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Colombeau solutions of a nonlinear stochastic predator–prey equation

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Abstract: The solution of a random semilinear hyperbolic system with singular initial data is sought as a random Colombeau distribution. The product of 2 additive white noises is well tackled within the theory of random Colombeau distributions. In the special case of a random predator–prey system, the exact Colombeau solution is obtained under some assumptions when the process is driven by doubly reflected Brownian motions.

Key words: Stochastic nonlinear predator–prey equation, generalized solutions, random Colombeau distributions

1. Introduction

This article is concerned basically with the stochastic version of the following deterministic semilinear hyperbolic system (Lotka–Volterra) in 2 variables [2]:

$$D_1u_1(x,t) = (\partial_t + c_1\partial_x)u_1(x,t) = \lambda_1 u_1(x,t)u_2(x,t)$$

$$D_2u_2(x,t) = (\partial_t + c_2\partial_x)u_2(x,t) = \lambda_2 u_1(x,t)u_2(x,t)$$

$$u_j(x,0) = \gamma_j(x)\,; \qquad j=1,2, \quad \lambda_1\lambda_2 < 0 \tag{1.1}$$

As a special case $\gamma_1(x) = \Delta_1\delta(x-\xi_1)$, $\gamma_2(x) = \Delta_2\delta(x-\xi_2)$ and $\lambda_1 = 1 = -\lambda_2$ represents a predator–prey system where the initial masses Δ_1 and Δ_1 of predators and preys have been concentrated at ξ_1 and ξ_2, respectively, in a one-dimensional space and they move in opposite directions towards each other with velocities c_1 and c_2.

The novelties that also constitute the main features of the present article are as follows:

I) For the stochastic case driven by white noise, a more realistic model could be

$$D_1u_1 = \lambda_1(u_1 + \dot{B}_1)(u_2 + \dot{B}_2)$$

$$D_2u_2 = \lambda_2(u_1 + \dot{B}_1)(u_2 + \dot{B}_2) \tag{1.2}$$

rather than assigning one additive noise to each equation. B : Wiener process (The Brownian motion).

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II) The solutions are expected to be as singular as initial data, e.g., in the case of the Dirac initial data they would be generalized functions. The product u_1u_2 on the right-hand side would then have to be interpreted within a theory of nonlinear distributions, like that of Colombeau algebras. Furthermore, as a consequence of the extra feature presented in I), we can not attribute any meaning to the product $\dot{B}_1\dot{B}_2$ by visualizing white noises as (linear) generalized processes (i.e. the distributional derivatives of nowhere differentiable Brownian paths).

Thus, the second important feature of the present paper is the use of the concept of random Colombeau distributions as worked out by Çapar and Aktuğlu [3],[4] and also by Oberguggenberger and Russo [6] for obtaining generalized solutions to (1.2) along with initial data.

2. Deterministic and random Colombeau algebras

2.1. Colombeau algebras

2.1.1. Inadequacy of Schwartz distributions

This inadequacy stems mainly from the lack of a suitable multiplication operation. The first remedy to get around this difficulty could be to imbed $\mathcal{D}'$ (Schwartz distributions) into a differential algebra. However, this also has limitations in accordance with the celebrated Schwartz impossibility result, which roughly states that any such imbedding of $\mathcal{D}'$ into an associative, commutative differential algebra has to give some concessions regarding the natural properties of the derivative and the product rules (like Leibniz, etc.). For instance, such an algebra can not extend the pointwise multiplication of continuous functions, and it also cannot possess any element δ satisfying $x\delta = 0$.

It is usually regarded that Colombeau's differential algebra wards off the consequences of the Schwartz impossibility result in an optimal manner and the concessions given are minimal in many respects.

In Colombeau's method, nonlinear distributions are formed by the classes of smooth regularizations and the space of generalized functions (the Colombeau algebra) is given as a factor algebra:

$$\mathcal{G}(\Re^n) = \mathcal{E}_M(\Re^n)/\mathcal{N}(\Re^n)$$

where $\mathcal{E}_M(\Re^n)$ is the algebra of moderate functions:

$$\mathcal{E}_M(\Re^n) = \{u \in [\mathcal{C}^\infty(\Re^n)]^{\mathcal{A}} \,|\, \forall K \subset\subset \Re^n,\, \forall \alpha \in N^n, \exists p \geq 0, \text{ such that } \forall \phi \in \mathcal{A}_p(\Re^n),$$

$$\sup_{x\in K} |\partial^\alpha u(\phi_\epsilon, x)| = O(\epsilon^{-p}) \text{ as } \epsilon \to 0\}$$

and where $\mathcal{N}(\Re^n)$ is the ideal of null germs:

$$\mathcal{N}(\Re^n) = \{u \in \mathcal{E}_M(\Re^n) \,|\, \forall K \subset\subset \Re^n,\, \forall \alpha \in N^n, \exists p \geq 0 \text{ such that } \forall q \geq p, \forall \phi \in \mathcal{A}_q(\mathcal{R}^n),$$

$$\sup_{x\in K} |\partial^\alpha u(\phi_\epsilon, x)| = O(\epsilon^{q-p}) \text{ as } \epsilon \to 0\}.$$

In both definitions $\phi_\epsilon(x) \equiv \epsilon^{-n}\phi(\frac{x}{\epsilon})$ and $\mathcal{A}_q, \mathcal{A}$ $(q \in N_+)$ refer to the index sets of test functions given as $\mathcal{A}_q = \{\phi \in \mathcal{D}(\Re^n) | \int \phi(x)dx = 1, \int x^\alpha \phi(x)dx = 0, 1 \leq |\alpha| \leq q\}$, $q \in N$, $\mathcal{A}_1 \equiv \mathcal{A} \supset \mathcal{A}_2 \supset \mathcal{A}_3 \supset \cdots\cdots.;\ \cap_q \mathcal{A}_q = \emptyset$. The set $[\mathcal{C}^\infty(\Re^n)]^{\mathcal{A}}$ will consist of elements $u(\phi, x)$, $\phi \in \mathcal{A}$.

A Colombeau generalized function T has the form

$$T = f_T(\text{a representative}) + \mathcal{N}.$$

Algebra and vector operations and differentiation are performed on the representatives.

The imbedding $\mathcal{D}'(\Re^n) \subset \mathcal{G}(\Re^n)$ is achieved by

$$T \longrightarrow \text{class } [\phi \to T * \phi], \ \ T \in \mathcal{D}'(\Re^n), \ \phi \in \mathcal{A}(\Re^n).$$

This is a vector space imbedding that preserves the distributional derivatives. With this imbedding, for instance, δ (the Dirac delta) is represented as $\delta = f_\delta + \mathcal{N}$, $f_\delta(\phi, x) = \phi(x)$ and $H(\text{Heaviside}) = f_H + \mathcal{N}$, $f_H(\phi, x) = \int_{-\infty}^{x} \phi(t)dt$; $\phi \in \mathcal{A}(\Re^n)$. $\mathcal{C}^\infty$ is imbedded as a faithful subalgebra, i.e. $T = h \in \mathcal{C}^\infty \Rightarrow f_T(\phi, x) = h(x) \ \ \forall \phi \in \mathcal{A}(\Re^n)$.

In Colombeau's theory $x^m \delta \neq 0$ for $m = 1, 2, \cdots$, in contrast to Schwartz's theory of linear distributions where $x\delta = 0$.

The space of continuous functions $\mathcal{C}^0$ can be imbedded as regular distributions, but then in accordance with the Schwartz impossibility result, the product of continuous functions in $\mathcal{G}$ will not correspond to their ordinary product. However, the 2 product rules are shown to be associated through an equivalence relation resulting from a specific coupled calculus and this is a bonus in favor of Colombeau's theory. The details of such features and also some of the other properties are found in references like [1],[2], and [8].

An element $T \in \mathcal{G}(\Re^n)$ is called **nonnegative** if it has a nonnegative representative.

Property. An important feature of Colombeau generalized functions is the following: as its construction suggests, it is only the behavior along delta sequences that matters; namely, for the quotient algebra only the values $f_T(\phi_\epsilon, x)$, $\phi \in \mathcal{A}_q$ for q large enough and $\epsilon > 0$ small enough depending on ϕ are significant. Therefore, in order to assert that a property holds for $T \in \mathcal{G}(\Re^n)$ it is sufficient to verify it on some representative f_T for large q and small ϵ. The following definition utilizes this feature.

A generalized function $T \in \mathcal{G}(\Re^n)$ is said to be of L^1_{loc}**-type** with respect to variable x_1 if it has a representative f_T with the following property:

For every $K \times K_{n-1} \subset \Re \times \Re^{n-1}$ compact, $\exists q \in N$ such that $\forall \phi \in \mathcal{A}_q(\Re^n), \exists M > 0, \eta > 0$ with $\sup_{x' \in K_{n-1}} \int_K |f_T(\phi_\epsilon, x', x_1)| dx_1 \leq M, \ 0 < \epsilon < \eta$, where $x' = (x_2, \cdots, x_n)$.

$T \in \mathcal{G}(\Re^n)$ is said to be of uniform L^1_{loc} type if in this definition $M > 0$ works out for every $\phi \in \mathcal{A}_q(\Re^n)$.

2.2. Association with Schwartz distributions

$T \in \mathcal{G}$ is said to be associated to $V \in \mathcal{D}'$, denoted $T \approx V$ if $\exists$ a representative f_T, $T = f_T + \mathcal{N}$ such that $\forall \psi \in \mathcal{D}, \exists q \in N$ with $\lim_{\epsilon \to 0} \int f_T(\phi_\epsilon, x)\psi(x)dx = \langle V, \psi \rangle, \forall \phi \in \mathcal{A}_q(\Re^n)$.

Interpretation: If we reduce the information contained in f_T to the level of the linear distribution theory, then T behaves like V.

2.3. Random Colombeau distributions

Given a probability space (Ω, Σ, μ) this is a mapping $T : \Omega \to \mathcal{G}(\Re^n)$ such that $\exists f_T : \mathcal{A}(\Re^n) \times \Re^n \times \Omega \to \Re$ with:

(i) For fixed $\phi \in \mathcal{A}(\Re^n)$, $(x, \omega) \to f_T(\phi, x, \omega)$ is jointly measurable on $\Re^n \times \Omega$,

(ii) a.s. in $\omega \in \Omega$, $\phi \to f_T(\phi, ., \omega)$ belongs to $\mathcal{E}_M$ and is a representative of $T(\omega)$. Denote random Colombeau generalized functions by $\mathcal{G}_\Omega(\mathcal{R}^n)$.

2.4. Classical generalized processes

These are weakly measurable maps $V : \Omega \to \mathcal{D}'(\Re^n)$. All such classical generalized processes are denoted by $\mathcal{D}'_\Omega(\Re^n)$. Then $V(\omega) * \phi(x) = \langle V(\omega), \phi(x - .)\rangle$ yields an imbedding $\mathcal{D}'_\Omega(\Re^n) \to \mathcal{G}_\Omega(\Re^n)$. Since it is measurable with respect to ω and smooth with respect to $x \in \Re^n$, hence it is jointly measurable.

The most important example of a classical generalized process is the **white noise** $\dot{B}$, $\langle \dot{B}, \phi\rangle = -\langle B, \phi'\rangle$, $(\phi \in \mathcal{D}$ or in n dimensions $(-1)^n\langle B, \dfrac{\partial^n \phi}{\partial x_1, \cdots, \partial x_n}\rangle$. Its Colombeau representative, e.g., in one-dimension, would be

$$f_{\dot{B}}(\phi, x, \omega) = -\int_\Re B(s, \omega)\phi'(x - s)ds.$$

For the white noise we usually use a special probability space where $\Omega :$ tempered distributions, $\mu :$ the Bochner measure.

3. Back to the stochastic semilinear hyperbolic system

3.1. Solution method

We write the system in a simpler form, without affecting the generality, by taking $c_1 = 1 = -c_2$ to get

$$\begin{aligned} D_1U_1 &= (\partial_t + \partial_x)U_1 = (U_1 + \dot{B}_1)(U_2 + \dot{B}_2) \\ D_2U_2 &= (\partial_t - \partial_x)U_2 = -(U_1 + \dot{B}_1)(U_2 + \dot{B}_2) \\ U_j(x, 0) &= \Gamma_j; \qquad j = 1, 2, \end{aligned} \tag{3.1}$$

where U_1, U_2 are generalized solutions in $\mathcal{G}_\Omega(\mathcal{R}^2)$, $\dot{B}_1, \dot{B}_2$ are space-time white noises, and Γ_1, $\Gamma_2 \in \mathcal{G}(\mathcal{R}^2)$.

Now $\dot{B}_1, \dot{B}_2 \in (S)^*$ (Hida distributions); their product is not defined in $(S)^*$ but is meaningful in the stochastic Colombeau distribution space $\mathcal{G}_\Omega$ along the lines introduced in [3] and [4].

We try to solve the last system at the representatives level u_1, u_2, $(U_1 = u_1 + \mathcal{N}$, $U_2 = u_2 + \mathcal{N})$ and then show that u_1, u_2 are moderate. Then their classes will constitute a generalized solution. More explicitly:

I) Find the classical solution $u_1(\phi, x, t, \omega), u_2(\phi, x, t, \omega)$ for fixed ϕ and ω:

$$\begin{aligned} D_1u_1 &= (u_1 + \alpha)(u_2 + \beta) \\ D_2u_2 &= -(u_1 + \alpha)(u_2 + \beta) \\ u_1(x, 0) &= \gamma_1(x),\ u_2(x, 0) = \gamma_2(x) \end{aligned} \tag{3.2}$$

where $\alpha = \alpha(\phi, x, t, \omega)$, $\beta = \beta(\phi, x, t, \omega)$ are representatives of white noises $\dot{B}_1$ and $\dot{B}_2$, respectively, as random Colombeau distributions and $\gamma_1(\phi, x)$ and $\gamma_2(\phi, x)$ are representatives of Γ_1 and Γ_2..

II) Show that $u_1 = u_1(\phi, x, t, \omega)$ and $u_2 = u_2(\phi, x, t, \omega)$ are in $\mathcal{E}_M$, i.e. are moderate, and also for fixed $\phi \in \mathcal{A}(\mathcal{R}^2)$, the mappings $(x, t, \omega) \to u_i(\phi, x, t, \omega)$ are jointly measurable on $\mathcal{R}^2 \times \Omega$. Then their class will form a random Colombeau solution to the system.

Since $D_1(-u_1) = D_2u_2$, $\exists X$ such that $D_1X = u_2, D_2X = -u_1$. To find the form of X,

$$D_1X = \frac{\partial X}{\partial t} + \frac{\partial X}{\partial x} = u_2 \qquad (\dagger)$$

$$D_2X = \frac{\partial X}{\partial t} - \frac{\partial X}{\partial x} = -u_1 \qquad (\dagger\dagger)$$

Adding the 2 equations:

$$2\frac{\partial X}{\partial t} = u_2 - u_1 \implies 2X(x,t) = \int_0^t [u_2(x,s) - u_1(x,s)]ds + A(x)\,. \qquad (\surd)$$

Differentiating with respect to x and interchanging the order of differentiation and integration, we have:

$$A'(x) = 2\frac{\partial X}{\partial x}(x,t) - \int_0^t [\frac{\partial u_2}{\partial x}(x,s) - \frac{\partial u_1}{\partial x}(x,s)]ds\,.$$

As this equality is true for all $t*$, , by putting $t = 0$ we get:

$$A'(x) = 2\frac{\partial X}{\partial x}(x,0)\,.$$

On the other hand by subtracting $(\dagger\dagger)$ from $(\dagger)$, we have $2\frac{\partial X}{\partial x} = u_1 + u_2$ or

$$2\frac{\partial X}{\partial x}(x,0) = u_1(x,0) + u_2(x,0) = \gamma_1(x) + \gamma_2(x)\ ,$$

and substituting for $A'(x)$

$$A'(x) = \gamma_1(x) + \gamma_2(x) \implies A(x) = \int_0^x [\gamma_1(r) + \gamma_2(r)]dr\ .$$

Thus, from $(\surd)$

$$X(x,t) = \frac{1}{2}\{\int_0^t [-u_1(x,s) + u_2(x,s)]ds + \int_0^x [\gamma_1(r) + \gamma_2(r)]dr\} \qquad (3.3)$$

(modulo a constant).

The initial conditions for X are (dependence of γ_i on $\phi\ \in \mathcal{A}$, $(i = 1,2)$ being suppressed):

$$X(x,0) = \frac{1}{2}\int_0^x [\gamma_1(r) + \gamma_2(r)]dr,\ X_t(x,0) = \frac{1}{2}[-\gamma_1(x) + \gamma_2(x)]$$

and X satisfies the differential equation

$$D_1D_2X = D_1(-u_1) = -(u_1 + \alpha)(u_2 + \beta) = -(-D_2X + \alpha)(D_1X + \beta) \text{ or}$$

$$D_1D_2X = (D_1X + \beta)(D_2X - \alpha),$$

Following Hasimoto's method [5], let $Y = e^{-X}, and\, Y$ then satisfies

$$Y_{tt} - Y_{xx} + (\alpha+\beta)Y_x + (\alpha-\beta)Y_t - \alpha\beta Y = 0. \tag{3.4}$$

Y has the following initial conditions:

$$Y(x,0) \doteq Y_1(x) = e^{-X(x,0)} = e^{-\frac{1}{2}\int_0^x(\gamma_1(r)+\gamma_2(r))dr}$$

$$Y_t(x,0) \doteq Y_2(x) = -X_t(x,0)e^{-X(x,0)} = \frac{1}{2}[\gamma_1(x)-\gamma_2(x)].\, e^{-\frac{1}{2}\int_0^x(\gamma_1(r)+\gamma_2(r))dr} \tag{3.5}$$

In general it is very difficult to find a solution to the hyperbolic system (3.4), (3.5) when $\alpha = \alpha(x,t)$ and $\beta = \beta(x,t)$ (the other variables ϕ and ω being suppressed). However, we make the natural assumption that noises propagate along the characteristics, allowing us to obtain a solution to the initial value problem (3.4) along with (3.5) in a nice closed form. That means we assume:

$$\alpha = \alpha(x-t) = -\int_{\mathcal{R}} B_1(s)\phi'(x-t-s)ds, \;\; \beta = \beta(x+t) = -\int_{\mathcal{R}} B_2(s)\phi'(x+t-s)ds, \;\; \phi \in \mathcal{A}. \tag{3.6}$$

Now transforming to the variables $\xi = x-t,\ \eta = x+t$ we get the following differential equation:

$$Y_{\xi\eta} - \frac{1}{2}(\beta Y_\xi + \alpha Y_\eta) + \frac{\alpha\beta}{4}Y = 0.$$

Substituting $Y = F(\xi)G(\eta)$ in the above equation, we find:

$$[F'(\xi) - \frac{1}{2}\alpha(\xi)F(\xi)][G'(\eta) - \frac{1}{2}\beta(\eta)G(\eta)] = 0$$

yielding

$$Y(\xi,\eta) = e^{\frac{1}{2}\int_0^\xi \alpha(u)du}G(\eta) + e^{\frac{1}{2}\int_0^\eta \beta(v)dv}F(\xi),$$

F, G : arbitrary C^1 functions. F and G are eliminated using the initial conditions Y_1 and Y_2. Then by transforming back to (x,t) variables and using (3.5) we find:

$$Y(x,t) = \exp[\frac{1}{2}\int_{x-t}^{x+t}\beta(v)dv\,]\,Y_1(x-t) + I(x-t)J(x+t)\int_{x-t}^{x+t}\frac{Y_2(u)+Y_1'(u)-\beta(u)Y_1(u)}{2I(u)J(u)}du$$

$$= \exp\{\frac{1}{2}[\int_{x-t}^{x+t}\beta(v)dv - \int_0^x(\gamma_1(r)+\gamma_2(r))dr\,]\}$$

$$-I(x-t)J(x+t)\int_{x-t}^{x+t}\frac{(\gamma_2(u)+\beta(u))\exp[-\frac{1}{2}\int_0^x(\gamma_1(r)+\gamma_2(r))dr\,]}{2I(u)J(u)}du,$$

where

$$I(u) = e^{\frac{1}{2}\int_0^u \alpha(s)ds}, \;\; J(u) = e^{\frac{1}{2}\int_0^u \beta(r)dr}. \tag{3.7}$$

3.2. Colombeau solution of the problem

Starting by (3.7), we can run the above procedure backwards and calculate

$$u_1 = -D_2(-\log Y), \quad u_2 = D_1(-\log Y)$$

as the solution at the representative platform. Here, however, we face a problem. Because of the noise terms α and β, Y may have negative values for certain ω and ϕ.

If for a moment we consider the no-noise case (i.e. $\alpha = \beta = 0$ in (3.7)), we retrieve the deterministic D'Alembert solution:

$$Y^d(x,t) = \frac{1}{2}[Y_1(x-t) + Y_1(x+t) + \int_{x-t}^{x+t} Y_2(u)du] \; ; \; (\, d \text{ stands for 'deterministic'}). \tag{3.8}$$

Using (3.3) consider the following quotient:

$$\frac{Y^d(x,t)}{\exp[-\frac{1}{2}\int_0^{x-t}(\gamma_1(r)+\gamma_2(r))dr]}$$

$$= \frac{1}{2}\left\{\exp[-\frac{1}{2}\int_0^{x-t}(\gamma_1(r)+\gamma_2(r))dr] + \exp[-\frac{1}{2}\int_0^{x+t}(\gamma_1(r)+\gamma_2(r))dr]\right.$$

$$\left. +\frac{1}{2}\int_{x-t}^{x+t}(\gamma_1(r)-\gamma_2(r))\exp[-\frac{1}{2}\int_0^{r}(\gamma_1(s)+\gamma_2(s))ds]dr\right\}$$

$$:\left\{\exp[-\frac{1}{2}\int_0^{x-t}(\gamma_1(r)+\gamma_2(r)dr)]\right\}$$

$$= \frac{1}{2}\left\{1+\exp[-\frac{1}{2}\int_{x-t}^{x+t}(\gamma_1(r)+\gamma_2(r))dr]\right.$$

$$\left. + \frac{1}{2}\int_{x-t}^{x+t}(\gamma_1(r)-\gamma_2(r))\exp[-\frac{1}{2}\int_{x-t}^{r}(\gamma_1(s)+\gamma_2(s))ds]dr\right\}$$

Substituting $v(r) = \int_{x-t}^{r}(\gamma_1(s)+\gamma_2(s))ds$, we find

$$= \frac{1}{2}\left\{1+\exp[-\frac{1}{2}\int_{x-t}^{x+t}(\gamma_1(r)+\gamma_2(r))dr] - \int_{x-t}^{x+t}\gamma_2(r)\exp[-\frac{1}{2}\int_{x-t}^{r}(\gamma_1(s)+\gamma_2(s))ds]dr\right.$$

$$\left. +\frac{1}{2}\int_{v=0}^{\int_{x-t}^{x+t}(\gamma_1(r)+\gamma_2(r)dr} e^{-\frac{v}{2}}dv\right\}$$

Hence:

$$\frac{Y^d(x,t)}{\exp[-\frac{1}{2}\int_0^{x-t}(\gamma_1(r)+\gamma_2(r)dr]} = 1 - \frac{1}{2}\int_{x-t}^{x+t}\gamma_2(r)\exp[-\frac{1}{2}\int_{x-t}^{r}(\gamma_1(s)+\gamma_2(s))ds]dr \tag{3.9}$$

3.2.1. Nonnegativity assumption

Now we make the assumption that Γ_1 and Γ_2 are of nonnegative L^1_{loc} type. Thus, the representatives γ_1 and γ_2 can be chosen as nonnegative, so that starting by (3.9) :

$$\frac{Y^d(x,t)}{\exp[-\frac{1}{2}\int_0^{x-t}(\gamma_1(r)+\gamma_2(r))dr]} \geq 1-\frac{1}{2}\int_{x-t}^{x+t}\gamma_2(r)\exp[-\frac{1}{2}\int_{x-t}^{r}\gamma_2(s)ds]dr$$

or by a further substitution:

$$Y^d(x,t) \geq \exp\{-\frac{1}{2}[\int_0^{x-t}\gamma_1(r)dr+\int_0^{x+t}\gamma_2(s)ds]\} \tag{3.10}$$

In view of the assumptions that γ_1 and γ_2 are of L^1_{loc} type, given any compact set $K \subset \Re^2$, for q large enough and $\eta > 0$ small enough , Y^d_ϵ is bounded away from zero uniformly for $(x,t) \in K$, i.e.

$$Y^d_\epsilon(x,t) \geq \exp\{-\frac{1}{2}[\int_0^{x-t}\gamma_1(\phi_\epsilon,r)dr+\int_0^{x+t}\gamma_2(\phi_\epsilon,s)ds]\}; \quad \phi \in \mathcal{A}_q,\ 0<\epsilon<\eta. \tag{3.11}$$

Besides the moderateness and nonnegativity of γ_1, γ_2 , and (3.9), $Y^d(x,t)$ is also moderate. Let its equivalence class be $W \in \mathcal{G}(\Re^2)$. Then $f_W = Y^d$.

3.2.2. Stochastic predator–prey model

In this case, the initial data are multiples of delta functions:

$$\Gamma_1(x) = \Delta_1\delta(x-\xi_1),\ \Gamma_2(x) = \Delta_2\delta(x-\xi_2); \Delta_1 \geq 0, \Delta_2 \geq 0,$$

where $\Gamma_1 = \gamma_1 + \mathcal{N}$, $\Gamma_2 = \gamma_2 + \mathcal{N}$. Suitable Colombeau representatives can be taken as:

$$\gamma_1(\phi,x) = \Delta_1\phi(x-\xi_1)\,,\ \gamma_2(\phi,x) = \Delta_2\phi(x-\xi_2);\ \ \phi \in \mathcal{A}.$$

It is easily seen that Γ_1 and Γ_2 are of uniformly L^1_{loc} type with $M_1 = \Delta_1$ and $M_2 = \Delta_2$. For q sufficiently large and ϵ sufficiently small, the estimate in (3.11) is also valid:

$$Y^d_\epsilon(x,t) \geq \exp[-\frac{\Delta_1}{2}\int_0^{x-t}\phi_\epsilon(x-\xi_1)dr]\exp[-\frac{\Delta_2}{2}\int_0^{x+t}\phi_\epsilon(s-\xi_2)ds]\,; \quad \phi \in \mathcal{A}_2. \tag{3.12}$$

Without loss of generality we may assume $0 < \xi_1 < \xi_2$ and $x-t > 0$. Consider 2 characteristic lines emanating from ξ_1 and ξ_2 , respectively, as shown below.

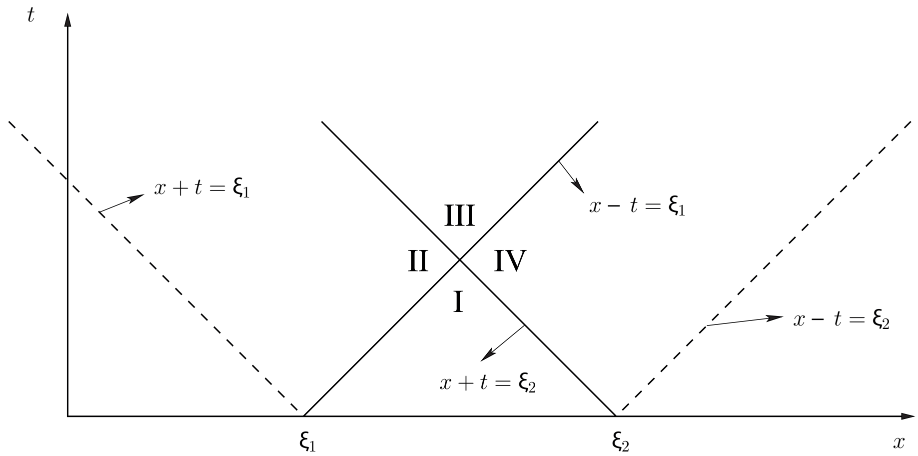

Figure. Characterization of the regions.

In region I : $\xi_1 < x-t < x+t < \xi_2$,
In region II : $x-t < \xi_1 < x+t < \xi_2$,
In region III : $\xi_1 < x-t < \xi_2 < x+t$,
In region IV : $x-t < \xi_1 < \xi_2 < x+t$.

In delta sequences $supp\{\phi_\epsilon(x-\xi_i)\} \to \{\xi_i\}$ as $\epsilon \to 0$, $(i = 1.2)$. Therefore, there exist sufficiently small $\epsilon_i > 0$, $(i = 1,2,3,4)$ such that:

$$\begin{aligned} &\text{In I}: \ Y^d_\epsilon(x,t) \geq e^{-\frac{\Delta_1}{2}}, \ \text{for} \ 0 < \epsilon < \epsilon_1; \\ &\text{In II}: \ Y^d_\epsilon(x,t) \geq e^0 = 1, \ \text{for} \ 0 < \epsilon < \epsilon_2; \\ &\text{In III}: \ Y^d_\epsilon(x,t) \geq e^{-\frac{1}{2}(\Delta_1+\Delta_2)}, \ \text{for} \ 0 < \epsilon < \epsilon_3; \\ &\text{In IV}: \ Y^d_\epsilon(x,t) \geq e^{-\frac{\Delta_2}{2}}, \ \text{for} \ 0 < \epsilon < \epsilon_4. \end{aligned} \tag{3.13}$$

Thus, for $0 < \epsilon < \eta(\phi)$, where $\eta = \min\{\epsilon_i : i = 1,2,3,4\}$:

$$Y^d_\epsilon x,t) \geq e^{-\frac{1}{2}(\Delta_1+\Delta_2)} \doteq k_0 > 0.$$

Note. Y^d, being moderate, is a representative of some generalized function $W \in \mathcal{G}$. In fact, in accordance with the property given in Section 2.1, there exists another representative, say $\tilde{Y}^d$, of W such that $|\tilde{Y}^d| \geq k_0$. More precisely, define an equivalence relation (descendance relation) in $\mathcal{A}$ (or in any $\mathcal{A}_q$) as follows: $\phi \approx \psi \iff \exists\epsilon > 0,\ \phi_\epsilon = \psi$ (e.g., symmetry by $\psi_{1/\epsilon} = \phi$). This equivalence relation partitions the index set $\mathcal{A}$. Let $[\phi]$ be an equivalence class with a fixed representative ϕ. For every $\psi \in [\phi]$, define

$$\tilde{Y}^d(x,t,\psi) = \begin{cases} Y^d(x,t,\phi_\delta) & \text{if} \quad \psi = \phi_\delta \quad \text{and} \quad 0 < \delta < \eta(\phi) \\ k_0 \quad \text{otherwise.} \end{cases}$$

$\tilde{Y}^d$ is clearly moderate, and $\tilde{Y}^d - Y^d \in \mathcal{N}(\Re^2)$ and satisfies $|\tilde{Y}^d| \geq q_0$ (3.14).

Now we show that Y^d is also bounded from above uniformly by another constant, k_1 (all symbols pertaining to the deterministic case):

$$u_{1,\epsilon}(x,t) = -D_2(-\log Y_\epsilon^d) = \frac{-Y_1'(x-t) + Y_2(x-t)}{Y_\epsilon^d(x,t)}.$$

By (3.3) and using the initial data of the predator–prey system :

$$u_{1,\epsilon}(x,t) = \frac{\gamma_1(\phi_\epsilon, x-t)\exp\left[-\frac{1}{2}\int_0^{x-t}(\gamma_1(\phi_\epsilon, r) + \gamma_2(\phi_\epsilon))dr\right]}{Y_\epsilon^d(x,t)}$$

$$= \frac{\Delta_1\phi_\epsilon(x-t-\xi_1)\exp\left[-\frac{1}{2}\int_0^{x-t}(\Delta_1\phi_\epsilon(r-\xi_1) + \Delta_2\phi_\epsilon(r-\xi_2)dr\right]}{Y_\epsilon^d}. \tag{3.15}$$

In view of the boundedness of Y^d, $|\int_0^t u_{1,\epsilon}(x,s)ds|$ is bounded uniformly in (x,t) by some multiple of the L^1 norm of $\Delta_1\phi_\epsilon$; a similar calculation applies to $u_{2,\epsilon} = -\frac{Y_1'(x+t) + Y_2(x+t)}{Y_\epsilon^d}$, showing that $|X_\epsilon|$ in (3.1), rewritten for the predator–prey initial data , is bounded in $L^\infty(\Re^2)$ independent of ϵ. Thus, $Y_\epsilon^d = \exp(-X_\epsilon)$ is also bounded from above (all results for large enough q and small enough ϵ). Hence:

$$k_0 \leq Y_\epsilon^d \leq k_1.$$

3.2.3. Colombeau solutions for bounded perturbations

In order to remedy the difficulty indicated at the beginning of Section 3.2 , we consider the Brownian motions involved with predator and prey populations as doubly reflected ones. This also conforms well to the cyclic nature of the predator–prey problems in which predator and prey populations fluctuate and swing between 2 bounds.

It is well-known that the biologist Umberto d'Ancona had brought a curious phenomenon to the attention of his father-in-law, the eminent mathematician Vito Volterra. In the post-World War I era the proportion of wild aquatic life (sharks, etc.) in the total fish-catch had considerably increased although there was much less fishing during the war. Volterra's modeling of the phenomenon led to the one of the first predator–prey equations. Here the preys were food fish. In other set-ups there may be foxes and rabbits, some plant species and herbivorous animals, parasites and hosting organisms, or even different industries linked to each other. The survival of predators depends entirely on the sufficient supply of prey. Thus, in a closed system, when the preys are consumed down to a certain level, predators start perishing. In the presence of fewer predators, then, preys begin to flourish. They multiply faster; in turn, with more food available, predators begin growing in number, resulting in a decline in the prey population, and then again a famine starts among predators, thus beginning a new cycle.

The state space will then be $[-b, b]$ for some positive constant b. Denote the doubly reflected Brownian motion with parameter b by $\hat{B}^b(t,\omega)$. Then the vertical position of the process immediately having been reflected downwards at b or reflected upwards at $-b$ is given by:

$$\hat{B}^b(t,\omega) = b - ||B(t,\omega) + b| - 2b|, \tag{3.16}$$

e.g., if $B(t,\omega) = b + \Delta$ $(0 < \Delta < 2b)$, then the formula yields $\hat{B}^b(t,\omega) = b + \Delta$, and if $B(t,\omega) = -b - \Delta$, it gives $\hat{B}^b(t,\omega) = -b + \Delta$. The white noise as a random Colombeau distribution resulting from $\hat{B}^b$ will have a representative

$$f^b_{\hat{B}\cdot}(\phi,x,\omega) = -\int_{\Re} \hat{B}^b(s,\omega)\phi'(x-s)ds. \tag{3.17}$$

We rewrite (3.4) replacing the representatives α and β by this type of representatives, i.e.

$$\hat{\alpha}^b(\phi,u,\omega) = -\int_{\Re} \hat{B_1}^b(s,\omega)\phi'(u-s)ds; \quad \hat{\beta}^b(\phi,u,\omega) = -\int_{\Re} \hat{B_2}^b(s,\omega)\phi'(u-s)ds. \tag{3.18}$$

Again we assume that the white noises are propagated along the characteristic lines.

The equation (3.7) rewritten for a doubly reflected Brownian motion and white noise representatives as given in (3.18) (ω and ϕ being suppressed) would be of the following form:

$$Y^b(x,t) = \exp\,[\frac{1}{2}\int_{x-t}^{x+t} \hat{\beta}^b(v)dv\,]Y_1(x-t) + I(x-t)J(x+t)\int_{x-t}^{x+t} \frac{Y_2(u)+Y'(u)-\hat{\beta}^b(u)Y_1(u)}{2I(u)J(u)}du;$$

where

$$I(u) = e^{\frac{1}{2}\int_0^u \hat{\alpha}^b(s)ds}, \; J(u) = e^{\frac{1}{2}\int_0 \hat{\beta}^b(r)dr}. \tag{3.19}$$

$Y^b(x,t)$ becomes a continuous function of b. We also have $\lim_{b\to 0} Y^b = \tilde{Y}^d$. As $\tilde{Y}^d$ is bounded by k_0 and k_1, $\exists b_0$ such that Y^{b_0} is also bounded by the same constants. Since $|\hat{\beta}^b(\phi,u,\omega)| \leq |\int_{\Re} \hat{B}^b_2(s,\omega)\phi'(u-s)ds| \leq b\int_{\Re}|\phi'(u-s)ds|$, b_0 can be determined independent of ω. On the other hand, in order to show that b_0 thus found works out for all test functions $\psi \in \mathcal{A}_q$, q being sufficiently large , we first change Y^b to another representative $\hat{Y}^b$ where all the test functions have their supports in $[x-t,x+t]$. This can be achieved by a technique similar to the one leading to (3.14). Then, as the path $\hat{B}_2(\omega,s)$ is continuous on the compact interval $[x-t,x+t]$, it can be approximated to any desired level of accuracy by the Weierstrass theorem so as to render the sign of $\hat{Y}^b$ still positive. If this is done by a polynomial Q_q of degree q where q is sufficiently large, for any pair of test functions ϕ, $\psi \in \mathcal{A}_q$ we have $\int Q_q(s)\phi'(u-s)ds = \int Q_q(s)\psi'(u-s)ds$, the last equality being due to the definition of test functions and their zero moment properties. In view of the nonnegativity of $Y_1(u)$ by (3.5), these integrals approximate the sign-determining factor $\hat{\beta}^b(u)$ in (3.19), and hence they can be evaluated as independent of the test function.

Then for such a small b , $0 < b < b_0$, we can reverse the procedure to calculate $u_1 = -D_2(-\log\hat{Y}^b)$, $u_2 = D_1(-\log\hat{Y}^b)$:

$$u_1(x,t) = \Big\{\exp\;[\tfrac{1}{2}\int_{x-t}^{x+t}\hat{\beta}^b(v)dv\;][\hat{\beta}^b(x-t)Y_1(x-t) - 2Y_1'(x-t)\;]$$

$$-\exp\,[\,\frac{1}{2}(\int_0^{x-t}\hat{\alpha}^b(u)du + \int_0^{x+t}\hat{\beta}^b(v)dv)\;]\hat{\alpha}^b(x-t) + \int_{x-t}^{x+t}\frac{Y_2(u)+Y_1'(u)-\hat{\beta}^b(u)Y_1(u)}{2I(u)J(u)}du$$

$$+\exp\,[\,\frac{1}{2}(\int_0^{x-t}\hat{\alpha}^b(u)du + \int_0^{x+b}\hat{\beta}^b(v)dv)\;]\frac{Y_2(x-t)+Y_1'(x-t)-\hat{\beta}^b(x-t)Y_1(x-t)}{I(x-t)J(x-t)}\Big\} : \hat{Y}^b(x,t)$$

$$u_2(x,t) = -\Big\{\exp\,[\,\frac{1}{2}\int_{x-t}^{x+t}\hat{\beta}^b(v)dv\;]\hat{\beta}^b(x+t)Y_1(x-t)$$

$$+\exp\Big[\,\frac{1}{2}\int_0^{x-t}\hat{\alpha}^b(u)du+\int_0^{x-t}\hat{\beta}^b(v)dv\,\Big]\hat{\beta}^b(x+t)\int_{x-t}^{x+t}\frac{Y_2(u)+Y_1'(u)-\hat{\beta}^bY_1(u)}{2I(u)J(u)}du$$

$$+\exp\Big[\,\frac{1}{2}\Big(\int_0^{x-t}\hat{\alpha}^b(u)du+\int_0^{x+t}\hat{\beta}^b(v)dv\Big)\,\Big]\frac{Y_2(x+t)+Y_1'(x+t)-\hat{\beta}^b(x+t)Y_1(x+t)}{I(x+t)J(x+t)}\Bigg\}:\hat{Y}^b(x,t) \qquad (3.20)$$

(Note: In these expressions $I(u)$ and $J(u)$ are formed by $\hat{\alpha}^b$ and $\hat{\beta}^b$.)

The quantities involved here, such as $Y_1, Y_2, \hat{\alpha}^b, \hat{\beta}^b$, are representatives of generalized functions and their smooth functions. Therefore, they are almost surely moderate, rendering u_1 and u_2 also as moderate functions. Besides, $\hat{\alpha}^b, \hat{\beta}^b$ are measurable with respect to ω. Thus, for fixed ϕ, being also smooth in $x, t \in \Re^2$, u_1 and u_2 are jointly measurable on $\Re^2 \times \Omega$, Their classes, U_1 and U_2, will be generalized solutions to the stochastic predator–prey system driven by doubly reflected Brownian motions.

We can round up the results of this section in the following theorem.

Theorem. The stochastic predator–prey system of (3.1) with delta function initial conditions has a unique solution in $\mathcal{G}_\Omega(\Re^2)$ when the driving white noises result from doubly reflected Brownian motions with sufficiently small amplitudes.

Note. For the uniqueness part, the proof will follow these lines: Let U_1 and U_2 be the equivalence classes in $\mathcal{G}(\Re^2)$ of solutions u_1 and u_2 given by (3.20). Let $\tilde{U}_1$ and $\tilde{U}_2$ be another pair of generalized functions with representatives $\tilde{u}_1$ and $\tilde{u}_2$, solution of the same problem. Use these representatives and the equation (3.3) to calculate $\tilde{X}(x,t)$. As $\tilde{\gamma}_1(x)$ and $\tilde{\gamma}_2(x)$ are the representatives of delta functions, they are moderate; accordingly $\tilde{X}(x,t)$ is moderate, so is $\tilde{Y}^b = \log(e^{-\tilde{X}})$. Hence, $\tilde{Y}^b$ and Y^b are governed by the same wave equation (3.4) and the initial conditions are moderate functions belonging to the same equivalence class. Therefore, their difference is estimated to belong to the ideal $\mathcal{N}(\Re^2)$. As a result their equivalence classes are equal, i.e. class $(\tilde{Y}^b) =$ class (Y^b) (see [7] for some details). In this way, class $(\tilde{X}^b) =$ class X^b in $\mathcal{G}(\Re^2)$. This implies in turn that $\tilde{U}_1 = U_1$ and $\tilde{U}_2 = U_2$ in $\mathcal{G}(\Re^2)$, i.e. $\tilde{u}_1 - u_1, \in \mathcal{N}(\Re^2)$ and $\tilde{u}_2 - u_2 \in \mathcal{N}(\Re^2)$.

4. Concluding remarks

A) "For $b > 0$ sufficiently small Y^b is also bounded away from 0" will not necessarily imply that the random perturbation is negligible in the true sense of the word. It all depends on the units involved. The differential equations are not dimensionless; recall that c_1 and c_2 were assumed for simplicity as $+1$ -1, which were velocities. At the microscopic level something very small may not be negligible. For instance, the predator–prey models are also used for the parasite–host organism interactions.

Besides, small random changes in the deterministic Y^d may produce considerable changes in u_1 and u_2 upon differentiation by the differential operators D_1 and D_2.

B) There could be another approach to get around the positivity requirement of $Y(x,t)$:

Consider small perturbations $\sigma_1(\omega)B_1(\omega)$ and $\sigma_2(\omega)B_2(\omega)$ defined pathwise and corresponding $Y^\sigma(\omega)$, where $0 < \sigma_i(\omega) < 1$, $(i = 1, 2)$ and $\sigma = \min\{\sigma_1, \sigma_2\}$. Considering the fact that Brownian paths are continuous, and therefore bounded on compact sets, the regularized white noise terms in (3.7) can be estimated for small σ_i and again following a continuity argument we can reach the conclusion that

$|Y^{\sigma} - Y^{d}|$ can be made small for sufficiently small σ, and therefore $Y^{\sigma(\omega)} > 0$ pathwise, σ depending on path. However, in order to assert that the solution constructed is in the sense of stochastic Colombeau algebras, we should show that $\sigma_i(\omega)$ can be determined via measurable selections. In this way, the joint measurability condition of a random Colombeau solution can be met. Of course, then the process $\sigma_i(\omega)B_i(t,\omega)$ will be no longer Gaussian.

C) In parallel to the concept of the associated Schwartz distribution of Section 2.2, we can coin the term "associated classical generalized process" (or associated Gelfand distribution) as follows:

$T(\omega) \in \mathcal{G}_{\Omega}(\Re^n)$ is associated to a classical generalized process $V(\omega) \in \mathcal{D}'_{\Omega}(\Re^n)$ if it has a representative $f_T(\omega)$ such that $\forall \psi \in \mathcal{D}(\Re^n)$, $\exists q \in N$ satisfying almost surely

$$\lim_{\epsilon \to 0} \int_{\Re^n} f_T(\phi_\epsilon, x, \omega)\psi(x)dx = \langle V(\omega), \psi \rangle, \ \phi \in \mathcal{A}_q(\Re^n).$$

Applying this definition to our case, we should investigate $\lim_{\epsilon\to 0}\{\int_{\Re^2} u_i(\phi_\epsilon, x, t, \omega)\psi(x,t)dxdt\}$; $\psi \in \mathcal{D}(\Re^2)$. However, in view of the messy expression (3.17) this seems to be an impossibly difficult task. For a short cut, one could try to find the associated linear process for Y^b if it exists and show that the limit and the differential operators D_i, $(i = 1, 2)$ commute.

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