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On the existence of nonzero injective covers and projective envelopes of modules

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Abstract: In general, the injective cover (projective envelope) of a simple module can be zero. A ring R is called a weakly left V-ring (strongly left Kasch ring) if every simple left R-module has a nonzero injective cover (projective envelope). It is proven that every nonzero left R-module has a nonzero injective cover if and only if R is a left Artinian weakly left V-ring. Dually, every nonzero left R-module has a nonzero projective envelope if and only if R is a left perfect right coherent strongly left Kasch ring. Some related rings and examples are considered.

Key words: Injective cover, projective envelope, weakly V-ring, strongly Kasch ring

1. Introduction

In 1953, Eckmann and Schopf proved that every module has an injective envelope. Bass considered the dual case in 1960 and characterized those rings over which every left module has a projective cover, namely, left perfect rings. Later, Enochs defined $\mathcal{C}$-(pre)envelopes and $\mathcal{C}$-(pre)covers, where $\mathcal{C}$ is an arbitrary class of modules (see [8]). Thus a $\mathcal{C}$-envelope ($\mathcal{C}$-cover) coincides with an injective envelope (projective cover) when $\mathcal{C}$ is the class of all injective (projective) modules. In 1981, Enochs [7] proved that a ring R is left Noetherian if and only if every left R-module has an injective cover. A dual result was given by Asensio Mayor and Martinez Hernandez in 1993. They showed that every left R-module has a projective envelope if and only if R is left perfect and right coherent [3, Proposition 3.5]. Ding and Chen also characterized left perfect and right coherent rings via the existence of projective preenvelopes (see [6, Proposition 3.14]).

In contrast to the case of injective envelopes (projective covers), an injective cover (projective envelope) of a nonzero module can be zero when it exists. Belshoff and Xu proved in 2001 that if every nonzero left R-module has a nonzero injective cover, then R is left Artinian. They also gave an example to show that the converse is not true in general (see [4]). The existence of nonzero projective (pre)envelopes of simple modules was investigated by Mao [12] in 2007.

In this paper, we will prove that every nonzero left R-module has a nonzero injective cover if and only if R is a left Artinian and every simple left R-module has a nonzero injective cover. Dually, every nonzero left R-module has a nonzero projective envelope if and only if R is left perfect right coherent and every simple left R-module has a nonzero projective envelope. We call R a weakly left V-ring (strongly left Kasch ring) if every simple left R-module has a nonzero injective cover (projective envelope). Some related rings and examples are considered.

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Throughout this paper R is an associative ring with identity 1_R and all modules are unitary.

Let $\mathcal{C}$ be a class of left R-modules. Recall [8] that a homomorphism $\varphi : C \to M$ with $C \in \mathcal{C}$ is called a *$\mathcal{C}$-precover* of M if, for any $C' \in \mathcal{C}$ and any homomorphism $\varphi' : C' \to M$, there exists a homomorphism $\alpha : C' \to C$ such that $\varphi\alpha = \varphi'$. If, in addition, any homomorphism $\beta : C \to C$ such that $\varphi\beta = \varphi$ is an automorphism of C, then φ is called a *$\mathcal{C}$-cover* of M. *$\mathcal{C}$-(pre)envelope* of M is defined dually. In the case when $\mathcal{C}$ is the class of all injective left R-modules, we will indicate $\mathcal{C}$-(pre)cover and $\mathcal{C}$-(pre)envelope by "injective (pre)cover" and "injective (pre)envelope", respectively. The terminology "projective (pre)cover" and "projective (pre)envelope" are adopted similarly when $\mathcal{C}$ is the class of all projective left R-modules.

Given a module M, $\mathrm{Add}M$ ($\mathrm{add}M$) denotes the class of modules that is isomorphic to a summand of a (finite) direct sum of copies of M. For instance, $\mathrm{Add}_R R$ ($\mathrm{add}_R R$) is the class of (finitely generated) projective left R-modules. The Jacobson radical, socle, and injective envelope of M are denoted by $\mathrm{Rad}(M)$, $\mathrm{Soc}(M)$, and $E(M)$, respectively. The Jacobson radical of R is written as $J(R)$. We refer the reader to [2, 8, 14] for more fundamental notions and results on modules and rings.

2. Nonzero injective covers

As we have mentioned above, if every nonzero left R-module has a nonzero injective cover, then R is left Artinian, but the converse does not hold in general. Note that over the ring in the counterexample given by Belshoff and Xu (see Example 2.4 below), there is a simple module that possesses no nonzero injective cover. It is natural to conjecture that if R is left Artinian and every simple left R-module admits a nonzero injective cover, then so does every nonzero left R-module.

Recall that a ring R is called a *left V-ring* [10] (*left GV-ring* [17]) if every (singular) simple left R-module is injective. This motivates us to give the following definition.

Definition 2.1 *A ring R is called a* weakly left V-ring *if every simple left R-module has a nonzero injective cover.* Weakly right V-ring *can be defined similarly. R is called a* weakly V-ring *if it is both a weakly left and a weakly right V-ring.*

Example 2.2 *(1) Every left V-ring is a weakly left V-ring. However, $\mathbb{Z}_4$, the ring of integers modulo 4, is a weakly left V-ring that is not even a left GV-ring. Note that the epimorphism $\mathbb{Z}_4 \to 2\mathbb{Z}_4$ is a nonzero injective cover of the unique (up to isomorphism) simple singular $\mathbb{Z}_4$-module $2\mathbb{Z}_4$ which is not injective.*

This example also shows that a commutative weak V-ring need not be von Neumann regular in contrast to the fact, due to Kaplansky, that a commutative ring is a V-ring if and only if it is von Neumann regular [18, Theorem 6].

(2) Every QF ring is a weakly left V-ring. Indeed, it is well known that a module over a QF ring is injective if and only if it is projective. Thus, over a QF ring, the injective cover of a simple module coincides with its projective cover, which obviously exists.

Conversely, let F be a field and $R = \prod_{i\in\mathbb{N}} F_i$, where each $F_i = F$. Then R is a weakly left V-ring but not QF. In fact, R is a non-Noetherian commutative von Neumann regular ring.

(3) A commutative Artinian ring is always a weakly left V-ring (see [4, Corollary]). The converse is obviously not true.

Since every nonzero homomorphism to a simple module is surjective and a simple module is projective or singular, the following result is obvious.

Proposition 2.3 *(1) If R is a weakly left V-ring then every projective simple left R-module is injective.*

(2) A ring R is a left V-ring if and only if it is both a left GV-ring and a weakly left V-ring.

(3) A left hereditary weakly left V-ring R is a left V-ring.

From Example 2.2(1) and the following example, one can see that the notions of left GV-rings and weakly left V-rings are independent and proper generalizations of left V-rings.

Example 2.4 *A left GV-ring need not be a weakly left V-ring. For instance, let R be the 2×2 upper triangular matrix ring over the field $\mathbb{Z}_2$. Then $I = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$ is the unique essential proper left ideal of R and I is a maximal left ideal of R. Every singular simple left R-module is isomorphic to R/I. It is easy to see that $Hom_R(I, R/I) = 0$, and hence R/I is injective. However, the projective simple left R-module $M = \begin{pmatrix} \mathbb{Z}_2 & 0 \\ 0 & 0 \end{pmatrix}$ is not injective (see the counterexample in [4]).*

Following [1], a ring R is called a *T-ring* if every nonzero left R-module has a nonzero socle. Now let us state the main result of this section.

Theorem 2.5 *The following are equivalent for a ring R:*

(1) Every nonzero left R-module has a nonzero injective cover.

(2) R is a left Artinian weakly left V-ring.

(3) R is a left Noetherian T-ring and a weakly left V-ring.

Proof It suffices to show (3)$\Rightarrow$(1). So suppose $\varphi : E \to M$ is an injective cover of a nonzero left R-module M. We may assume that M has a simple submodule S with a nonzero injective cover $f : E_0 \to S$ by hypothesis. It follows that if factors through φ, where $i : S \to M$ is the including map, i.e. $\varphi\alpha = if$ for some $\alpha \in \mathrm{Hom}_R(E_0, E)$.

$$\begin{array}{ccc} E_0 & \xrightarrow{f} & S \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle i} \\ E & \xrightarrow{\varphi} & M \end{array}$$

So we have $\varphi \neq 0$ since $if \neq 0$. □

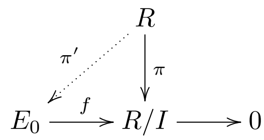
Corollary 2.6 *Every nonzero module over a QF ring R has a nonzero injective cover.*

Theorem 2.7 *The following are equivalent for a ring R:*

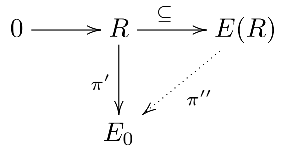
(1) R is a weakly left V-ring.

(2) For each maximal left ideal I of R, there is a maximal submodule K of $E(R)$ such that $I = R \cap K$ and $E(R)/K$ has a nonzero injective cover, where $E(R)$ is the injective envelope of ${}_RR$.

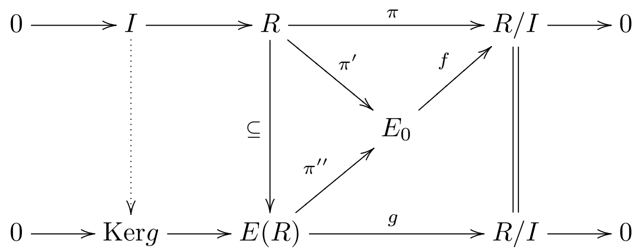
Proof (1)$\Rightarrow$(2). For every maximal left ideal I of R, let $f : {}_RE_0 \to R/I$ be a nonzero injective cover of R/I. It is clear that f is surjective. Hence the canonical epimorphism $\pi : R \to R/I$ can be lifted to E_0, i.e., there exists $\pi' : R \to E_0$ such that the following diagram commutes.



Now the injectivity of E_0 guarantees that π' can be extended to $E(R)$, i.e. there is a homomorphism $\pi'' : E(R) \to E_0$ such that



commutes. Let $g = f\pi''$. Then we have the following commutative diagram.

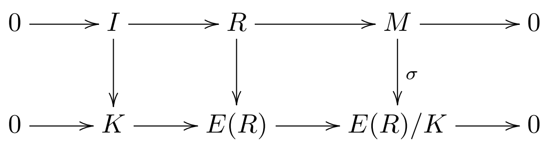


Now it is easy to see that $K = \text{Ker}g$ is as required.

(2)$\Rightarrow$(1). For any simple left R-module M, there is an exact sequence

$$0 \to I \to R \to M \to 0,$$

where I is a maximal left ideal of R. By (2), we have the following commutative diagram:

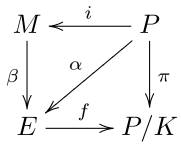


We claim that $\sigma \neq 0$. If this is not the case, then $R \subseteq K$. Hence $I = R \cap K = R$, a contradiction. Therefore, $M \cong E({}_RR)/K$ and has a nonzero injective cover. □

It is well known that a ring R is a left V-ring if and only if every left (cyclic) R-module M has Jacobson radical $\text{Rad}(M) = 0$ [13, Theorem 2.1]. As far as a weakly left V-ring is concerned, we have the following.

Proposition 2.8 *Suppose that R is a weakly left V-ring and M is a left R-module. Then $Rad(M)$ contains no nonzero projective submodule.*

Proof Let P be a nonzero projective submodule of M. Then P has a maximal submodule K. Hence P/K has a nonzero injective cover $f: E \to P/K$, which is surjective by hypothesis. Now the projectivity of P guarantees that the canonical epimorphism $\pi : P \to P/K$ can be lifted to some $\alpha : P \to E$ and the injectivity of E implies that $\alpha = \beta i$ for some $\beta \in \mathrm{Hom}_R(M, E)$, where $i : P \to M$ is the including map. We thus have the following commutative diagram:



It follows that $\mathrm{Ker}(f\beta)$ is a maximal submodule of M. Note that P is not contained in $\mathrm{Ker}(f\beta)$. Therefore, $\mathrm{Rad}(M)$ contains no nonzero projective submodule. □

3. Nonzero projective envelopes

In this section we investigate rings R over which every (simple) left R-module has a nonzero projective envelope. It is easy to see that such rings must be left Kasch, i.e. every simple left R-module embeds in R [20]. Thus, we provide the following definition.

Definition 3.1 *A ring R is called a* strongly left Kasch ring *if every simple left R-module has a nonzero projective envelope.*

Theorem 3.2 *The following are equivalent for a ring R:*

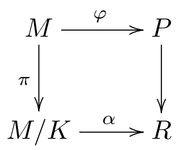
(1) Every nonzero left R-module has a nonzero projective envelope.

(2) R is a left perfect right coherent strongly left Kasch ring.

(3) R is a left perfect right coherent left Kasch ring.

Proof (1)⇒(2)⇒(3) is clear.

(3)⇒(1). Let M be a nonzero left R-module. Since R is left perfect, M has a maximal submodule K and a simple factor module M/K. Now it is easy to obtain a commutative diagram:



where $\varphi : M \to P$ is a projective envelope of M, $\pi : M \to M/K$ is the natural epimorphism, and $\alpha : M/K \to R$ is a monomorphism. It follows that $\varphi \neq 0$. □

Remark 3.3 *(1) Every QF ring satisfies the conditions in the above theorem, but the converse is not true in general (see Example 3.4).*

(2) Let R be the 2×2 upper triangular matrix ring over the field $\mathbb{Z}_2$. Then R is Artinian and hence every R-module has a projective envelope. Note that the right annihilator of the maximal left ideal $I = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$

is zero. For any $f \in Hom_R(R/I, R)$*, we have* $f(1+I) = 0$ *since* $f(1+I)$ *is in the right annihilator of* I*. It follows that* $Hom_R(R/I, R) = 0$*. Therefore,* R *is not a left Kasch ring and the simple left* R*-module* R/I *admits only a zero projective envelope.*

(3) Every strongly left Kasch ring is left SPP (i.e. every simple left module has a projective preenvelope [12]). The ring R *in (2) is left SPP but it is not a left Kasch ring. In addition, note that any homomorphism from a simple* $\mathbb{Z}$*-module to a projective* $\mathbb{Z}$*-module is zero; hence,* $\mathbb{Z}$ *is a left SPP ring and every simple* $\mathbb{Z}$*-module has a zero projective (pre)envelope. Thus,* $\mathbb{Z}$ *is not a (strongly) left Kasch ring.*

(4) Let $M = Rx$ *be a simple left* R*-module with a nonzero projective envelope* $\varphi : M \to P$*. Suppose that* $P \oplus Q = R^{(I)}$ *and* $\pi_i : R^{(I)} \to R$ *is the canonic projection for each* $i \in I$*. It follows that* $\pi_i\varphi(x) \neq 0$ *for some* i*. From this, one can see that every strongly left Kasch ring must be left Kasch, but the converse is not true. See Example 3.10.*

We will focus on strongly left Kasch rings in what follows.

Example 3.4 *Let* $R = \mathbb{Z}_2 \propto (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ *be the trivial extension of* $\mathbb{Z}_2$ *by* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$*. It is easy to see that* R *is a commutative Artinian Kasch ring. Thus* R *is a strongly left Kasch ring by Theorem 3.2.*

We can construct a nonzero projective envelope for a simple left R*-module as follows. Let* $x = 0 \propto (1,0) = \begin{pmatrix} 0 & (1,0) \\ 0 & 0 \end{pmatrix}$ *and, similarly,* $y = 0 \propto (0,1)$*. Note that every simple left* R*-module is isomorphic to* Rx*. It is enough to show that*

$$\varphi : Rx \to R \oplus R; \quad rx \mapsto (rx, ry) \quad (r \in R)$$

is a nonzero projective envelope of Rx*. Indeed,* φ *is a nonzero* R*-homomorphism and* $R \oplus R$ *is projective. For any* $f \in Hom_R(Rx, P)$*, where* P *is a projective* R*-module, suppose that* P *is a direct summand of a free* R*-module* $R^{(I)}$ *with the natural inclusion map* $\alpha : P \to R^{(I)}$ *and the projection* $\pi : R^{(I)} \to P$*. Now, let* $\alpha f(x) = (x_i) \in R^{(I)}$*. There are 3 finite subsets,* J*,* K*, and* L*, of the index set* I *such that*

$$x_i = \begin{cases} x, & i \in J; \\ y, & i \in K; \\ x+y, & i \in L; \\ 0, & else. \end{cases}$$

Now, we can define $g : R \oplus R \to R^{(I)}$ *such that, for any* $s \in R$*, the* i*th component of* $g(s,0)$ *is*

$$\begin{cases} s, & i \in J \cup L; \\ 0, & i \notin J \cup L; \end{cases}$$

and the i*th component of* $g(0,s)$ *is*

$$\begin{cases} s, & i \in K \cup L; \\ 0, & i \notin K \cup L. \end{cases}$$

It follows that $\alpha f = g\varphi$ *and hence* $f = \pi\alpha f = \pi g \varphi$*. This shows that* φ *is a projective preenvelope of* M*.*

Finally, write the endomorphisms of the left R-module $R \oplus R$ as 2×2 matrixes. Assume $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ $End_R(R \oplus R)$ satisfies $\varphi = h\varphi$. Then we have

$$(x, y) = (x, y) \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

which implies

$$x(a + 1_R) + yc = 0 = xb + y(d + 1_R).$$

Consequently, $x(a + 1_R) = yc = xb = y(d + 1_R) = 0$. One can verify that a and d are invertible in R, $b, c \in J(R)$ and h is invertible in $End_R(R \oplus R)$.

Recall that a ring R is said to be *left mininjective* if every R-homomorphism from a minimal left ideal of R to ${}_RR$ extends to one from ${}_RR$ to ${}_RR$ [9, 15]. We note that the ring R in the above example satisfies the conditions in Theorem 3.2. But R is not left mininjective (and hence not QF). In fact, one can verify that $f : Rx \to R$ such that $f(rx) = ry$ (for all $r \in R$) is a well-defined R-homomorphism, which does not extend to ${}_RR$.

To investigate the case that a left mininjective ring is strongly left Kasch we prove a lemma that is of interest in its own right.

Lemma 3.5 *The following are equivalent for a left R-module M:*

(1) M is quasi-injective.

(2) For any submodule $K \leq M$, the inclusion map $K \hookrightarrow M$ is an $addM$-preenvelope of K.

(3) For any essential submodule $K \leq M$, the inclusion map $K \hookrightarrow M$ is an $addM$-preenvelope of K.

(4) For any essential submodule $K \leq M$, the inclusion map $K \hookrightarrow M$ is an $addM$-envelope of K.

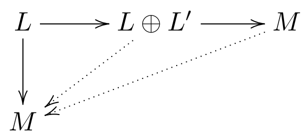
(5) For any submodule $K \leq M$, there exists a direct summand L of M such that K is essential in L and the inclusion map $K \hookrightarrow L$ is an $addM$-preenvelope of K.

(6) For any submodule $K \leq M$, there exists a direct summand L of M such that K is essential in L and the inclusion map $K \hookrightarrow L$ is an $addM$-envelope of K.

Proof (1)$\Leftrightarrow$(2). See [22, Proposition 3.2(1)].

(2)$\Rightarrow$(3) is trivial.

(3)$\Rightarrow$(1). It is well known that, for every submodule L of M, there exists $L' \leq M$ such that $L \cap L' = 0$ and $L \oplus L'$ is essential in M. Therefore, for any homomorphism $f : L \to M$, we have the following commutative diagram:



(3)$\Rightarrow$(4). Let g be an endomorphism of M such that $g(x) = x$ for all $x \in K$. Then g is monic since K is essential in M. This means Img is isomorphic to M. It follows that Img is a direct summand of M since M is quasi-injective (see [14, Proposition 2.1]), but $K \subseteq$ Img and K is essential in M. Therefore, Im$g = M$, i.e. g is an automorphism of M.

(4) ⇒ (3) is clear.

(1) ⇒ (5). Note that a direct summand of a quasi-injective module is quasi-injective and every quasi-injective module is extending (see [14, Proposition 1.17 and 2.1]).

(5) ⇒ (1) is easy.

(5) ⇔ (6) is similar to (3) ⇔ (4). □

Recall that a ring R is said to be *left min-CS* provided that every minimal left ideal is essential in a direct summand of ${}_RR$ [9, 15]. R is called *left C2* if a left ideal of R is a direct summand whenever it is isomorphic to a direct summand of ${}_RR$ (see [14]).

It is easy to see that the inclusion maps $K \hookrightarrow M$ and $K \hookrightarrow L$ in Lemma 3.5 are also $\mathrm{Add}\,M$-envelopes in case K is finitely generated (see [22, Lemma 3.1]). Therefore, by a similar argument to (1) ⇒ (5) in the proof of the above lemma, we have the following proposition.

Proposition 3.6 *Suppose that R is left mininjective left min-CS and left C2. Then R is a left Kasch ring if and only if R is a strongly left Kasch ring. In this case, every simple left R-module M has an essential projective envelope of the form $M \to P$, where P is a direct summand of ${}_RR$.*

We illustrate Proposition 3.6 by the following example.

Example 3.7 *(Camillo's Example [5, 21, 16]) Let R be the ring of polynomials in countably many indeterminates $\{x_1, x_2, \cdots\}$ over the field $\mathbb{Z}_2$ with (i) $x_i^3 = 0$ for all i; (ii) $x_i x_j = 0$ for all $i \neq j$; and (iii) $x_i^2 = x_j^2$ for all i, j. For any $f(x_1, x_2, \cdots) \in R$, if the constant term of f is 0, then it is easy to see that $f^3 = 0$ and $(1+f)(1+f+f^2) = 1$. Thus R is a commutative local ring with unique maximal (left) ideal $J(R) = \{f(x_1, x_2, \cdots) \in R \mid$ the constant term of f is 0$\}$. Moreover, R has a unique minimal (left) ideal $Rx_1^2 = Rx_2^2 = \cdots$, which is essential in R and is isomorphic to $R/J(R)$ as left R-modules. This shows that R is min-CS, Kasch and C2. (Note that every Kasch ring is C2 by [16, Proposition 1.46].) If α is a nonzero R-homomorphism from Rx_1^2 to R, then α is the inclusion map and extends to R obviously. This means that R is left mininjective. Therefore, R satisfies the conditions in Proposition 3.6. The inclusion map $Rx_1^2 \to R$ is the projective envelope of Rx_1^2.*

Now, let $S = R \times R$. Then every left S-module can be written as $U \times V$ with obvious addition and scalar multiplication, where U and V are left R-modules. It is straightforward to verify that S also satisfies the conditions in Proposition 3.6. The inclusion map $Rx_1^2 \times 0 \to R \times 0$ is the projective envelope of $Rx_1^2 \times 0$, where $R \times 0$ is a proper direct summand of ${}_SS$.

To see that the hypothesis "left min-CS and left C2" in Proposition 3.6 is not superfluous, we need the following results.

Proposition 3.8 *The following are equivalent for a ring R:*

(1) R is left SPP.

(2) The dual module of any simple left R-module is finitely generated.

(3) Every homogeneous component of Soc$({}_RR)$ is a finite direct sum of minimal left ideals.

Proof (1) ⇔ (2) has been shown in [12, Theorem 3.1] (see also [22, Theorem 5.1]).

(2)⇒(3). Let $\oplus_{i\in I}Rx_i$ be a homogeneous component of Soc($_RR$), where I is an index set and each Rx_i is a minimal left ideal of R. For any $x \in \{x_i \mid i \in I\}$ and any $f \in \mathrm{Hom}_R(Rx, R)$, there exists a finite subset J of I such that $\mathrm{Im}f \subseteq \oplus_{i\in J}Rx_i$. Let $\{f_j \mid j = 1, 2, \cdots, n\}$ be a generating set of the dual module $(Rx)^*$. Then we have $\oplus_{i\in I}Rx_i = \Sigma_{f\in M^*}\mathrm{Im}f = \Sigma_{j=1}^{n}\mathrm{Im}f_j$. This forces I to be a finite set.

(3)⇒(2). Note that every homomorphism from a simple left R-module M to R is of the form

$$M \xrightarrow{f} \oplus_{i=1}^{n}Rx_i \xrightarrow{g} R$$

where $\oplus_{i=1}^{n}Rx_i$ is a homogeneous component of Soc($_RR$) and $g : \oplus_{i=1}^{n}Rx_i \to R$ is the inclusion map. It follows that $M^* = g\mathrm{Hom}_R(M, \oplus_{i=1}^{n}Rx_i)$ is a finitely generated right R-module. □

Corollary 3.9 *If R is a strongly left Kasch ring then every homogeneous component of Soc($_RR$) is a finite direct sum of minimal left ideals.*

The following example shows that the hypothesis "left min-CS and left C2" in Proposition 3.6 is not superfluous.

Example 3.10 *(Björk's Example [16, 19]). Let F be a field with an isomorphism $F \to \overline{F}$ via $a \mapsto \overline{a}$, where $\overline{F}$ is a subfield of F and F is an infinite dimensional $\overline{F}$-space with basis $\{e_1, e_2, \cdots\}$. Let R be an F-algebra with F-basis $\{1, t\}$ such that $t^2 = 0$ and $at = t\overline{a}$ for all $a \in F$. One can verify that R is a left mininjective left Kasch ring with a unique maximal left ideal Rt. However, $\oplus_{i=1}^{\infty}Rt\overline{e_i}$ = Soc($_RR$) is an infinite direct sum of mutual isomorphic minimal left ideals of R. Thus, R is not a strongly left Kasch ring in view of Corollary 3.9.*

The following result is motivated by [12, Proposition 3.9].

Proposition 3.11 *Let R be a left mininjective ring such that the injective envelope of every simple left R-module is projective. Then R is a strongly left Kasch ring. In this case, the projective envelope of a simple left R-module coincides with its injective envelope.*

Proof It suffices to prove the last statement. Let M be a simple left R-module with injective envelope $E(M)$. Since $E(M)$ is projective, we can make a monomorphism $\alpha : M \to R$ via $M \hookrightarrow E(M) \hookrightarrow R^{(I)} \to R$. By [22, Corollary 3.3(4)], $\alpha : M \to R$ is a projective preenvelope of M since R is left mininjective. It follows that $M \hookrightarrow E(M)$ is a projective preenvelope of M. Finally, by a similar argument to (3)⇒(4) in the proof of Lemma 3.5, we can see that $M \hookrightarrow E(M)$ is a projective envelope of M. □

Remark 3.12 *(1) It is well known that a left R-module U is a cogenerator if and only if, for any simple module $_RV$, U contains a copy of $E(V)$ (see [11, Theorem 19.8]). Thus, if $_RR$ is a cogenerator, then the injective envelope of every simple left R-module is projective. By Proposition 3.11, every left mininjective left cogenerator ring is a strongly left Kasch ring (see also [12, Proposition 3.9]). In particular, every left PF ring (i.e. a left injective left cogenerator) is a strongly left Kasch ring.*

(2) The ring R in Example 3.7 is a left mininjective strongly left Kasch ring, which is neither left self-injective nor a left cogenerator.

In fact, every element in $J(R)$ *is of the form*

$$ax_1^2 + a_1x_1 + a_2x_2 + \cdots \quad (a, a_i \in \mathbb{Z}_2)$$

with finitely many nonzero a_i *s. There is an* R*-homomorphism* $\alpha : J(R) \to R$ *such that* $\alpha(x_i) = x_i(1 + x_i)$ *for each* i*. Assume that* α *extends to* $\beta : R \to R$*, i.e.* $\beta|_{J(R)} = \alpha$*. Let* $\beta(1) = bx_1^2 + b_0 + b_1x_1 + b_2x_2 + \cdots$ *with* b *and each* b_i *in* $\mathbb{Z}_2$*. There exists* i *such that* $b_j = 0$ *for all* $j \geq i$*. So we have*

$$x_i + x_i^2 = \alpha(x_i) = \beta(x_i) = x_i\beta(1) = x_i(bx_1^2 + b_0 + b_1x_1 + \cdots + b_{i-1}x_{i-1}) = b_0x_i,$$

a contradiction. This shows that ${}_RR$ *is not injective.*

Moreover, R *is not a left cogenerator since* R *can not contain a copy of* $E(Rx_1^2)$*, the injective envelope of the simple left* R*-module* Rx_1^2*. Otherwise,* $E(Rx_1^2)$ *is projective and the inclusion map* $Rx_1^2 \hookrightarrow E(Rx_1^2)$ *is a projective envelope of* Rx_1^2*. This implies that* $E(Rx_1^2)$ *is isomorphic to* ${}_RR$ *(since* $Rx_1^2 \hookrightarrow R$ *is also a projective envelope of* Rx_1^2*). But this contradicts the fact that* ${}_RR$ *is not injective.*

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