spanS \cap spanS' = \{0\}$ since $B = S \cup S'$ is linearly independent. Thus, $\mathcal{L}$ is a Boolean algebra. □

Let $\mathcal{L}(V)$ be a finite distributive lattice of subspace of the vector space V where $\dim V = n$. If $\mathcal{A} = Alg\mathcal{L}$, we know that $\mathcal{A}$ is (isomorphic with) a structural matrix algebra $M_n\,(F,\rho)$. Furthermore, $\mathcal{L}(V)$ and $Alg\,\mathcal{L}$ are reflexive. If $\mathcal{L}(V)$ is a Boolean algebra (or equivalently if $Alg\mathcal{L}$ is semisimple), then $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$, where $\{V_1, V_2, \ldots, V_m\}$ is the complete set of join irreducible elements of $\mathcal{L}$. If $\mathcal{A}_k = \mathcal{A}\,|_{V_k}$, then $\mathcal{A} = \bigoplus \mathcal{A}_k$. Consequently, $A \in \mathcal{A}$ has a block diagonal representation:

$$A = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \cdots & A_{m-1} & 0 \\ 0 & 0 & \cdots & 0 & A_m \end{bmatrix}.$$

We have that $Lat\mathcal{A}_k = \{0, V_k\}$, so that $\mathcal{A}_k$ is transitive on V_k. If $\dim V_k = n_k$, then $\mathcal{A}_k$ is $n_k \times n_k$. $\mathcal{A}_k$ is more than transitive, however, since $\mathcal{A}_k$ is (isomorphic with) $AlgV_k$. Thus, $\mathcal{A}_k$ is isomorphic with $M_{n_k}\,(F)$, $k = 1, ..., m$ and

$$\mathcal{A} = Alg\mathcal{L} = M_{n_1}\,(F,\rho) \oplus \cdots \oplus M_{n_m}\,(F,\rho)\,.$$

Now consider $\Phi \in Aut\,(\mathcal{A})$. We know from [2] Theorem A that $\Phi = \Theta \circ \Pi$ where Θ is inner and Π is a permutation. The equivalence classes of ρ are the set

$$[i] := \{j \mid (i,j) \in \rho \text{ and } (j,i) \in \rho\}.$$

Then the admissible permutations must permute the indices of 2 classes, that is, must permute the diagonal blocks. But we can, for instance, permute 2 such diagonal blocks only when $n_p = n_q$ (compare [2, Theorem B]).

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