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Some results on cyclic codes over the ring $R_{2,m}$

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Abstract: Let $R_{k,m}$ be the ring $\mathbb{F}_{2^m}[u_1, u_2, \cdots, u_k]/\langle u_i^2, u_iu_j - u_ju_i\rangle$. In this paper, cyclic codes of arbitrary length n over the ring $R_{2,m}$ are completely characterized in terms of unique generators and a way for determination of these generators is investigated. A $\mathbb{F}_{2^m}$-basis for these codes is also derived from this representation. Moreover, it is proven that there exists a one-to-one correspondence between cyclic codes of length $2n$, n odd, over the ring $R_{k-1,m}$ and cyclic codes of length n over the ring $R_{k,m}$. By determining the complete structure of cyclic codes of length 2 over $R_{2,m}$, a mass formula for the number of these codes is given. Using this and the mentioned correspondence, the number of ideals of the rings $R_{2,m}$ and $R_{3,m}$ is determined. As a corollary, the number of cyclic codes of odd length n over the rings $R_{2,m}$ and $R_{3,m}$ is obtained.

Key words: Cyclic codes, codes over $R_{2,m}$

1. Introduction

Codes over rings have been studied extensively after the publishing of the work done in [12], in which the authors looked at linear codes over $\mathbb{Z}_4$ and their binary images. Since then, many different types of rings have been studied in connection with the coding theory. Among various types of codes, cyclic codes form an important class of codes due to their rich algebraic structure. Given a ring R, these codes are in correspondence with ideals in the polynomial ring $R[x]/\langle x^n - 1\rangle$, where n is the length of the code.

Many of the works in the literature deal with the case wherein R is a finite chain ring [1–9,12–18]. However, recently, the authors of [19] considered the ring $\mathbb{F}_2[u, v]/\langle u^2, v^2, uv - vu\rangle$, which is a local ring but not a chain ring, and studied general linear codes over that. This work was continued in [20], [10], and [11].

In [20], the authors considered cyclic codes over $\mathbb{F}_2[u, v]/\langle u^2, v^2, uv - vu\rangle$ and obtained a partial characterization for them by presenting a set of generators for these codes. Though a few conditions on the polynomials involved in the generators were given in [20], the classification of them is still incomplete and the generators are not necessarily unique. The cardinality of codes and the basis for them are also not known.

Let $R_{k,m}$ be the ring $\mathbb{F}_{2^m}[u_1, u_2, \cdots, u_k]/\langle u_i^2, u_iu_j - u_ju_i\rangle$ with the convention that $R_{0,m} = \mathbb{F}_{2^m}$. In this work, we introduce a unique set of generators for cyclic codes over the ring $R_{2,m}$ and present a way for determination of these generators. A $\mathbb{F}_{2^m}$-basis for these codes is also derived from this representation. We also determine all distinct cyclic codes of length 2 over this ring and give a mass formula for the number of them. Note that the number of distinct cyclic codes of length 2 over $R_{1,m}$ can be obtained from the results

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of [5]. Next, we show that there exists a one-to-one correspondence between cyclic codes of length $2n$, n odd, over the ring $R_{k-1,m}$ and cyclic codes of length n over the ring $R_{k,m}$. Taking n to be 1, ideals of the ring $R_{k,m}$ correspond bijectively to cyclic codes of length 2 over $R_{k-1,m}$. Hence, we have determined the number of ideals of the rings $R_{2,m}$ and $R_{3,m}$. As a corollary, the number of cyclic codes of odd length n over the rings $R_{2,m}$ and $R_{3,m}$ is obtained.

The paper is organized as follows. In Section 2, a complete classification is given for cyclic codes of arbitrary length n over $R_{2,m}$. An algorithm generating all distinct cyclic codes of a given length over $R_{2,m}$ is also presented there. All distinct cyclic codes of length 2 over $R_{2,m}$ and a mass formula for the number of them can be found in this section. In Section 3, a one-to-one correspondence between cyclic codes of length $2n$, n odd, over the ring $R_{k-1,m}$ and cyclic codes of length n over the ring $R_{k,m}$ is introduced and some consequences of this correspondence are investigated. The paper is closed with a conclusion section.

2. Cyclic codes over $R_{2,m}$

Let $\sigma : R_{k,m}^n \longrightarrow R_{k,m}^n$ be the map sending $(c_0, c_1, \ldots, c_{n-1})$ to $(c_{n-1}, c_0, c_1, \ldots, c_{n-2})$. σ is called cyclic shift permutation. A linear code of length n over $R_{k,m}$, that is a $R_{k,m}$-submodule of the ring $R_{k,m}^n$, is said to be cyclic if it is invariant under the cyclic shift permutation σ. We use the natural correspondence between cyclic codes of length n over $R_{k,m}$ and ideals of the ring $R_{k,m}[x]/\langle x^n - 1\rangle$, which sends $(c_0, c_1, \ldots, c_{n-1})$ to $c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}$. In this paper, we denote the ring $R_{k,m}[x]/\langle x^n - 1\rangle$ by $R_{k,m,n}$. For an ideal I of $R_{k,m,n}$ we define the residue and the torsion ideals as those defined in [11]:

$$Res(I) = \{\mathbf{a} \in R_{k-1,m} | \exists \mathbf{b} \in R_{k-1,m} : \mathbf{a} + u_k\mathbf{b} \in I\},$$

and

$$Tor(I) = \{\mathbf{a} \in R_{k-1,m} | u_k\mathbf{a} \in I\}.$$

Recall from [11] that we have $|I| = |Res(I)||Tor(I)|$. It is easy to see that if I is an ideal of $R_{k,m,n}$ then both $Res(I)$ and $Tor(I)$ are ideals of $R_{k-1,m,n}$.

In this section we completely determine the structure of ideals of the ring $R_{2,m,n}$, i.e. cyclic codes of length n over $R_{2,m}$. The ring $R_{2,m}$ is represented by $\mathbb{F}_{2^m}[u,v]/\langle u^2, v^2, uv - vu\rangle$ in this section. Let I be an ideal of the ring $R_{2,m,n}$. We associate 4 ideals,

$$\begin{aligned}
I_1 &:= Res(Res(I)) = I \bmod \langle u, v\rangle,\\
I_2 &:= Tor(Res(I)) = \{f(x) \in \mathbb{F}_{2^m}[x] | uf(x) \in I \bmod v\}\\
I_3 &:= Res(Tor(I)) = \{f(x) \in \mathbb{F}_{2^m}[x] | vf(x) \in I \bmod uv\}\\
I_4 &:= Tor(Tor(I)) = \{f(x) \in \mathbb{F}_{2^m}[x] | uvf(x) \in I\},
\end{aligned}$$

to I. These are ideals of the ring $R_{0,m,n}$ and hence for any $1 \le j \le 4$ we have $I_j = \langle f_j(x)\rangle$, where $f_j(x) \mid x^n - 1$ in $\mathbb{F}_{2^m}[x]$. We also have $I_1 \subseteq I_j$ for $2 \le j \le 4$, $I_2 \subseteq I_4$, and $I_3 \subseteq I_4$. Hence, $f_4(x) \mid f_j(x)$ for $1 \le j \le 3$, $f_2(x) \mid f_1(x)$, and $f_3(x) \mid f_1(x)$.

According to Theorem 3.6 of [20], any ideal I can be generated by polynomials of the forms

$$\begin{aligned}
A_1(x) &= f_1(x) + uf_{1,2}(x) + vf_{1,3}(x) + uvf_{1,4}(x),\\
A_2(x) &= uf_2(x) + vf_{2,3}(x) + uvf_{2,4}(x),\\
A_3(x) &= vf_3(x) + uvf_{3,4}(x),\\
A_4(x) &= uvf_4(x).
\end{aligned}$$

Note that when $f_j(x) = x^n - 1$ we set $A_j(x) := 0$. Some conditions on the polynomials $f_{i,j}(x)$ were given in [20]. Here we prove that we can choose these generators such that $f_{i,j}(x) = 0$ or $\deg(f_{i,j}(x)) < \deg(f_j(x))$. We prove this for $i = 1$ and $2 \leq j \leq 4$. The remaining cases are easy. Assume that $A_1(x) \neq 0$ and $\deg(f_{1,2}(x)) \geq \deg(f_2(x))$. Dividing $f_{1,2}(x)$ by $f_2(x)$, we have $f_{1,2}(x) = q(x)f_2(x) + r(x)$, where $\deg(r(x)) < \deg(f_2(x))$ or $r(x) = 0$. Now

$$A_1(x) - q(x)A_2(x) = f_1(x) + ur(x) + v(f_{1,3}(x) - q(x)f_{2,3}(x)) + uv(f_{1,4}(x) - q(x)f_{2,4}(x)) \in I.$$

If $\deg(f_{1,3}(x) - q(x)f_{2,3}(x)) \geq \deg(f_3(x))$, then dividing it by $f_3(x)$ we have

$$f_{1,3}(x) - q(x)f_{2,3}(x) = f_3(x)q'(x) + r'(x),$$

where $\deg(r'(x)) < \deg(f_3(x))$. Therefore,

$$A_1(x) - q(x)A_2(x) - q'(x)A_3(x) = f_1(x) + ur(x) + vr'(x) + uv(f_{1,4}(x) - q(x)f_{2,4}(x) - q'(x)f_{3,4}(x)) \in I.$$

Finally, dividing $f_{1,4}(x) - q(x)f_{2,4}(x) - q'(x)f_{3,4}(x)$ by $f_4(x)$ we have $f_{1,4}(x) - q(x)f_{2,4}(x) - q'(x)f_{3,4}(x) = f_4(x)q''(x) + r''(x)$, where $\deg(r''(x)) < \deg(f_4(x))$ and hence

$$A_1(x) - q(x)A_2(x) - q'(x)A_3(x) - q''(x)A_4(x) = f_1(x) + ur(x) + vr'(x) + uvr''(x) \in I.$$

Now this later polynomial has the desired property and it is easy to check that we can replace it with $A_1(x)$. Next we prove that these polynomials are unique. Again we prove it only for $A_1(x)$. Assume that $A_1(x) = f_1(x) + uf_{1,2}(x) + vf_{1,3}(x) + uvf_{1,4}(x)$ and $B_1(x) = f_1(x) + uf'_{1,2}(x) + vf'_{1,3}(x) + uvf'_{1,4}(x)$ are 2 polynomials with the mentioned property in I. Hence,

$$A_1(x) - B_1(x) = u(f_{1,2}(x) - f'_{1,2}(x)) + v(f_{1,3}(x) - f'_{1,3}(x)) + uv(f_{1,4}(x) - f'_{1,4}(x)) \in I.$$

Since $f_{1,2}(x) - f'_{1,2}(x) \in I_2$ and $\deg(f_{1,2}(x) - f'_{1,2}(x)) < \deg(f_2(x))$, we have $f_{1,2}(x) - f'_{1,2}(x) = 0$ and hence $f_{1,2}(x) = f'_{1,2}(x)$. Similarly, we have $f_{1,j}(x) = f'_{1,j}(x)$ for $3 \leq j \leq 4$. Therefore, $A_1(x) = B_1(x)$. Let us summarize:

Theorem 1 *Any ideal I of the ring $R_{2,m,n}$ is uniquely generated by the polynomials*

$$\begin{aligned} A_1(x) &= f_1(x) + uf_{1,2}(x) + vf_{1,3}(x) + uvf1,4(x), \\ A_2(x) &= uf_2(x) + vf_{2,3}(x) + uvf_{2,4}(x), \\ A_3(x) &= vf_3(x) + uvf_{3,4}(x), \\ A_4(x) &= uvf_4(x), \end{aligned}$$

where $I_j := \langle f_j(x) \rangle$ for $1 \leq j \leq 4$ and $f_{i,j}(x) = 0$ or $\deg(f_{i,j}(x)) < \deg(f_j(x))$.

For an ideal I of the ring $R_{2,m,n}$, the form described in Theorem 1 is referred to as the unique form of I. Checking the conditions $I_j := \langle f_j(x) \rangle$ is the main and difficult task in obtaining the unique form. We start with the following proposition to obtain more practical conditions instead.

Proposition 1 *Let*

$$\begin{aligned} I = \langle & f_1(x) + uf_{1,2}(x) + vf_{1,3}(x) + uvf_{1,4}(x), \\ & uf_2(x) + vf_{2,3}(x) + uvf_{2,4}(x), \\ & vf_3(x) + uvf_{3,4}(x), \\ & uvf_4(x)\rangle \end{aligned}$$

be an ideal of the ring $R_{2,m,n}$ in the unique form. Then we must have

$$f_4(x)|f_1(x), \quad f_4(x)|f_2(x), \quad f_4(x)|f_3(x), \quad f_3(x)|f_1(x), \quad f_2(x)|f_1(x), \quad f_1(x)|x^n-1. \tag{1}$$

$$f_2(x) \mid f_{1,2}(x)\frac{(x^n-1)}{f_1(x)}. \tag{2}$$

$$f_3(x) \mid \frac{(x^n-1)}{f_1(x)}\left(f_{1,3}(x) + \frac{f_{1,2}(x)}{f_2(x)}f_{2,3}(x)\right). \tag{3}$$

$$f_3(x)|\frac{f_1(x)}{f_2(x)}f_{2,3}(x). \tag{4}$$

$$f_4(x)|f_{2,3}(x). \tag{5}$$

$$f_4(x)|\frac{(x^n-1)}{f_3(x)}f_{3,4}(x). \tag{6}$$

$$f_4(x)|\frac{(x^n-1)}{f_2(x)}\left(f_{2,4}(x) + \frac{f_{2,3}(x)}{f_3(x)}f_{3,4}(x)\right). \tag{7}$$

$$f_4(x)|\left(f_{1,2}(x) + \frac{f_1(x)}{f_3(x)}f_{3,4}(x)\right). \tag{8}$$

$$f_4(x)|\left(f_{1,3}(x) + \frac{f_1(x)}{f_2(x)}f_{2,4}(x) + \frac{f_1(x)}{f_2(x)f_3(x)}f_{2,3}(x)f_{3,4}(x)\right). \tag{9}$$

$$f_4(x)|\frac{(x^n-1)}{f_1(x)}\left(f_{1,4}(x) + \frac{f_{1,2}(x)}{f_2(x)}f_{2,4}(x) + \frac{f_{1,3}(x) + \frac{f_{1,2}(x)}{f_2(x)}f_{2,3}(x)}{f_3(x)}f_{3,4}(x)\right). \tag{10}$$

Proof Conditions given in (1) are clear since we have $I_1 \subseteq I_j$ for $2 \leq j \leq 4$ and $I_j \subseteq I_4$ for $2 \leq j \leq 3$. Condition (2) follows from the facts that

$$\frac{(x^n-1)}{f_1(x)}\left(f_1(x) + uf_{1,2}(x) + vf_{1,3}(x) + uvf_{1,4}(x)\right) =$$

$$u\frac{(x^n-1)}{f_1(x)}f_{1,2}(x) + v\frac{(x^n-1)}{f_1(x)}f_{1,3}(x) + uv\frac{(x^n-1)}{f_1(x)}f_{1,4}(x) \in I$$

and $I_2 = \langle f_2(x)\rangle$. Similarly, we have

$$\frac{(x^n-1)}{f_1(x)}\left(f_1(x) + uf_{1,2}(x) + vf_{1,3}(x) + uvf_{1,4}(x)\right) + \frac{(x^n-1)}{f_1(x)}\frac{f_{1,2}(x)}{f_2(x)}\left(uf_2(x) + vf_{2,3}(x) + uvf_{2,4}(x)\right) =$$

$$v\left(\frac{(x^n-1)}{f_1(x)}(f_{1,3}(x) + \frac{f_{1,2}(x)}{f_2(x)}f_{2,3}(x))\right) + uv\left(\frac{(x^n-1)}{f_1(x)}((f_{1,4}(x) + \frac{f_{1,2}(x)}{f_2(x)}f_{2,4}(x))\right) \in I,$$

which imply condition (3). Also,

$$u\left(f_1(x)+uf_{1,2}(x)+vf_{1,3}(x)+uvf_{1,4}(x)\right)+\frac{f_1(x)}{f_2(x)}\left(uf_2(x)+vf_{2,3}(x)+uvf_{2,4}(x)\right)=$$

$$v\left(\frac{f_1(x)}{f_2(x)}f_{2,3}(x)\right)+uv\left(f_{1,3}(x)+\frac{f_1(x)}{f_2(x)}f_{2,4}(x)\right)\in I$$

imply condition (4),

$$u\left(uf_2(x)+vf_{2,3}(x)+uvf_{2,4}(x)\right)=uvf_{2,3}(x)\in I$$

imply condition (5),

$$\frac{(x^n-1)}{f_3(x)}\left(vf_3(x)+uvf_{3,4}(x)\right)\in I$$

imply condition (6),

$$\frac{(x^n-1)}{f_2(x)}\left(uf_2(x)+vf_{2,3}(x)+uvf_{2,4}(x)\right)+\frac{(x^n-1)}{f_2(x)}\frac{f_{2,3}(x)}{f_3(x)}\left(vf_3(x)+uvf_{3,4}(x)\right)=$$

$$uv\frac{(x^n-1)}{f_2(x)}\left(f_{2,4}(x)+\frac{f_{2,3}(x)}{f_3(x)}f_{3,4}(x)\right)\in I$$

imply condition (7),

$$v\left(f_1(x)+uf_{1,2}(x)+vf_{1,3}(x)+uvf_{1,4}(x)\right)+\frac{f_1(x)}{f_3(x)}\left(vf_3(x)+uvf_{3,4}(x)\right)=$$

$$uv\left(f_{1,2}(x)+\frac{f_1(x)}{f_3(x)}f_{3,4}(x)\right)\in I$$

imply condition (8),

$$u\left(f_1(x)+uf_{1,2}(x)+vf_{1,3}(x)+uvf_{1,4}(x)\right)+\frac{f_1(x)}{f_2(x)}\left(uf_2(x)+vf_{2,3}(x)+uvf_{2,4}(x)\right)+$$

$$\frac{f_1(x)f_{2,3}(x)}{f_2(x)f_3(x)}\left(vf_3(x)+uvf_{3,4}(x)\right)=$$

$$uv\left(f_{1,3}(x)+\frac{f_1(x)}{f_2(x)}f_{2,4}(x)+\frac{f_1(x)}{f_2(x)f_3(x)}f_{2,3}(x)f_{3,4}(x)\right)\in I$$

imply condition (9), and finally

$$\frac{(x^n-1)}{f_1(x)}\left(f_1(x)+uf_{1,2}(x)+vf_{1,3}(x)+uvf_{1,4}(x)\right)+\frac{(x^n-1)}{f_1(x)}\frac{f_{1,2}(x)}{f_2(x)}\left(uf_2(x)+vf_{2,3}(x)+uvf_{2,4}(x)\right)+$$

$$\frac{(x^n-1)}{f_1(x)}\frac{f_{1,3}(x)+\frac{f_{1,2}(x)}{f_2(x)}f_{2,3}(x)}{f_3(x)}\left(vf_3(x)+uvf_{3,4}(x)\right)=$$

$$\frac{(x^n-1)}{f_1(x)}\left(f_{1,4}(x)+\frac{f_{1,2}(x)}{f_2(x)}f_{2,4}(x)+\frac{f_{1,3}(x)+\frac{f_{1,2}(x)}{f_2(x)}f_{2,3}(x)}{f_3(x)}f_{3,4}(x)\right)\in I$$

imply condition (10). The proof is now completed. □

Theorem 2 *Let*

$$I =< f_1(x)+uf_{1,2}(x)+vf_{1,3}(x)+uvf_{1,4}(x),$$
$$uf_2(x)+vf_{2,3}(x)+uvf_{2,4}(x),$$
$$vf_3(x)+uvf_{3,4}(x),$$
$$uvf_4(x) >$$

be an ideal of the ring $R_{2,m,n}$ such that $deg(f_{i,j}(x)) < deg(f_j(x))$ and also $f_j(x)$ and $f_{i,j}(x)$ satisfy in conditions given in (1) through (10) described in Proposition 1. Then I is in the unique form.

Proof It is enough to show $I_j = \langle f_j(x)\rangle$. Suppose that $m(x)$ is an arbitrary element of I. Hence, we can write

$$m(x) = (a_1(x)+ua_2(x)+va_3(x)+uva_4(x))(f_1(x)+uf_{1,2}(x)+vf_{1,3}(x)+uvf_{1,4}(x))$$
$$+(b_1(x)+ub_2(x)+vb_3(x))(uf_2(x)+vf_{2,3}(x)+uvf_{2,4}(x))$$
$$+(c_1(x)+uc_2(x))(vf_3(x)+uvf_{3,4}(x))+d_1(x)(uvf_4(x)) =$$
$$a_1(x)f_1(x)+u(a_1(x)f_{1,2}(x)+a_2(x)f_1(x)+b_1(x)f_2(x))+$$
$$v(a_1(x)f_{1,3}(x)+a_3(x)f_1(x)+b_1(x)f_{2,3}(x)+c_1(x)f_3(x))+$$
$$uv(a_1(x)f_{1,4}(x)+a_2(x)f_{1,3}(x))+a_3(x)f_{1,2}(x)+a_4(x)f_1(x)+b_1(x)f_{2,4}(x)+$$
$$b_2(x)f_{2,3}(x)+b_3(x)f_2(x)+c_1(x)f_{3,4}(x)+c_2(x)f_3(x)+f_4(x)d_1(x)),$$

where $a_i(x), b_i(x), c_i(x), d_1(x) \in \mathbb{F}_{2^m}[x]$.

Clearly we have $I_1 = \langle f_1(x)\rangle$. If $m(x)$ be of the form $uf'_2(x)+vf'_{2,3}(x)+uvf'_{2,4}(x)$, then we must have $a_1(x)f_1(x) = 0$ and so $\frac{(x^n-1)}{f_1(x)}|a_1(x)$. Also by conditions (1) and (3) of Proposition 1, we have

$$f_2(x)|a_1(x)f_{1,2}(x)+a_2(x)f_1(x)+b_1(x)f_2(x),$$

which implies $I_2 = \langle f_2(x)\rangle$. If $m(x)$ be of the form $vf'_3(x)+uvf'_{3,4}(x)$, then we must have $a_1(x)f_1(x) = 0$ implying $\frac{(x^n-1)}{f_1(x)}|a_1(x)$ and also $a_1(x)f_{1,2}(x)+a_2(x)f_1(x)+b_1(x)f_2(x) = 0$ implying

$$b_1(x) = \frac{a_1(x)f_{1,2}+a_2(x)f_1(x)}{f_2(x)}.$$

Therefore,

$$a_1(x)f_{1,3}(x)+a_3(x)f_1(x)+b_1(x)f_{2,3}(x)+c_1(x)f_3(x) = a_1(x)f_{1,3}(x)$$
$$+a_3(x)f_1(x)+(\frac{a_1(x)f_{1,2}+a_2(x)f_1(x)}{f_2(x)})f_{2,3}(x)+c_1(x)f_3(x).$$

By conditions given in (1), (3), and (4) of Proposition 1, we have

$$f_3(x)|a_1(x)f_{1,3}(x)+a_3(x)f_1(x)+b_1(x)f_{2,3}(x)+c_1(x)f_3(x),$$

and hence $I_3=\langle f_3(x)\rangle$. Finally, if $m(x)$ be of the form $uvf_4^{'}(x)$, then we conclude $a_1(x)f_1(x)=0$, implying $\frac{(x^n-1)}{f_1(x)}|a_1(x)$, $a_1(x)f_{1,2}(x)+a_2(x)f_1(x)+b_1(x)f_2(x)=0$ implying

$$b_1(x)=\frac{a_1(x)f_{1,2}+a_2(x)f_1(x)}{f_2(x)},$$

and

$$a_3(x)f_1(x)+a_1(x)f_{1,3}(x)+b_1(x)f_{2,3}(x)+c_1(x)f_3(x)=0$$

implying

$$c_1(x)=\frac{a_3(x)f_1(x)+a_1(x)f_{1,3}(x)+b_1(x)f_{2,3}(x)}{f_3(x)}.$$

Now we have

$$\begin{aligned}&a_1(x)f_{1,4}(x)+a_2(x)f_{1,3}(x)+a_3(x)f_{1,2}(x)+a_4(x)f_1(x)+b_1(x)f_{2,4}(x)+b_2(x)f_{2,3}(x)+\\&b_3(x)f_2(x)+c_1(x)f_{3,4}(x)+c_2(x)f_3(x)+f_4(x)d_1(x)=\\&a_1(x)f_{1,4}(x)+a_2(x)f_{1,3}(x))+a_3(x)f_{1,2}(x)+a_4(x)f_1(x)+(\tfrac{a_1(x)f_{1,2}+a_2(x)f_1(x)}{f_2(x)})f_{2,4}(x)+\\&b_2(x)f_{2,3}(x)+b_3(x)f_2(x)+(\tfrac{a_3(x)f_1(x)+a_1(x)f_{1,3}(x)+\frac{a_1(x)f_{1,2}+a_2(x)f_1(x)}{f_2(x)}f_{2,3}(x)}{f_3(x)})f_{3,4}(x)\\&+c_2(x)f_3(x)+f_4(x)d_1(x).\end{aligned}$$

By conditions (1), (2), (8), and (10) of Proposition 1, we have

$$\begin{aligned}&f_4(x)|a_4(x)f_1(x)+b_2(x)f_{2,3}(x)+b_3(x)f_2(x)\\&+c_2(x)f_3(x)+f_4(x)d_1(x)+a_1(x)f_{1,4}(x)+a_3(x)f_{1,2}(x)\\&+\frac{a_1(x)f_{1,2}(x)f_{2,4}(x)}{f_2(x)}+\frac{a_1(x)f_{1,3}(x)f_{3,4}(x)}{f_3(x)}\\&+\frac{a_3(x)f_1(x)f_{3,4}(x)}{f_3(x)}+\frac{a_1(x)f_{1,2}(x)f_{2,3}(x)f_{3,4}(x)}{f_2(x)f_3(x)}.\end{aligned}$$

Also, condition (9) implies

$$f_4(x)|a_2(x)f_{1,3}(x)+a_2(x)\frac{f_1(x)f_{2,4}(x)}{f_2(x)}+a_2(x)\frac{f_1(x)f_{2,3}(x)f_{3,4}(x)}{f_2(x)f_3(x)}.$$

Therefore, $I_4=\langle f_4(x)\rangle$ and the proof is completed. □

Theorem 3 *Let n be odd and I be an ideal of the ring $R_{2,m,n}$. Then the unique form of I is*

$$I=<f_1(x),uf_2(x)+vf_{2,3}(x),vf_3(x),uvf_4(x)>.$$

Proof The proof follows from Proposition 1 and the facts $deg(f_{i,j}(x)) < deg(f_j(x))$ and $(f_2(x), \frac{(x^n-1)}{f_1(x)}) = (\frac{(x^n-1)}{f_3(x)}, f_4(x)) = (\frac{(x^n-1)}{f_2(x)}, f_4(x)) = 1$. □

Proposition 2 *Let* $I = \langle A(x), B(x), C(x), D(x) \rangle$, *where*

$$\begin{aligned} A(x) &= f_1(x) + uf_{1,2}(x) + vf_{1,3}(x) + uvf_{1,4}(x), \\ B(x) &= uf_2(x) + vf_{2,3}(x) + uvf_{2,4}(x), \\ C(x) &= vf_3(x) + uvf_{3,4}(x), \\ D(x) &= uvf_4(x) \end{aligned}$$

is an ideal of $R_{2,m,n}$ *in its unique form. Then the set*

$$\left\{ \begin{array}{l} A(x), xA(x), \cdots, x^{n-a-1}A(x), \\ B(x), xB(x), \cdots, x^{n-b-1}B(x), \\ C(x), xC(x), \cdots, x^{n-c-1}C(x), \\ D(x), xD(x), \cdots, x^{n-d-1}D(x) \end{array} \right\}$$

forms an $\mathbb{F}_{2^m}$*-basis for* I, *where* $a = \deg(f_1(x))$, $b = \deg(f_2(x))$, $c = \deg(f_3(x))$, *and* $d = \deg(f_4(x))$.

Proof Assume that $\alpha(x), \beta(x), \gamma(x)$, and $\delta(x)$ are polynomials in $\mathbb{F}_{2^m}[x]$, such that $\deg(\alpha) < n - a$, $\deg(\beta) < n - b$, $\deg(\gamma) < n - c$, $\deg(\delta) < n - d$, and we have

$$\alpha(x)A(x) + \beta(x)B(x) + \gamma(x)C(x) + \delta(x)D(x) = 0.$$

Therefore, we must have $\alpha(x)f_1(x) = 0$, which implies that $\alpha(x) = 0$ since $\deg(\alpha) < n - a$. Similar arguments show that we must have $\beta(x) = \gamma(x) = \delta(x) = 0$ and the proof is now completed. □

Example 1 *Let* $n = 14$, $f_1(x) = (x+1)(x^3+x^2+1)^2(x^3+x+1)^2$, *and* $f_2(x) = f_3(x) = f_4(x) = (x+1)(x^3+x^2+1)(x^3+x+1)^2$. *Clearly these polynomials satisfy condition (1) of Proposition 1. Next, we will find ideals* I *of the ring* $R_{2,1,14}$ *such that* $I_i = \langle f_i(x) \rangle$ *for* $1 \leq i \leq 4$. *Such ideals are in the following form:*

$$\begin{aligned} I = \langle f_1(x) + uf_{1,2}(x) + vf_{1,3}(x) + uvf_{1,4}(x), \\ uf_2(x) + vf_{2,3}(x) + uvf_{2,4}(x), \\ vf_3(x) + uvf_{3,4}(x), \\ uvf_4(x) \rangle. \end{aligned}$$

According to $deg(f_{i,j}(x)) < deg(f_j(x))$, *the polynomials* $f_{i,j}(x)$ *have degree of at most 9. Now we check conditions (2) through (10) of Proposition 1.*
Condition (2): $(x+1)(x^3+x^2+1)(x^3+x+1)^2 | (x+1)f_{1,2}(x)$. *Thus,*

$$f_{1,2}(x) = \alpha_1(x^3+x^2+1)(x^3+x+1)^2 \qquad \alpha_1 \in \mathbb{F}_2.$$

Condition (3): $(x+1)(x^3+x^2+1)(x^3+x+1)^2 | (x+1)f_{1,3}(x)$. *Therefore,*

$$f_{1,3}(x) = \alpha_2(x^3+x^2+1)(x^3+x+1)^2 \qquad \alpha_2 \in \mathbb{F}_2.$$

Condition (5): $(x+1)(x^3+x^2+1)(x^3+x+1)^2 | f_{2,3}(x)$. *Hence,* $f_{2,3}(x)=0$.
Condition (6): $(x+1)(x^3+x^2+1)(x^3+x+1)^2 | (x+1)(x^3+x^2+1)f_{3,4}(x)$. *Consequently,*

$$f_{3,4}(x) = (x^3+x+1)^2 \sum_{i=0}^{3} b_i x^i.$$

Condition (7): $(x+1)(x^3+x^2+1)(x^3+x+1)^2 | (x+1)(x^3+x^2+1)f_{2,4}(x)$. *So,*

$$f_{2,4}(x) = (x^3+x+1)^2 \sum_{i=0}^{3} a_i x^i.$$

Condition (8): $(x+1)(x^3+x^2+1)(x^3+x+1)^2 | \alpha_1(x^3+x^2+1)(x^3+x+1)^2 + (x^3+x^2+1)f_{3,4}(x)$. *Thus,*

$$\alpha_1 = b_0+b_1+b_2+b_3.$$

Condition (9): $(x+1)(x^3+x^2+1)(x^3+x+1)^2 | \alpha_2(x^3+x^2+1)(x^3+x+1)^2 + (x^3+x^2+1)f_{3,4}(x)$. *Hence,*

$$\alpha_2 = a_0+a_1+a_2+a_3.$$

Condition (10):

$$(x+1)(x^3+x^2+1)(x^3+x+1)^2 | (x+1)f_{1,4}(x) + \alpha_1(x^3+x+1)^2 \sum_{i=0}^{3} a_i x^i + \alpha_2(x^3+x+1)^2 \sum_{i=0}^{3} b_i x^i.$$

Therefore, $(x^3+x+1)^2 \mid f_{1,4}(x)$. *Also, from relations obtained from conditions (8) and (9), we can deduce that*

$$\alpha_1 \sum_{i=0}^{3} a_i x^i + \alpha_2 \sum_{i=0}^{3} b_i x^i = (x+1)(u_3x^2 + (u_3+u_2)x + (u_3+u_2+u_1)),$$

where $u_i = \alpha_1 a_i + \alpha_2 b_i$ *for* $0 \le i \le 3$. *Now summarizing the above results we can conclude that* $f_{1,4}(x) = \alpha(x^3+x+1)^2(x^3+(1+u_3)x^2+(u_2+u_3)x+(1+u_1+u_2+u_3))$ *where* $\alpha \in \mathbb{F}_{2^m}$. *Therefore,*

$$\begin{aligned} I \;=\; & (x^3+x+1)^2 \left((x+1)(x^3+x^2+1)^2 + (x^3+x^2+1)(u\alpha_1+v\alpha_2) + uv\alpha(x^3+(1+u_3)x^2 \right. \\ & \left. +(u_2+u_3)x+(1+u_1+u_2+u_3)) \right), u(x^3+x+1)^2 \left((x+1)(x^3+x^2+1) + v\sum_{i=0}^{3} a_i x^i \right), \\ & v(x^3+x+1)^2 \left((x+1)(x^3+x^2+1) + u\sum_{i=0}^{3} b_i x^i \right), uv(x+1)(x^3+x^2+1)(x^3+x+1)^2 \end{aligned}$$

Therefore, we have 512 *distinct ideals in this case. Searching among all 512 ideals, the Gray images of many of them, such as one with* $\alpha=1$, $a_0=b_0=b_1=a_3=1$ *and* $a_1=a_2=b_2=b_3=0$, *have minimum distance 20, and hence we obtain a [56,13,20] code that has the best minimum distance among all codes of that size.*

Now we will determine all distinct ideals of the ring $R_{2,m,2}$ in the next theorem.

Theorem 4 *Ideals of the ring $R_{2,m,2}$ are precisely the following ideals:*

$$\langle 0\rangle, \langle uv(x-1)\rangle, \langle uv\rangle, \langle v, uv\rangle, \langle v(x-1), uv\rangle, \langle v(x-1)+uv\zeta_1, uv(x-1)\rangle,$$
$$\langle u(x-1), v, uv\rangle, \langle u(x-1)+v\zeta_1, v(x-1), uv\rangle, \langle u(x-1)+v\zeta_1(x-1), uv\rangle,$$
$$\langle u(x-1)+v\zeta_1(x-1)+uv\zeta_2, uv(x-1)\rangle, \langle u(x-1)+uv\zeta_1, v(x-1)+uv\zeta_2, uv(x-1)\rangle,$$
$$\langle u, v, uv\rangle, \langle u+v\zeta_1, v(x-1), uv\rangle, \langle u+v(\zeta_1(x-1)+\zeta_2), uv\rangle,$$
$$\langle (x-1), u, v, uv\rangle, \langle (x-1)+u\zeta_1, u(x-1), v, uv\rangle, \langle (x-1)+v\zeta_1, u+v\zeta_2, v(x-1), uv\rangle,$$
$$\langle (x-1)+u\zeta_1+v\zeta_2, u(x-1), v(x-1), uv\rangle,$$
$$\langle (x-1)+u\zeta_1+v\zeta_2+uv\zeta_3, u(x-1)+uv\zeta_2, v(x-1)+uv\zeta_1, uv(x-1)\rangle, \langle 1\rangle,$$

where $\zeta_j \in \mathbb{F}_{2^m}$ for $1 \le j \le 3$.

Proof First note that we have $x^2-1=(x-1)^2$ in $\mathbb{F}_{2^m}[x]$ and any polynomial of degree less than 2 over $\mathbb{F}_{2^m}$ can be written as $a(x-1)+b$, where $a, b \in \mathbb{F}_{2^m}$. According to Theorem 1, any ideal of the ring $R_{2,m,2}$ can be uniquely generated by polynomials

$$A_1(x) = (x-1)^{a_1} + uf_{1,2}(x) + vf_{1,3}(x) + uvf_{1,4}(x),$$
$$A_2(x) = u(x-1)^{a_2} + vf_{2,3}(x) + uvf_{2,4}(x),$$
$$A_3(x) = v(x-1)^{a_3} + uvf_{3,4}(x),$$
$$A_4(x) = uv(x-1)^{a_4},$$

where $I_j = \langle (x-1)^{a_j}\rangle$ and $f_{i,j}(x) \in \mathbb{F}_{2^m}[x]$ and $\deg(f_{i,j}(x)) < a_j$. Moreover, we have $0 \le a_1, a_2, a_3, a_4 \le 2$, $a_2 \le a_1$, $a_3 \le a_1$, $a_4 \le a_2$, and $a_4 \le a_3$. Also, if $a_t = 2$ then we have $A_t(x) = 0$. Next, we shall call the sequence a_1, a_2, a_3, a_4 the type of the ideal. We argue on the type of an ideal and determine all distinct ideals in the unique form. Since the arguments for all cases are similar, we only argue on the ideals of type $1,1,1,1$. Ideals of this type are of the form

$$I = \langle (x-1)+u\zeta_1+v\zeta_2+uv\zeta_3, u(x-1)+v\zeta_4+uv\zeta_5, v(x-1)+uv\zeta_6, uv(x-1)\rangle.$$

Now condition (5) of Proposition 1 implies that $(x-1) \mid \zeta_4$ and hence we must have $\zeta_4 = 0$. Moreover, condition (8) of the proposition implies that $(x-1) \mid (\zeta_2 - \zeta_5)$ and therefore we must have $\zeta_2 = \zeta_5$. Also, condition (9) implies $(x-1) \mid (\zeta_1 - \zeta_6)$. Consequently, we must have $\zeta_1 = \zeta_6$. Note that other conditions given in Proposition 1 are clearly satisfied. Therefore, according to Theorem 2, I is of the unique form

$$I = \langle (x-1)+u\zeta_1+v\zeta_2+uv\zeta_3, u(x-1)+uv\zeta_2, v(x-1)+uv\zeta_1, uv(x-1)\rangle,$$

where $\zeta_j \in \mathbb{F}_{2^m}$ for $1 \le j \le 3$ and the proof is now completed. □

Counting all of the above ideals, we have the following corollary.

Corollary 1 *There are $9+5(2^m)+5(2^{2m})+2^{3m}$ distinct ideals in the ring $R_{2,m,2}$.*

3. A one-to-one correspondence

In this section we show that there exists a one-to-one correspondence between cyclic codes of length n, n odd, over $R_{k,m}$ and cyclic codes of length $2n$ over $R_{k-1,m}$. For unifying the proof statements and simplicity, for $1 \le k \le 3$, we denote by

$$\frac{\mathbb{F}_{2^m}[u_1, u_2, \cdots, u_k]}{\langle u_i^2, u_iu_j - u_ju_i\rangle}$$

the ring $R_{k,m}$. Let us start the section with the following lemma.

Lemma 1 *The ring $R_{k,m}$ is isomorphic to the ring*

$$\frac{R_{k-1,m}[w]}{\langle w^2-1, wu_i-u_iw\rangle}.$$

Proof Set

$$S := \frac{R_{k-1,m}[w]}{\langle w^2-1, wu_i-u_iw\rangle}$$

and write

$$R_{k,m} = \frac{R_{k-1,m}[u_k]}{\langle u_k^2, u_ku_i-u_iu_k\rangle}.$$

Now define the map $\varphi : R_{k,m} \longrightarrow S$ by the role $\varphi(\alpha+u_k\beta) = (\alpha-\beta)+w\beta$. Since $\varphi(u_k) = (w-1)$ and in S we have $(w-1)^2 = 0$, it is easy to verify that the map φ is a ring isomorphism. □

Recall that we denote the ring $R_{k,m}[x]/\langle x^n-1\rangle$ by $R_{k,m,n}$.

Proposition 3 *Assume that n is odd. Then we have*

$$R_{k,m,n} \cong \frac{\frac{R_{k-1,m}[w]}{\langle w^2-1\rangle}[x]}{\langle x^n-w\rangle}.$$

Proof Define the map

$$\eta : R_{k,m,n} \longrightarrow \frac{\frac{R_{k-1,m}[w]}{\langle w^2-1\rangle}[x]}{\langle x^n-w\rangle}$$

by the role $\eta(f(x)) = f(wx)$. Since $(wx)^n-1 = wx^n-1 = w(x^n-w)$, we have $x^n-1 \mid f(x)$ if and only if $x^n-w \mid f(wx)$. Now it is straightforward to show that η is a ring isomorphism. □

Theorem 5 *There exists a one-to-one correspondence between ideals of the ring*

$$\frac{\frac{R_{k-1,m}[w]}{\langle w^2-1\rangle}[x]}{\langle x^n-w\rangle}$$

and ideals of the ring $R_{k-1,m,2n}$.

Proof Set

$$S := \frac{\frac{R_{k-1,m}[w]}{\langle w^2-1\rangle}[x]}{\langle x^n-w\rangle}$$

and define the map $\delta : S \longrightarrow R_{k-1,m,2n}$ by the role

$$\begin{aligned}\delta((a_0+wb_0)+(a_1+wb_1)x+\cdots+(a_{n-1}+wb_{n-1})x^{n-1}) \quad &= \\ a_0+a_1x+\cdots+a_{n-1}x^{n-1}+b_0x^n+b_1x^{n+1}+\cdots+b_{c-1}x^{2n-1}.&\end{aligned}$$

Let I be an ideal of the ring S. We have

$$\begin{aligned}(a_0+wb_0)+(a_1+wb_1)x+\cdots+(a_{n-1}+wb_{n-1})x^{n-1}\in I &\iff\\ w(a_{n-1}+wb_{n-1})+(a_0+wb_0)x+\cdots+(a_{n-2}+wb_{n-2})x^{n-1}\in I &\iff\\ (b_{n-1}+wa_{n-1})+(a_0+wb_0)x+\cdots+(a_{n-2}+wb_{n-2})x^{n-1}\in I &\iff\\ b_{n-1}+a_0x+\cdots+a_{n-2}x^{n-1}+a_{n-1}x^n+b_0x^{n+1}+\cdots+b_{n-2}x^{2n-1}\in\delta(I).\end{aligned}$$

Therefore, I is an ideal of S if and only if $\delta(I)$ is an ideal of $R_{k-1,m,2n}$. □

Corollary 2 *I is an ideal of the ring $R_{k,m,n}$ if and only if $\psi(I)$ is an ideal of the ring $R_{k-1,m,2n}$, where $\psi=\varphi\circ\eta\circ\delta$.*

Corollary 3 *There are $9+5(2^m)+5(2^{2m})+2^{3m}$ distinct ideals in the ring $R_{3,m}$.*

Remark 1 *We should note that the determination of all ideals of the ring $R_{k,m}$ was introduced in [11] as a challenging open problem. The previous corollary solves it for $k=3$.*

Example 2 *Here we list all 47 distinct ideals of the ring $R_{3,1}$. Note that for the ring $R_{3,m}$ we set $u:=u_1$, $v:=u_2$ and $w:=u_3$.*

$$\begin{array}{l}\langle 0\rangle,\ \langle 1\rangle,\ \langle u\rangle,\ \langle v\rangle,\ \langle w\rangle,\ \langle u,v\rangle,\langle u,w\rangle,\langle v,w\rangle,\langle u,v,w\rangle,\\ \langle uv\rangle,\ \langle uw\rangle,\ \langle vw\rangle,\ \langle uvw\rangle,\ \langle u+v\rangle,\ \langle u+w\rangle,\langle v+w\rangle,\\ \langle u+v+w\rangle,\ \langle uv,uw\rangle,\ \langle uv,vw\rangle,\ \langle uw,vw\rangle,\\ \langle uv,vw,uw\rangle,\ \langle uv+uw\rangle,\ \langle uv+vw\rangle,\ \langle uw+vw\rangle,\\ \langle uv+vw+uw\rangle,\ \langle w+v,u+v\rangle,\ \langle uw+uv,vw+uv\rangle,\ \langle u,vw\rangle,\\ \langle v,uw\rangle,\ \langle w,uv\rangle,\ \langle u+vw\rangle,\ \langle v+uw\rangle,\ \langle w+uv\rangle,\\ \langle u,v+w\rangle,\ \langle v,u+w\rangle,\ \langle w,u+v\rangle,\ \langle uw,v+w\rangle,\\ \langle vw,u+w\rangle,\ \langle vw,u+v\rangle,\ \langle uv+v+w\rangle,\ \langle uv+u+w\rangle,\\ \langle vw+u+v\rangle,\ \langle uv,wv+uw\rangle,\ \langle vw,uw+uv\rangle,\ \langle uw,uw+uv\rangle,\\ \langle u+v+w,wv,uw\rangle,\ \langle u+v+w+uv\rangle.\end{array}$$

If $x^n-1=p_1(x)p_2(x)\cdots p_r(x)$ be the factorization of x^n-1 over $\mathbb{F}_{2^m}$ into basic irreducible pairwise coprime polynomials, then we have

$$R_{k,m,n}\cong\bigoplus_{i=1}^{r}R_{k,m_i},$$

where $m_i=\deg(g_i(x))$. Noting that, from the results of [5], the number of ideals of the ring $R_{1,m,2}$, i.e. the number of ideals of the ring $R_{2,m}$, is $5+2^m$, we have the following corollary.

Corollary 4 *Let n be an odd number and m_i be as above. Then there are*

$$\prod_{i=1}^{r}(5+2^{m_i})$$

distinct cyclic codes of length n *over* $R_{2,m}$ *and there are*

$$\prod_{i=1}^{r}\left(9+5(2^{m_i})+5(2^{2m_i})+2^{3m_i}\right)$$

distinct cyclic codes of length n *over* $R_{3,m}$.

4. Conclusion

A unique set of generators for cyclic codes over the ring $R_{2,m}$ was introduced and a way for determination of these generators was presented. All distinct cyclic codes of length 2 over this ring were determined and a mass formula for the number of them was given. A one-to-one correspondence between cyclic codes of length $2n$, n odd, over the ring $R_{k-1,m}$ and cyclic codes of length n over the ring $R_{k,m}$ was introduced. The number of ideals of the rings $R_{2,m}$ and $R_{3,m}$ was determined. As a corollary, the number of cyclic codes of odd length n over the rings $R_{2,m}$ and $R_{3,m}$ was obtained.

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