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On quotients of ith affine surface areas

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Abstract: Following the volume difference function, we first introduce the notion of the *affine surface area quotient function*. We establish Brunn–Minkowski type inequalities for the affine surface area quotient function, which in special cases yield some well-known results.

Key words: Volume difference function, affine surface area quotient function, Blaschke sum, Brunn–Minkowski inequality

1. Introduction and statement of results

The well-known classical Brunn–Minkowski inequality can be stated as follows:

If K and L are convex bodies in $\mathbb{R}^n$, then (see, e.g., [16])

$$V(K+L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \tag{1.1}$$

with equality if and only if K and L are homothetic. Here, $+$ is the usual Minkowski sum.

Let K and L be star bodies in $\mathbb{R}^n$, then the dual Brunn–Minkowski inequality states that (see [8])

$$V(K\tilde{+}L)^{1/n} \leq V(K)^{1/n} + V(L)^{1/n}, \tag{1.2}$$

with equality if and only if K and L are dilates. Here, $\tilde{+}$ is the radial Minkowski sum.

A vector addition was defined on $\mathbb{R}^n$ which we call radial Minkowski addition, as follows. If $x, y \in \mathbb{R}^n$, then $x\tilde{+}y$ is defined to be the usual vector sum of x, y provided x, y both lie in a 1-dimensional subspace of $\mathbb{R}^n$ and as the zero vector otherwise. If K, L are star bodies and $\lambda, \mu \in \mathbb{R}$, then the radial Minkowski linear combination, $\lambda K\tilde{+}\mu L$, is defined by $\lambda K\tilde{+}\mu L = \{\lambda x\tilde{+}\mu y : x \in K, y \in L\}$. The expression $K\tilde{+}L$ is called the radial Minkowski sum of the star bodies K and L (see [5]).

In 2004, Leng [6] defined the volume difference function of compact domains D and K, where $D \subseteq K$, by

$$D_V(K, D) = V(K) - V(D).$$

The following Brunn–Minkowski type inequality for volume difference functions was also established by Leng [6].

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Theorem A *If $K, L,$ and D are compact domains, $D \subseteq K, D' \subseteq L$, D' is a homothetic copy of D, then*

$$(V(K+L)-V(D+D'))^{1/n} \geq (V(K)-V(D))^{1/n}+(V(L)-V(D'))^{1/n}, \tag{1.3}$$

with equality if and only if K and L are homothetic and $(V(K),V(D)) = \mu(V(L),V(D'))$, where μ is a constant.

Recently, Lv [13] introduced the *dual volume difference function* for star bodies and established the following dual Brunn–Minkowski type inequality for dual volume difference functions:

Theorem B *If K, D, and D' are star bodies in $\mathbb{R}^n$, and $D \subseteq K, D' \subseteq L$, L is a dilation of K, then*

$$(V(K\tilde{+}L)-V(D\tilde{+}D'))^{1/n} \geq (V(K)-V(D))^{1/n}+(V(L)-V(D'))^{1/n}, \tag{1.4}$$

with equality if and only if D and D' are dilates and $(V(K),V(D)) = \mu(V(L),V(D'))$, where μ is a constant.

In fact, some more general versions on these types of inequalities were proved in [6] and [13], respectively.

In 2005, Zhao [19] defined the volume sum function of star bodies D and K, by

$$S_V(K,D) = V(K)+V(D).$$

The Minkowski inequality for volume sums of mixed intersection bodies was also established in [19].

Motivated by the work of Leng, Lv, and Zhao, we give the following definition:

Definition 1.1 Let K be a convex body in $\mathbb{R}^n$, then the affine surface area quotient function of K, $Q_{\Omega_{i,j}}(K)\,(i,j\in\mathbb{R})$, can be defined by

$$Q_{\Omega_{i,j}}(K) = \frac{\Omega_i(K)}{\Omega_j(K)}. \tag{1.5}$$

If $i=0$ and $j=n$ in (1.5), then we get the affine surface area quotient of the convex body K and the unit n-ball B:

$$Q_{\Omega_{0,n}}(K) = \frac{\Omega(K)}{\Omega(B)},$$

where $\Omega(B)$ is the surface area of the unit n-ball B.

A convex body K is said to have a continuous curvature function,

$$f(K,\cdot): S^{n-1} \to [0,\infty),$$

if for each $L \in \mathbb{C}^n$, the mixed volume $V_1(K,L)$ has the integral representation

$$V_1(K,L) = \frac{1}{n}\int_{S^{n-1}} f(K,u)h(L,u)dS(u),$$

see [9].

The subset of $\mathcal{K}^n$ consisting of all bodies that have a positive continuous curvature function will be denoted by κ^n.

The Brunn–Minkowski inequality for the affine surface area of a Blaschke sum was established by Lutwak [10] as follows:

If $K,\ L \in \kappa^n$, then for $i < -1$,

$$\Omega_i(K\ddot{+}L)^{\frac{n+1}{n-i}} \leq \Omega_i(K)^{\frac{n+1}{n-i}} + \Omega_i(L)^{\frac{n+1}{n-i}}, \tag{1.6}$$

with equality if and only if K and L are homothetic, where $\ddot{+}$ is Blaschke addition (see Section 2).

The first aim of this paper is to establish the following Brunn–Minkowski type inequality for the affine surface area quotient functions.

Theorem 1.2 *If $K,\ L \in \kappa^n$ and $i \leq -1 \leq j \leq n,\ i,j \in \mathbb{R}$, then for $i \neq j$*

$$\left(\frac{\Omega_i(K\ddot{+}L)}{\Omega_j(K\ddot{+}L)}\right)^{\frac{n+1}{j-i}} \leq \left(\frac{\Omega_i(K)}{\Omega_j(K)}\right)^{\frac{n+1}{j-i}} + \left(\frac{\Omega_i(L)}{\Omega_j(L)}\right)^{\frac{n+1}{j-i}}. \tag{1.7}$$

In fact, Theorem 1.2 is a special case of Theorem 3.2 established in Section 3.

Remark 1.3 Taking $j = n$ and $i < -1$ in (1.7), and in view of the fact that $\int_{S^{n-1}} dS(u) = n\omega_n$ is a constant (where ω_n is the volume of the unit n-ball), (1.7) changes to (1.6).

The class ν^n is defined as follows:

$$\nu^n = \{K \in \kappa^n : f(K,\cdot)^{-1/(n+1)} = h(Q,\cdot) \text{ for some } Q \in \mathcal{K}^n\}.$$

The class ν^n has been extensively investigated by Petty [15].

In 1990, Lutwak [9] established the following Brunn–Minkowski type inequality for the affine surface area:

If $K,\ L \in \nu^n$, then

$$\Omega(K\check{+}L)^{-\frac{1}{n}} \geq \Omega(K)^{-\frac{1}{n}} + \Omega(L)^{-\frac{1}{n}}, \tag{1.8}$$

with equality if and only if K and L are homothetic, where $\check{+}$ is the Lutwak linear sum (see Section 2).

In 2005, an inverse form of (1.8) was established in [20] as follows:

If $K,\ L \in \nu^n$, then for $i > n+1$,

$$\Omega_i(K\check{+}L)^{-\frac{1}{n-i}} \leq \Omega_i(K)^{-\frac{1}{n-i}} + \Omega_i(L)^{-\frac{1}{n-i}}, \tag{1.9}$$

with equality if and only if K and L are homothetic.

The second aim of this paper is to establish the following Brunn–Minkowski type inequality for affine surface area quotient functions.

Theorem 1.4 *If $K,\ L \in \kappa^n$ are symmetric and $i \geq n+1 \geq j \geq n,\ i,j \in \mathbb{R}$, then for $i \neq j$*

$$\left(\frac{\Omega_i(K\check{+}L)}{\Omega_j(K\check{+}L)}\right)^{\frac{1}{i-j}} \leq \left(\frac{\Omega_i(K)}{\Omega_j(K)}\right)^{\frac{1}{i-j}} + \left(\frac{\Omega_i(L)}{\Omega_j(L)}\right)^{\frac{1}{i-j}}. \tag{1.10}$$

In fact, Theorem 1.4 is a special case of Theorem 3.3 established in Section 3.

Remark 1.5 Taking $j = n$ in (1.10) and in view of the fact that $\int_{S^{n-1}} dS(u) = n\omega_n$ is a constant, (1.10) changes to (1.9).

2. Notations and preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$ $(n > 2)$. Let $\mathbb{C}^n$ denote the set of nonempty convex figures (compact, convex subsets) and $\mathcal{K}^n$ denote the subset of $\mathbb{C}^n$ consisting of all convex bodies (compact, convex subsets with nonempty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors and the letter B for the unit ball centered at the origin. The surface of B is S^{n-1}. The volume of the unit n-ball is denoted by ω_n.

We use $V(K)$ for the n-dimensional volume of a convex body K. Let $h(K,\cdot) : S^{n-1} \to \mathbb{R}$ denote the support function of $K \in \mathcal{K}^n$; i.e. for $u \in S^{n-1}$,

$$h(K,u) = Max\{u \cdot x : x \in K\},$$

where $u \cdot x$ denotes the usual inner product of u and x in $\mathbb{R}^n$.

Let δ denote the Hausdorff metric on $\mathcal{K}^n$, i.e. for $K, L \in \mathcal{K}^n$, $\delta(K,L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

Associated with a nonempty compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K,\cdot) : S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$ by

$$\rho(K,u) = Max\{\lambda \geq 0 : \lambda u \in K\}.$$

If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Let $\mathcal{S}^n$ denote the set of star bodies in $\mathbb{R}^n$. Let $\tilde{\delta}$ denote the radial Hausdorff metric as follows: if $K, L \in \mathcal{S}^n$, then $\tilde{\delta}(K,L) = |\rho_K - \rho_L|_\infty$.

2.1. Blaschke linear combination and mixed affine surface area

For $K \in \kappa^n$, we have (see [9])

$$\int_{S^{n-1}} u f(K,u) dS(u) = 0. \tag{2.1}$$

Suppose $K, L \in \kappa^n$ and $\lambda, \mu \geq 0$ (not both zero). From (2.1) it follows that the function $\lambda f(K,\cdot) + \mu f(L,\cdot)$ satisfies the hypothesis of Minkowski's existence theorem (see [3]). The solution of the Minkowski problem for this function is denoted by $\lambda \cdot K \ddot{+} \mu \cdot L$; that is,

$$f(\lambda \cdot K \ddot{+} \mu \cdot L, \cdot) = \lambda f(K,\cdot) + \mu f(L,\cdot), \tag{2.2}$$

where the linear combination $\lambda \cdot K \ddot{+} \mu \cdot L$ is called a Blaschke linear combination.

The relationship between Blaschke and Minkowski scalar multiplication is given by

$$\lambda \cdot K = \lambda^{1/(n-1)} K. \tag{2.3}$$

The affine surface area of K, $\Omega(K)$, is defined by

$$\Omega(K) = \int_{S^{n-1}} f(K,u)^{\frac{n}{n+1}} dS(u), \quad K \in \kappa^n. \tag{2.4}$$

It is well known that this functional is invariant under unimodular affine transformations. For $K, L \in \kappa^n$, and $i \in \mathbb{R}$, the ith mixed affine surface area of K and L, $\Omega_i(K,L)$, is defined by (see [10])

$$\Omega_i(K,L) = \int_{S^{n-1}} f(K,u)^{\frac{n-i}{n+1}} f(L,u)^{\frac{i}{n+1}} dS(u). \tag{2.5}$$

For $K \in \kappa^n$, we define the ith affine surface area of K, $\Omega_i(K)$, by (see also [15])

$$\Omega_i(K) = \int_{S^{n-1}} f(K,u)^{\frac{n-i}{n+1}} dS(u), \quad i \in \mathbb{R}. \tag{2.6}$$

2.2. Lutwak linear combination

The class ν^n is defined as follows:

$$\nu^n = \{K \in \kappa^n : f(K,\cdot)^{-\frac{1}{n+1}} = h(Q,\cdot) \text{ for some } Q \in \mathcal{K}^n\}. \tag{2.7}$$

Lutwak defined a new linear combination $\lambda \circ K \check{+} \mu \circ L \in \nu^n$. (This is called Lutwak linear combination throughout the article.) For centrally symmetric $K, L \in \nu^n$ (see [9]),

$$f(K\check{+}L,\cdot)^{-1/(n+1)} = f(K,\cdot)^{-1/(n+1)} + f(L,\cdot)^{-1/(n+1)}. \tag{2.8}$$

For our purposes it is more natural to use the set of all symmetric convex bodies in κ^n and to just define the addition by (2.8).

3. Main results

An extension of Beckenbach's inequality (see [2], p.27) was obtained by Dresher [4] by means of moment-space techniques:

Lemma 3.1 (The Beckenbach–Dresher inequality) *Let $p \geq 1 \geq r \geq 0$ and f, g be measurable and nonnegative functions. If ϕ is a distribution function, then for $p \neq r$*

$$\left(\frac{\int_{\mathbb{E}} |f+g|^p d\phi}{\int_{\mathbb{E}} |f+g|^r d\phi}\right)^{\frac{1}{p-r}} \leq \left(\frac{\int_{\mathbb{E}} f^p d\phi}{\int_{\mathbb{E}} f^r d\phi}\right)^{\frac{1}{p-r}} + \left(\frac{\int_{\mathbb{E}} g^p d\phi}{\int_{\mathbb{E}} g^r d\phi}\right)^{\frac{1}{p-r}}, \tag{3.1}$$

Here $\mathbb{E}$ is a bounded measurable subset of $\mathbb{R}^n$.

We will need the above inequality in Lemma 3.1 to prove our main theorems. Our main results are given in the following theorems.

Theorem 3.2 *If $K, L \in \kappa^n$ and $0 \leq r \leq n+1 \leq p$, $p, r \in \mathbb{R}$, then for $p \neq r$*

$$\left(\frac{\Omega_{n-p}(K\ddot{+}L)}{\Omega_{n-r}(K\ddot{+}L)}\right)^{\frac{n+1}{p-r}} \leq \left(\frac{\Omega_{n-p}(K)}{\Omega_{n-r}(K)}\right)^{\frac{n+1}{p-r}} + \left(\frac{\Omega_{n-p}(L)}{\Omega_{n-r}(L)}\right)^{\frac{n+1}{p-r}}. \tag{3.2}$$

Proof From (2.2), (2.3), and (2.6), we have

$$\Omega_{n-p}(K\ddot{+}L) = \int_{S^{n-1}} f(K\ddot{+}L,u)^{\frac{p}{n+1}} dS(u) = \int_{S^{n-1}} (f(K,u)+f(L,u))^{\frac{p}{n+1}} dS(u) \tag{3.3}$$

and

$$\Omega_{n-r}(K\ddot{+}L) = \int_{S^{n-1}} (f(K,u)+f(L,u))^{\frac{r}{n+1}} dS(u). \tag{3.4}$$

Since $0 \leq r \leq n+1 \leq p$, we have

$$0 \leq \frac{r}{n+1} \leq 1 \leq \frac{p}{n+1}. \tag{3.5}$$

From (3.3), (3.4), and (3.5) and in view of the Beckenbach–Dresher inequality for integrals, we obtain that

$$\left(\frac{\Omega_{n-p}(K\ddot{+}L)}{\Omega_{n-r}(K\ddot{+}L)}\right)^{\frac{n+1}{p-r}} = \left(\frac{\int_{S^{n-1}}(f(K,u)+f(L,u))^{p/(n+1)}dS(u)}{\int_{S^{n-1}}(f(K,u)+f(L,u))^{r/(n+1)}dS(u)}\right)^{\frac{1}{\frac{p}{n+1}-\frac{r}{n+1}}}$$

$$\leq \left(\frac{\int_{S^{n-1}}f(K,u)^{\frac{p}{n+1}}dS(u)}{\int_{S^{n-1}}f(K,u)^{\frac{r}{n+1}}dS(u)}\right)^{\frac{n+1}{p-r}} + \left(\frac{\int_{S^{n-1}}f(L,u)^{\frac{p}{n+1}}dS(u)}{\int_{S^{n-1}}f(L,u)^{\frac{r}{n+1}}dS(u)}\right)^{\frac{n+1}{p-r}}$$

$$= \left(\frac{\Omega_{n-p}(K)}{\Omega_{n-r}(K)}\right)^{\frac{n+1}{p-r}} + \left(\frac{\Omega_{n-p}(L)}{\Omega_{n-r}(L)}\right)^{\frac{n+1}{p-r}}.$$

Let $p=n-i$ and $r=n-j$, then

$$r\leq n+1\leq p \Leftrightarrow i\leq -1\leq j, \quad 0\leq r\Leftrightarrow j\leq n.$$

Namely,

$$i\leq -1\leq j\leq n. \tag{3.6}$$

Taking $p=n-i$ and $r=n-j$ in (3.2) and using (3.6), (3.2) changes to the inequality in Theorem 1.2 stated in Section 1.

Taking $p=n-i$ and $r=0$ in (3.2), (3.2) changes to (1.6) stated in Section 1.

Taking $p=n+1$ and $r=n$ in (3.2) and in view of (2.4), (3.2) changes to the following result.

$$\left(\frac{\Omega_{-1}(K\ddot{+}L)}{\Omega(K\ddot{+}L)}\right)^{n+1} \leq \left(\frac{\Omega_{-1}(K)}{\Omega(K)}\right)^{n+1} + \left(\frac{\Omega_{-1}(L)}{\Omega(L)}\right)^{n+1}. \tag{3.7}$$

Taking $p=2n$ and $r=n$ in (3.2) and in view of (2.4), (3.2) changes to the following result.

$$\left(\frac{\Omega_{-n}(K\ddot{+}L)}{\Omega(K\ddot{+}L)}\right)^{\frac{n+1}{n}} \leq \left(\frac{\Omega_{-n}(K)}{\Omega(K)}\right)^{\frac{n+1}{n}} + \left(\frac{\Omega_{-n}(L)}{\Omega(L)}\right)^{\frac{n+1}{n}}.$$

Taking $p=2n$ and $r=n+1$ in (3.2), (3.2) changes to the following result.

$$\left(\frac{\Omega_{-n}(K\ddot{+}L)}{\Omega_{-1}(K\ddot{+}L)}\right)^{\frac{n+1}{n-1}} \leq \left(\frac{\Omega_{-n}(K)}{\Omega_{-1}(K)}\right)^{\frac{n+1}{n-1}} + \left(\frac{\Omega_{-n}(L)}{\Omega_{-1}(L)}\right)^{\frac{n+1}{n-1}}.$$

□

Theorem 3.3 *If $K,L\in\kappa^n$ are symmetric and $0\geq r\geq -1\geq p$, $p,r\in\mathbb{R}$, then for $p\neq r$*

$$\left(\frac{\Omega_{n-p}(K\check{+}L)}{\Omega_{n-r}(K\check{+}L)}\right)^{\frac{1}{r-p}} \leq \left(\frac{\Omega_{n-p}(K)}{\Omega_{n-r}(K)}\right)^{\frac{1}{r-p}} + \left(\frac{\Omega_{n-p}(L)}{\Omega_{n-r}(L)}\right)^{\frac{1}{r-p}}. \tag{3.8}$$

Proof From (2.6) and (2.8), we have

$$\Omega_{n-p}(K\check{+}L)=\int_{S^{n-1}}f(K\check{+}L,u)^{p/(n+1)}dS(u)=\int_{S^{n-1}}(f(K,u)^{-\frac{1}{n+1}}+f(L,u)^{-\frac{1}{n+1}})^{-p}dS(u) \tag{3.9}$$

and

$$\Omega_{n-r}(K\check{+}L) = \int_{S^{n-1}} (f(K,u)^{-\frac{1}{n+1}} + f(L,u)^{-\frac{1}{n+1}})^{-r} dS(u). \tag{3.10}$$

Since $0 \geq r \geq -1 \geq p$, we have

$$0 \leq -r \leq 1 \leq -p. \tag{3.11}$$

From (3.9), (3.10), and (3.11) and in view of the Beckenbach–Dresher inequality for integrals, we obtain that

$$\left(\frac{\Omega_{n-p}(K\check{+}L)}{\Omega_{n-r}(K\check{+}L)}\right)^{\frac{1}{r-p}} = \left(\frac{\int_{S^{n-1}}(f(K,u)^{-\frac{1}{n+1}} + f(L,u)^{-\frac{1}{n+1}})^{-p} dS(u)}{\int_{S^{n-1}}(f(K,u)^{-\frac{1}{n+1}} + f(L,u)^{-\frac{1}{n+1}})^{-r} dS(u)}\right)^{\frac{1}{-p-(-r)}}$$

$$\leq \left(\frac{\int_{S^{n-1}} f(K,u)^{\frac{p}{n+1}} dS(u)}{\int_{S^{n-1}} f(K,u)^{\frac{r}{n+1}} dS(u)}\right)^{\frac{1}{r-p}} + \left(\frac{\int_{S^{n-1}} f(L,u)^{\frac{p}{n+1}} dS(u)}{\int_{S^{n-1}} f(L,u)^{\frac{r}{n+1}} dS(u)}\right)^{\frac{1}{r-p}}$$

$$= \left(\frac{\Omega_{n-p}(K)}{\Omega_{n-r}(K)}\right)^{\frac{1}{r-p}} + \left(\frac{\Omega_{n-p}(L)}{\Omega_{n-r}(L)}\right)^{\frac{1}{r-p}}.$$

Let $p = n - i$ and $r = n - j$, then

$$r \geq -1 \geq p \Leftrightarrow i \geq n+1 \geq j, \quad 0 \geq r \Leftrightarrow j \geq n.$$

Namely,

$$i \geq n+1 \geq j \geq n. \tag{3.12}$$

Taking $p = n - i$ and $r = n - j$ in (3.8) and using (3.12), (3.8) changes to the inequality in Theorem 1.4 stated in Section 1.

Taking $p = -n$ and $r = -1$ in (3.8), (3.8) changes to the following result.

$$\left(\frac{\Omega_{2n}(K\check{+}L)}{\Omega_{n+1}(K\check{+}L)}\right)^{\frac{1}{n-1}} \leq \left(\frac{\Omega_{2n}(K)}{\Omega_{n+1}(K)}\right)^{\frac{1}{n-1}} + \left(\frac{\Omega_{2n}(L)}{\Omega_{n+1}(L)}\right)^{\frac{1}{n-1}}. \tag{3.13}$$

Finally, we remark that the Aleksandrov–Fenchel inequality for volume difference was established in [21]. Inequalities for the volume differences of radial Blaschke–Minkowski homomorphisms were established in [22]. Inequalities for the volume sum function were given in [23,24]. Moreover, some interrelated results have appeared in [1,11,12,14,17,18]. □

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