On the Pollard decomposition method applied to some Jacobi–Sobolev expansions

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Abstract: Let $\{q_n^{(\alpha,\beta)}\}_{n\geq 0}$ be the sequence of polynomials orthonormal with respect to the Sobolev inner product

$$\langle f,g \rangle_S := \int_{-1}^{1} f(x)g(x)w^{(\alpha,\beta)}(x)dx + \int_{-1}^{1} f'(x)g'(x)w^{(\alpha+1,\beta+1)}(x)dx,$$

where $w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$, $x \in [-1,1]$ and $\alpha, \beta > -1$. This paper explores the convergence in the $W^{1,p}\left((-1,1),(w^{(\alpha,\beta)},w^{(\alpha+1,\beta+1)})\right)$ norm of the Fourier expansion in terms of $\{q_n^{(\alpha,\beta)}\}_{n\geq 0}$ with $1 < p < \infty$, using the Pollard decomposition method. Numerical examples concerning the comparison between the approximation of functions in L^2 norm and $W^{1,2}$ norm are also presented.

Key words: Sobolev orthogonal polynomials, weighted Sobolev spaces, Fourier expansions, Sobolev–Fourier expansions

1. Introduction

The study of the behavior of the L^p-convergence of orthogonal series of polynomials began in 1946 and it was due to Pollard [25], who proposed the use of methods from functional analysis in order to give sufficient and necessary conditions that guarantee the L^p-convergence of an orthogonal series of polynomials given. However, this first work gave evidence to think that such tools of functional analysis did not seem be a general method in order to determine when an orthonormal basis of polynomials is also a basis in L^p. Therefore, Pollard decided to study the behavior of the L^p-convergence of orthogonal series of polynomials separately. He began with the Legendre polynomials [26], showing that the sequence $\{p_n\}_{n\geq 0}$ of orthonormal Legendre polynomials is a basis of $L^p(-1,1)$, whenever $\frac{4}{3} < p < 4$ (this interval is the best possible, cf. [23]) and it is not a basis of $L^p(-1,1)$ when $p < \frac{4}{3}$ or $p > 4$. We can summarize the first part of Pollard's proof as follows.

Let $\{p_n\}_{n\geq 0}$ be the sequence of orthonormal Legendre polynomials, $f \in L^p(-1,1)$ for $1 \leq p < \infty$, and the n-th Fourier partial sum $S_n(f,x)$, given by

$$S_n(f,x) = \sum_{k=0}^{n} \hat{f}(k)p_k(x), \quad \text{where} \quad \hat{f}(k) = \int_{-1}^{1} f(y)p_k(y)dy. \tag{1.1}$$

Pollard was interested in determining the values of p for which the following limit holds.

$$\lim_{n \to \infty} \int_{-1}^{1} |f(x) - S_n(f, x)|^p \, dx = 0. \tag{1.2}$$

In order to determine such values of p, Pollard took into account the following facts:

(A) The n-th Fourier partial sum (1.1) induces a linear operator $S_n : L^p(-1, 1) \to L^p(-1, 1)$, given by

$$(S_n f)(x) := S_n(f, x), \quad \text{for } x \in (-1, 1),$$

and the validity of (1.2) is equivalent to the uniform boundedness of the operator S_n. Furthermore, the operator S_n has the following integral representation:

$$(S_n f)(x) = \int_{-1}^{1} f(y) K_n(x, y) dy, \tag{1.3}$$

where $K_n(x, y)$ is the n-th Dirichlet kernel, given by

$$K_n(x, y) = \sum_{k=0}^{n} p_k(x) p_k(y). \tag{1.4}$$

(B) The Christoffel–Darboux formula allows us to express the n-th Dirichlet kernel as

$$K_n(x, y) = u_n \left(\frac{p_{n+1}(x) p_n(y) - p_n(x) p_{n+1}(y)}{x - y} \right), \tag{1.5}$$

where $u_n = \frac{k_n}{k_{n+1}}$, k_n being the leading coefficient of $p_n(x)$.

(C) Using the 3-term recurrence formula satisfied by the Legendre polynomials $\{P_n\}_{n \geq 0}$ and (1.5), the n-th Dirichlet kernel is

$$K_n(x, y) = \alpha_n T_1(n, x, y) + \alpha_n T_2(n, x, y) - T_3(n, x, y), \tag{1.6}$$

where the sequence $\{\alpha_n\} = \{\frac{n+2}{4n+6}\}$, and

$$
\begin{aligned}
T_1(n, x, y) &:= (n+1) P_{n+1}(x) \left(\frac{P_n(y) - P_{n+2}(y)}{x - y} \right), \\
T_2(n, x, y) &:= -T_1(n, y, x), \\
T_3(n, x, y) &:= \left(\frac{n+1}{2} \right) P_{n+1}(x) P_{n+1}(y).
\end{aligned}
$$

(D) In order to obtain the uniform boundedness of the operator S_n it suffices to show the uniform boundedness of the operators $W_{j,n} : L^p(-1, 1) \to L^p(-1, 1)$, given by

$$(W_{j,n} f)(x) := \int_{-1}^{1} f(y) T_j(n, x, y) dy, \quad j = 1, 2, 3. \tag{1.7}$$

(E) The Legendre polynomials satisfy the following pointwise estimates (see [32]):

$$|P_{n+1}(x)| \leq C(1-x^2)^{-1/4}, \tag{1.8}$$

$$|P_{n+2}(x) - P_n(x)| \leq C(1-x^2)^{1/4}, \tag{1.9}$$

for $x \in (-1, 1)$. The operators $W_{j,n}$, $j = 1, 2$ can be expressed in terms of the Hilbert transform H as follows:

$$(W_{1,n}f)(x) = P_{n+1}(x)H((P_n - P_{n+2})f; x), \tag{1.10}$$

$$(W_{2,n}f)(x) = -(P_n(x) - P_{n+2}(x))H((P_{n+1})f; x). \tag{1.11}$$

Finally, using the pointwise estimates (1.8) and (1.9) and the boundedness of the Hilbert transform with certain weights, Pollard obtained the values of p for which the operators $W_{j,n}$ are uniformly bounded in the L^p norm or, equivalently, the values of p for which (1.2) has a sense.

Since then, the L^p-convergence of orthogonal series of polynomials has been investigated by many authors in various contexts and forms. See, for instance, the subsequent works of Pollard himself [27, 28, 29], Wing [35], Newman and Rudin [23], Muckenhoupth [21], Badkov [2], Meaney [19], Máté et al. [18], and Varona [33, 34] and the references therein, and, more recently, Stempak [31] and the remarkable works of Mastroianni and Notarangelo [16, 17] about the L^p-convergence of Fourier sums with exponential weights in $(-1, 1)$ for $1 \leq p \leq \infty$.

Although the corresponding study of Fourier series of orthogonal polynomials in the setting of Sobolev orthogonality is most recent, in the last decades it has attracted considerable attention, mainly in subjects concerning the comparison with the standard theory and the search of algorithms for computing Fourier–Sobolev series in terms of Sobolev orthogonal polynomials [8], the study of asymptotic properties of the Fourier expansions of orthogonal polynomials with certain (discrete or nondiscrete) Sobolev inner products (cf. [6, 10, 12, 13]), the divergence of certain Legendre–Sobolev series [5], Cohen-type inequalities for Laguerre–Sobolev expansions [7, 24], and $W^{1,p}$-convergence for $1 < p < \infty$ [15]. Despite these efforts, most of the progress attained is for the special type of Sobolev inner products, but, to the best of our knowledge, there still is a considerable number of open problems [11, 14].

The aim of the present paper is to follow steps (A) through (E) of Pollard (or his decomposition method) in order to study the behavior of some operators associated with the Fourier–Sobolev expansions associated with the following Jacobi–Sobolev inner product:

$$\langle f, g \rangle_S := \int_{-1}^{1} f(x)g(x)w^{(\alpha,\beta)}(x)dx + \int_{-1}^{1} f'(x)g'(x)w^{(\alpha+1,\beta+1)}(x)dx, \tag{1.12}$$

where $w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$, $x \in [-1, 1]$ and $\alpha, \beta > -1$.

Notice that this approach makes sense because the orthonormal polynomials with respect to (1.12) are essentially Jacobi polynomials and hence they satisfy 3-term recurrence relations and similar pointwise estimates like (1.8) and (1.9).

The structure of the paper is as follows. Section 2 introduces the notation as well as some basic background to be needed further on. Section 3 is focused on the study of the $W^{1,p}$-convergence of some operators related to the Legendre–Sobolev expansions, i.e. $\alpha = \beta = 0$ in (1.12) using the Pollard decomposition method. We also

show some numerical experiments about the comparison between the approximation of functions in L^2 norm and $W^{1,2}$ norm.

2. Previous definitions and notations

The Jacobi polynomials are classical orthogonal polynomials, including several families of orthogonal polynomials like Chebyshev, Legendre, and Gegenbauer (ultraspherical) polynomials. There are many equivalent definitions of $P_n^{(\alpha,\beta)}$, the Jacobi polynomials of the degree n and order (α, β), with $\alpha, \beta > -1$, and one of them is as follows (the Rodrigues formula).

$$P_n^{(\alpha,\beta)}(x) = (1-x)^{-\alpha}(1+x)^{-\beta}\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\left\{(1-x)^{\alpha+n}(1+x)^{n+\beta}\right\}, \quad x \in (-1,1), \tag{2.13}$$

and the value in the end points of the interval $[-1,1]$ is given by

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}, \quad P_n^{(\alpha,\beta)}(-1) = (-1)^n\binom{n+\beta}{n}.$$

Historically, one of the seminal reasons to study Jacobi polynomials $P_n^{(\alpha,\beta)}$ is the fact that their zeros $x_{n,k}^{(\alpha,\beta)}$, $k = 1, \ldots, n$, have a very interesting electrostatic interpretation. Indeed, they are the equilibrium points of n unit charges in $(-1,1)$ in the field generated by charges $\frac{\alpha+1}{2}$ at 1 and $\frac{\beta+1}{2}$ at -1, where the charges repel each other according to the interaction law under a logarithmic potential. Furthermore, the Jacobi polynomials constitute a complete orthogonal system in $L^2([-1,1], w^{(\alpha,\beta)}(x)dx)$ and an explicit representation for them is given by hypergeometric functions as follows:

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(n+\alpha+\beta+1)_k(k+\alpha+1)_{n-k}\left(\frac{x-1}{2}\right)^k, \tag{2.14}$$

where $(a)_k$, $k > 0$, and $(a)_0 = 1$ is the Pochhammer symbol, (see [32], p. 62, or [1], p. 7).

On the other hand, these polynomials satisfy the following properties:

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x), \tag{2.15}$$

i.e. they are either even or odd functions when $\alpha = \beta$ according to the parity of their degrees.

A 3-term recurrence relation ([32], p. 71) holds

$$\kappa_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(x) = (2n+\alpha+\beta-1)(\nu_n^{(\alpha,\beta)}x - \alpha^2 - \beta^2)P_{n-1}^{(\alpha,\beta)}(x) - \gamma_n^{(\alpha,\beta)}P_{n-2}^{(\alpha,\beta)}(x), \quad n \geq 2, \tag{2.16}$$

where $\kappa_n^{(\alpha,\beta)} = 2n(n+\alpha+\beta)(2n+\alpha+\beta-2)$, $\gamma_n^{(\alpha,\beta)} = 2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)$, $\nu_n^{(\alpha,\beta)} = (2n+\alpha+\beta)(2n+\alpha+\beta-2)$ and

$$P_0^{(\alpha,\beta)}(x) = 1, \quad P_1^{(\alpha,\beta)}(x) = \frac{\alpha+\beta+2}{2}x + \frac{\alpha-\beta}{2}.$$

The derivatives of the Jacobi polynomials are, up to a constant factor, Jacobi polynomials with a unit shift in the parameters. Indeed, from (2.14) we get

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{2.17}$$

Also, the importance of Jacobi polynomials follows from the fact that $P_n^{(\alpha,\beta)}$ is the only polynomial solution (up to a constant factor [32, p. 61, Theorem 4.2.2]) of the homogeneous, second-order differential equation

$$(1-x^2)y''(x) + (\beta - \alpha - (\alpha + \beta + 2)x)y'(x) + n(n + \alpha + \beta + 1)y(x) = 0. \tag{2.18}$$

When $\alpha > -\frac{1}{2}$ and $\beta \leq \alpha$, the maximum of $P_n^{(\alpha,\beta)}(x)$ in $[-1,1]$ can be explicitly computed (see [32], p. 168). Indeed,

$$\sup_{-1\leq x\leq 1}\left|P_n^{(\alpha,\beta)}(x)\right| = P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!}, \quad \alpha > -\frac{1}{2}, \quad \beta \leq \alpha. \tag{2.19}$$

The square of the L^2 norm of $P_n^{(\alpha,\beta)}$ with respect to the measure $w^{(\alpha,\beta)}(x)dx$ in $[-1,1]$ is given by

$$
\begin{aligned}
h_n^{(\alpha,\beta)} &:= \int_{-1}^1 \left(P_n^{(\alpha,\beta)}(x)\right)^2 w^{(\alpha,\beta)}(x)dx \\
&= \frac{2^{\alpha+\beta+1}}{(2n+\alpha+\beta+1)} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}.
\end{aligned}
\tag{2.20}
$$

For n large enough, the Jacobi polynomials behave like Bessel functions, according to the well-known Mehler–Heine type of asymptotic formula:

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right) = \left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z). \tag{2.21}$$

In the following, $C, C_1, C_2, \ldots$ will denote positive constants, independent of n, x, y and the function f and its derivatives. The same symbol does not necessarily denote the same constant in different occurrences. We write $C(p)$ in order to show that C depends on p.

Let $\{p_n^{(\alpha,\beta)}\}_{n\geq 0}$ be the sequence of orthonormal Jacobi polynomials. It is well known that the following estimates are satisfied (see [32], p. 196, or [22, 34]).

Theorem 2.1 *There exists a positive constant C, such that*

$$|p_n^{(\alpha,\beta)}(x)| \leq C\left(1-x+\frac{1}{n^2}\right)^{-(\alpha/2+1/4)}\left(1+x+\frac{1}{n^2}\right)^{-(\beta/2+1/4)}, \quad \text{for } \alpha, \beta > -1. \tag{2.22}$$

If $\alpha = \beta = 0$, then there exists a positive constant C, such that

$$|p_n'(x)| \leq C(n+1)(1-x^2)^{-3/4}. \tag{2.23}$$

Theorem 2.2 *Let $\{P_n\}_{n\geq 0}$ be the sequence of orthogonal Legendre polynomials; then there exist constants $C_1, C_2, C_3 > 0$ such that*

$$(n+1)^{1/2}|P_{n+1}(x)| \leq C_1(1-x^2)^{-1/4}, \tag{2.24}$$

$$(n+1)^{1/2}|P_{n+2}(x) - P_n(x)| \leq C_2(1-x^2)^{1/4}, \tag{2.25}$$

$$(n+1)^{-1/2}|P_{n+2}'(x) - P_n'(x)| \leq C_3(1-x^2)^{-1/4}, \tag{2.26}$$

as well as

$$|P_n'(x)| \leq n(n+1)/2. \tag{2.27}$$

When the Jacobi–Sobolev inner product (1.12) is considered, it is very easy to verify that the orthonormal Jacobi polynomials $\{p_n^{(\alpha,\beta)}\}_{n\geq 0}$ satisfy the following Sobolev orthogonality relationship:

$$\langle p_n^{(\alpha,\beta)}, p_m^{(\alpha,\beta)}\rangle_S = (1 + n(n + \alpha + \beta + 1))\delta_{n\,m}, \quad n, m \geq 0. \tag{2.28}$$

We can then easily deduce that the polynomials $\{q_n^{(\alpha,\beta)}\}_{n\geq 0}$, defined by

$$q_n^{(\alpha,\beta)}(x) = (1 + n(n + \alpha + \beta + 1))^{-1/2} p_n^{(\alpha,\beta)}(x), \quad n \geq 0, \quad \alpha, \beta > -1, \tag{2.29}$$

are orthonormal with respect to the Jacobi–Sobolev inner product (1.12), and they form a complete orthonormal system in the Sobolev space (cf. [8]).

$$W^{1,2}((-1,1)) = \left\{ f : (-1,1) \to \mathbb{R} : f \in AC_{loc}(-1,1), \; f, \sqrt{1-x^2} f' \in L^2((-1,1), w^{(\alpha,\beta)} dx) \right\}.$$

When $\alpha = \beta = 0$, it is well known that

$$\begin{aligned} p_n(x) &= p_n^{(0,0)}(x) = \sqrt{\frac{2n+1}{2}} \; P_n^{(0,0)}(x) \\[2mm] &= \sqrt{\frac{2n+1}{2}} \sum_{j=0}^{[n/2]} \frac{(-1)^j (2n-2j)!}{2^n j!(n-j)!(n-2j)!} x^{n-2j}, \quad n \geq 0, \end{aligned}$$

and from (2.28) the polynomials

$$q_n^{(0,0)}(x) = q_n(x) = \sqrt{\frac{2n+1}{2n(n+1)+2}} \sum_{j=0}^{[n/2]} \frac{(-1)^j (2n-2j)!}{2^n j!(n-j)!(n-2j)!} x^{n-2j}, \quad n \geq 0, \tag{2.30}$$

are orthonormal with respect to the Legendre–Sobolev inner product

$$\langle p, q \rangle_S = \int_{-1}^{1} p(x)q(x)dx + \int_{-1}^{1} p'(x)q'(x)(1-x^2)dx. \tag{2.31}$$

Definition 2.1 *Let* $1 < p < \infty$. *If* (a,b) *is a fixed interval with* $-\infty \leq a < b \leq \infty$, *we say that a weight function* $w : (a,b) \to \mathbb{R}$ *belongs to the class* $A_p((a,b))$, *if there exists a constant* $C > 0$ *such that*

$$\left(\int_I \omega(x)dx \right) \left(\int_I \omega^{-1/(p-1)}(x)dx \right)^{p-1} \leq C|I|^p, \tag{2.32}$$

for every subinterval $I \subseteq (a,b)$, *with* C *independent of* I.

Definition 2.2 *Let* w *be a weight function on* $[-1,1]$ *and* $1 < p < \infty$. *The finite Hilbert transform,* H, *associates to each function* $f \in L^p((-1,1), w)$ *the function*

$$H(f;x) := PV \int_{-1}^{1} \frac{f(y)}{x-y} dy = PV \int_{-1}^{1} \frac{f(x-t)}{t} dt, \quad x \in (-1,1),$$

where the above integrals are considered in Cauchy's principal value sense [30].

Theorem 2.3 *Let $1 < p < \infty$. The following statements are equivalent:*

(i) $w \in A_p((a,b))$.

(ii) *The operator $H : L^p((a,b), w) \to L^p((a,b), w)$ is bounded, i.e. there exists $C(p) > 0$, such that*

$$\|Hf\|_{L^p((a,b),w)} = \left(\int_a^b \left| PV \int_a^b \frac{f(y)}{x-y} dy \right|^p w(x) dx \right)^{1/p} \leq C(p) \|f\|_{L^p((a,b),w)}.$$

Theorem 2.4 *If $1 < p < \infty$ and $f \in L^p((a,b))$ is a differentiable function, then $H(f; \cdot)$ is also differentiable, and*

$$\frac{d}{dx} H(f; \cdot) = H(f'; \cdot). \tag{2.33}$$

Furthermore, in the case $a = -1$, $b = 1$, it is possible to add the assumption $f(-1) = f(1) = 0$.

For details of the proof of the above result, see [30] and the proof of Theorem 3.2 in [9] or [20].

As an immediate consequence of the 2 previous theorems, we have:

Corollary 2.1 *Let $1 < p < \infty$ and $w \in A_p((a,b))$. For $f \in L^p((a,b), w)$ suppose that $f', \frac{d}{dx} H(f; \cdot) \in L^p((a,b), w)$; then there exists $C(p) > 0$ such that*

$$\left\| \frac{d}{dx} H(f; \cdot) \right\|_{L^p((a,b),w)} \leq C(p) \|f'\|_{L^p((a,b),w)}. \tag{2.34}$$

3. $W^{1,p}$-convergence of some Legendre–Sobolev expansions via the Pollard decomposition method

In the following, we consider the Fourier expansions in terms of Legendre–Sobolev polynomials $\{q_n\}_{n \geq 0}$ given in (2.30). Notice that these polynomials are essentially Legendre polynomials and therefore they satisfy a similar 3-term recurrence relation like (2.16), but with some different parameters according to its normalization. Thus, the Pollard decomposition method could be applied in this setting.

In fact, we will show that some elements of the Pollard scheme can be used in order to study the $W^{1,p}$-convergence of certain operators related to the orthogonal series of Legendre–Sobolev polynomials. However, in our case, it will be not enough to give a representation of the corresponding n-th Dirichlet–Sobolev kernel in terms of other bounded operators, because such a kernel does not satisfy a Christoffel–Darboux formula like (1.5).

Now, we summarize some properties of the polynomials $\{q_n\}_{n \geq 0}$.

Lemma 3.1 *Let $\{q_n\}_{n \geq 0}$ be the sequence of orthonormal polynomials with respect to the inner product (2.31). Then,*

(i) *the polynomials $\{q_n\}_{n \geq 0}$ satisfy the following 3-term recurrence relation:*

$$xq_n(x) = \vartheta_{n+1} q_{n+1}(x) + v_n q_{n-1}(x), \quad n \geq 1, \tag{3.35}$$

where

$$\vartheta_{n+1} = \left(\frac{(n+1)^2}{(2n+1)(2n+3)}\right)^{1/2} \left(\frac{n^2+3n+3}{n^2+n+1}\right)^{1/2}, \quad n \geq 1,$$

$$\upsilon_n = \left(\frac{n^2}{(2n+1)(2n-1)}\right)^{1/2} \left(\frac{n^2-n+1}{n^2+n+1}\right)^{1/2}, \quad n \geq 1,$$

$q_0(x) = \frac{1}{\sqrt{2}}, \quad q_1(x) = \frac{\sqrt{2}}{2}x.$

(ii) *Alternatively, they satisfy the following 3-term recurrence relation:*

$$xq_n(x) = A_n q_{n+1}(x) + B_n q_{n-1}(x), \quad n \geq 1, \tag{3.36}$$

where

$$A_n = \langle xq_n, q_{n+1}\rangle_S, \quad B_n = \langle xq_n, q_{n-1}\rangle_S, \quad q_{-1}(x) = 0 \quad and \quad q_0(x) = 1.$$

Proof   The relation (3.35) is a straightforward consequence of (2.16). The relation (3.36) follows from the properties (2.15) and (2.17), for $\alpha = \beta = 0$, and the orthogonality with respect to the standard inner product of the orthonormal Legendre polynomials $\{p_n\}_{n\geq 0}$.   □

The operator $\mathcal{S}_n$ associated with the n-th Fourier–Sobolev partial sum has the following integral representation:

$$(\mathcal{S}_n f)(x) = \mathcal{S}_n(f,x) = \int_{-1}^{1} \mathcal{K}_n^{(0,0)}(x,y)f(y)dy + \int_{-1}^{1} \mathcal{K}_n^{(0,1)}(x,y)f'(y)(1-y^2)dy, \tag{3.37}$$

where $\mathcal{K}_n^{(0,0)}(x,y)$ is the n-th Dirichlet–Sobolev kernel, given by

$$\mathcal{K}_n^{(0,0)}(x,y) = \mathcal{K}_n(x,y) = \sum_{k=0}^{n} q_k(x)q_k(y), \quad x,y \in [-1,1], \tag{3.38}$$

and $\mathcal{K}_n^{(0,1)}(x,y) = \frac{\partial}{\partial y}\mathcal{K}_n(x,y) = \sum_{k=1}^{n} q_k(x)q_k'(y)$.

Since the integral representation (3.37) takes place, we need to find alternative expressions for the kernels $\mathcal{K}_n^{(0,0)}(x,y)$ and $\mathcal{K}_n^{(0,1)}(x,y)$. Furthermore, the recurrence relation (3.35) does not allow us to deduce, a priori, a Christoffel–Darboux formula for $\mathcal{K}_n^{(0,0)}(x,y)$. Similarly, in the case of the recurrence relation (3.36), the fact that the multiplication operator M_x is not symmetric with respect to the Sobolev inner product will be the main obstacle in order to find a Christoffel–Darboux formula for $\mathcal{K}_n^{(0,0)}(x,y)$.

In such a way, using (1.4), (1.5), and (3.35), some alternative expressions for the kernels $\mathcal{K}_n^{(0,0)}(x,y)$ and $\mathcal{K}_n^{(0,1)}(x,y)$ can be deduced as follows.

Lemma 3.2 *Let $\{q_n\}_{n\geq 0}$ be the sequence of orthonormal polynomials with respect to the inner product (2.31) and $\mathcal{K}_n^{(0,0)}(x,y)$, $\mathcal{K}_n^{(0,1)}(x,y)$ the kernels of the integral representation (3.37), respectively. Then,*

(i) the kernel $\mathcal{K}_n^{(0,0)}(x,y)$ admits the following Christoffel–Darboux formula type:

$$(x-y)\mathcal{K}_n^{(0,0)}(x,y) = \vartheta_0 q_1(x)q_0(y) - \vartheta_0 q_0(x)q_1(y) + \cdots + \vartheta_n q_{n+1}(x)q_n(y) \tag{3.39}$$
$$-\vartheta_n q_n(x)q_{n+1}(y) + v_0 q_0(x)q_0(y) - v_0 q_1(x)q_0(y)$$
$$+ \cdots + v_{n-1}q_{n-1}(x)q_n(y) - v_{n-1}q_n(x)q_{n-1}(y),$$

$$\mathcal{K}_n^{(0,0)}(x,y) = u_n\left(\frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y}\right) - \mathcal{R}_n^{(0,0)}(x,y), \tag{3.40}$$

where $\mathcal{R}_n^{(0,0)}(x,y) = \sum_{k=1}^n (k^2+k)q_k(x)q_k(y)$.

(ii) The kernel $\mathcal{K}_n^{(0,1)}(x,y)$ can be expressed as

$$\mathcal{K}_n^{(0,1)}(x,y) = u_n\left(\frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{(x-y)^2}\right) \tag{3.41}$$
$$+ u_n\left(\frac{p_{n+1}(x)p_n'(y) - p_n(x)p_{n+1}'(y)}{x-y}\right) - \mathcal{R}_n^{(0,1)}(x,y).$$

(iii) If the sequence α_n and the operators $T_j(n,x,y)$, $j = 1,2,3$, are defined by (1.6), then the kernel $\mathcal{K}_n^{(0,0)}(x,y)$ satisfies the following Pollard decomposition type:

$$\mathcal{K}_n^{(0,0)}(x,y) = (\alpha_n T_1(n,x,y) + \alpha_n T_2(n,x,y) - T_3(n,x,y)) - \mathcal{R}_n^{(0,0)}(x,y). \tag{3.42}$$

(iv) There exist constants $C_1, C_2, C_3 > 0$, such that

$$\left|\mathcal{R}_n^{(0,0)}(x,y)\right| \leq C_1(n+1)(1-x^2)^{-1/4}(1-y^2)^{-1/4}, \tag{3.43}$$

$$\left|\mathcal{R}_n^{(0,1)}(x,y)\right| \leq C_2 n(n+1)^2(1-x^2)^{-1/4}(1-y^2)^{-3/4}, \tag{3.44}$$

$$\left|\mathcal{R}_n^{(1,0)}(x,y)\right| \leq C_3 n(n+1)^2(1-x^2)^{-3/4}(1-y^2)^{-1/4}. \tag{3.45}$$

Proof

(i) (3.39) is an immediate consequence of (1.6) and (3.35). Taking $\alpha = \beta = 0$ in (2.29), the n-th Dirichlet kernel $K_n(x,y)$ can be written in terms of the polynomials $q_n(x)$ as follows:

$$K_n(x,y) = \sum_{k=0}^n (1 + k(k+1))q_k(x)q_k(y),$$

or, equivalently,

$$K_n(x,y) = \mathcal{K}_n^{(0,0)}(x,y) + \sum_{k=1}^n (k^2+k)q_k(x)q_k(y).$$

Therefore, using the Christoffel–Darboux formula (1.5), the relation (3.40) is obtained.

(ii) It is suffices to take the partial derivative with respect to y in (3.40).

(iii) It is an immediate consequence of (1.6) and (3.40).

(iv) Taking $\alpha = \beta = 0$ in (2.29) and using the pointwise estimates (2.22) and (2.23), the estimates (3.43), (3.44), and (3.45) are deduced.

$\square$

For $1 < p < \infty$ let us consider the weighted Sobolev space $W^{1,p}\left((-1,1),(1,1-x^2)\right)$ given by

$$W^{1,p}\left((-1,1),(1,1-x^2)\right) = \left\{ f : (-1,1) \to \mathbb{R} : f \in AC_{loc}(-1,1), \, f, (1-x^2)^{1/p} f' \in L^p((-1,1)) \right\}.$$

The Pollard type decomposition (3.42) yields

(3.46)

$$
\begin{aligned}
\mathcal{S}_n(f,x) \;=\; & \alpha_n(n+1)P_{n+1}(x)\int_{-1}^{1}\left(\frac{P_n(y)-P_{n+2}(y)}{x-y}\right)f(y)dy \\[2mm]
& +\alpha_n(n+1)\left(P_{n+2}(x)-P_n(x)\right)\int_{-1}^{1}\frac{P_{n+1}(y)}{x-y}f(y)dy \\[2mm]
& -\left(\frac{n+1}{2}\right)P_{n+1}(x)\int_{-1}^{1}P_{n+1}(y)f(y)dy - \int_{-1}^{1}\mathcal{R}_n^{(0,0)}(x,y)f(y)dy \\[2mm]
& +\int_{-1}^{1}\mathcal{K}_n^{(0,1)}(x,y)f'(y)(1-y^2)dy.
\end{aligned}
$$

The integrals in the first 2 terms on the right-hand side of (3.46) exist almost everywhere in Cauchy's principal value sense. However, this property does not hold for the last one in (3.46) when we consider (3.41). In order to avoid this inconvenience and according to [3], we split the square $(-1,1)\times(-1,1)$ in 7 regions:

$$
\begin{aligned}
A_0(\varepsilon) \;&=\; \{(x,y) : |x-y| < \varepsilon\}, \\
A_1(\varepsilon) \;&=\; \{(x,y) : 1-\varepsilon \le x < 1, \, -1 < y \le x+\varepsilon-1\}, \\
A_2(\varepsilon) \;&=\; \{(x,y) : 1-\varepsilon \le x \le \varepsilon, \, x+1-\varepsilon \le y < 1\}, \\
A_3(\varepsilon) \;&=\; \{(x,y) : -1 < x \le \varepsilon-1, \, x+1-\varepsilon \le y < 1\}, \\
A_4(\varepsilon) \;&=\; \{(x,y) : -\varepsilon \le x \le \varepsilon-1, \, -1 < y \le x+\varepsilon-1\}, \\
A_5(\varepsilon) \;&=\; \{(x,y) : -1+\varepsilon < x < 1-\varepsilon, \, -1 < y \le x+\varepsilon-1\}, \\
A_6(\varepsilon) \;&=\; \{(x,y) : -1+\varepsilon < x < 1-\varepsilon, \, x+1-\varepsilon \le y < 1\},
\end{aligned}
$$

where $0 < \varepsilon < 1$. So, for $x \ne y$ we have

$$\left|\mathcal{K}_n^{(0,0)}(x,y)\right| = \sum_{k=0}^{6}\left|K_n(x,y)-\mathcal{R}_n^{(0,0)}(x,y)\right|\chi_{A_k(\varepsilon)}(x,y).$$

Then, as a consequence of Lemma 3.2 and the L^p-boundedness of the operators T_j, $j = 1, 2, 3$, see [26], for $4/3 < p < 2$, the operators defined by

$$\mathcal{T}_k(f, x) = \int_{-1}^{1} f(y) \left| \mathcal{K}_n^{(0,0)}(x, y) \right| \chi_{A_k(\varepsilon)}(x, y) dy,$$

$$\mathcal{W}_k(f', x) = \int_{-1}^{1} f'(y) \left| \mathcal{K}_n^{(0,1)}(x, y) \right| \chi_{A_k(\varepsilon)}(x, y)(1 - y^2) dy,$$

for $k = 1, 2, \ldots, 6$ are bounded in $L^p((-1, 1))$ and $L^p((-1, 1), 1 - x^2)$, respectively. Therefore, there exists a positive constant $C(p)$, such that

$$\int_{I_\varepsilon} |\mathcal{S}_n(f, x)|^p \, dx \le C(p) \, \|f\|_{W^{1,p}(I_\varepsilon, (1, 1 - x^2))}^p, \tag{3.47}$$

where $I_\varepsilon = \{x \in (-1, 1) : (x, y) \in (-1, 1) \times (-1, 1) \setminus A_0(\varepsilon)\}$. Similarly, for $k = 1, 2, \ldots, 6$ we define the operators $\mathcal{L}_k$ and $\mathcal{H}_k(f', x)$ as follows:

$$\mathcal{L}_k(f, x) = \int_{-1}^{1} f(y) \left| \mathcal{K}_n^{(1,0)}(x, y) \right| \chi_{A_k(\varepsilon)}(x, y) dy,$$

$$\mathcal{H}_k(f', x) = \int_{-1}^{1} f'(y) \left| \mathcal{K}_n^{(1,1)}(x, y) \right| \chi_{A_k(\varepsilon)}(x, y)(1 - y^2) dy.$$

Then, using Lemma 3.2, the L^p-boundedness of the operators T_j, $j = 1, 2, 3$, and the following integral representation,

$$\frac{d}{dx} \mathcal{S}_n(f, x) = \int_{-1}^{1} \mathcal{K}_n^{(1,0)}(x, y) f(y) dy + \int_{-1}^{1} \mathcal{K}_n^{(1,1)}(x, y) f'(y)(1 - y^2) dy, \tag{3.48}$$

we obtain the L^p-boundedness (resp. $L^p((-1, 1), 1 - x^2)$-boundedness) of the operators $\mathcal{L}_k(f, x)$ (resp. $\mathcal{H}_k(f', x)$). As a consequence, there exists a positive constant $C(p)$, such that

$$\int_{I_\varepsilon} \left| \frac{d}{dx} \mathcal{S}_n(f, x) \right|^p (1 - x^2) dx \le C(p) \, \|f\|_{W^{1,p}(I_\varepsilon, (1, 1 - x^2))}^p. \tag{3.49}$$

Finally, from the inequalities (3.47) and (3.50), the convergence of the Fourier–Sobolev in the $W^{1,p}(I_\varepsilon, (1, 1 - x^2))$ norm is deduced.

The above remarks can be summarized in the following 2 theorems.

Theorem 3.1 *Let $1 < p < \infty$, $0 < \varepsilon < 1$ and let $\mathcal{S}_n(f, x)$ be the n-th partial sum of the Legendre–Sobolev Fourier expansion of $f(x)$ with respect to the orthonormal polynomials $\{q_n\}_{n \ge 0}$. Assume that $2 \le p < 8/3$. Then there exists a positive constant $C(p)$, such that*

$$\|\mathcal{S}_n f\|_{W^{1,p}(I_\varepsilon, (1, 1 - x^2))} \le C(p) \, \|f\|_{W^{1,p}(I_\varepsilon, (1, 1 - x^2))}. \tag{3.50}$$

Theorem 3.2 *Assume that* $\alpha, \beta \geq -1/2$, $1 < p < \infty$, $0 < \varepsilon < 1$, *and let* $\mathcal{S}_n^{\alpha,\beta}(f, x)$ *denote the* n-*th partial sum of the Fourier expansion of* $f(x)$ *in terms of* $\{q_n^{(\alpha,\beta)}\}_{n \geq 0}$. *Assume that* $2 < p < 4 \min \left\{ \frac{\alpha+2}{2\alpha+3}, \frac{\beta+2}{2\beta+3} \right\}$. *Then there exists a positive constant* $C(p)$, *such that*

$$\|\mathcal{S}_n^{\alpha,\beta} f\|_{W^{1,p}\left(I_\varepsilon, (w^{(\alpha,\beta)}, w^{(\alpha+1,\beta+1)})\right)} \leq C(p) \|f\|_{W^{1,p}\left(I_\varepsilon, (w^{(\alpha,\beta)}, w^{(\alpha+1,\beta+1)})\right)}. \tag{3.51}$$

Corollary 3.1 *If* $\alpha, \beta \geq -1/2$ *and* $2 \leq p < 4 \min \left\{ \frac{\alpha+2}{2\alpha+3}, \frac{\beta+2}{2\beta+3} \right\}$, *then for* $0 < \varepsilon < 1$ *small enough, the Fourier expansion in terms of* $\{q_n^{(\alpha,\beta)}\}_{n \geq 0}$ *converges in the* $W^{1,p}\left(I_\varepsilon, (w^{(\alpha,\beta)}, w^{(\alpha+1,\beta+1)})\right)$ *norm.*

We finish this section by providing 2 illustrative numerical examples (with the help of MAPLE) about the approximation errors $\|f - S_n(f)\|_{L^2}$ and $\|f - \mathcal{S}_n(f)\|_{W^{1,2}}$, respectively, with $f \in W^{1,2}\left((-1,1), (1, 1 - x^2)\right)$. Since the polynomials $\{q_n\}_{n \geq 0}$ are, up to constant factor, orthonormal Legendre polynomials, it is not a surprise that such polynomials adopt the same behavior of the last ones when we wish to approximate a function f by its Fourier–Sobolev partial sum and, simultaneously, to approximate its derivative by the derivative of the Fourier–Sobolev partial sum of f. Consequently, if the derivative of f is steep, then the quality of the derivative of the Fourier–Sobolev partial sum of f in the standard L^2 norm deteriorates, just like what happens in the case of Fourier partial sums (see [8] or Figures 1 and 2 below).
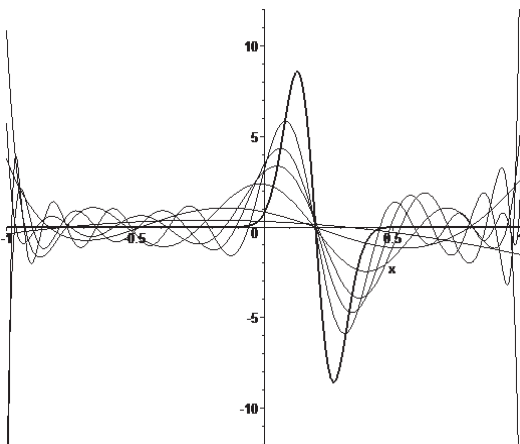

Figure 1. Graphics of f' (bold) and of the derivatives of the partial sums of degrees $n = 3, 6, 9, 12, 15, 18$ of the Fourier–Legendre expansion of f.
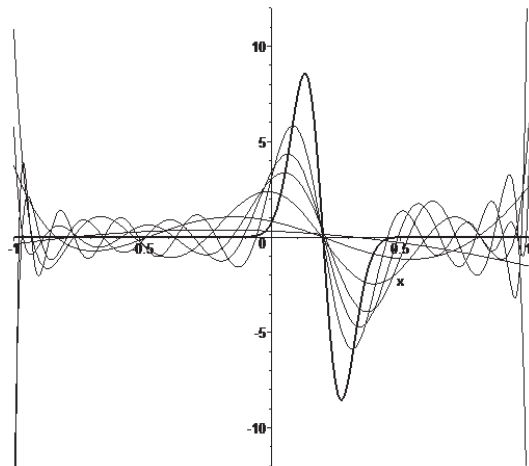
Figure 2. Graphics of f' (bold) and of the derivatives of the partial sums of degrees $n = 3, 6, 9, 12, 15, 18$ of the Legendre–Sobolev Fourier expansion of f.

However, it is very remarkable that the above situation changes when the Sobolev inner product considered is different from (1.12). For instance, if we consider the Sobolev inner products associated to the (M, N)-coherent pair of measures [4, 8], then the quality of the derivative of the Fourier–Sobolev partial sum of f in the standard L^2 norm is better than the quality of the derivative of the Fourier partial sum of f in the same norm.

Let $f : (-1, 1) \to \mathbb{R}$ be the function considered in [8, Section 4], defined by

$$f(x) := e^{-100\left(x - \frac{1}{5}\right)^2}, \quad x \in (-1, 1),$$

and similarly, let $g : (-1, 1) \to \mathbb{R}$ be defined by

$$g(x) := e^{-10\left(x - \frac{1}{2}\right)^2}, \quad x \in (-1, 1).$$

Obviously, $f, g \in W^{1,2}\left((-1, 1), (1, 1 - x^2)\right)$. Tables 1 and 2 display the approximation errors $\|f - S_n(f)\|_{L^2}$, $\|f - \mathcal{S}_n(f)\|_{W^{1,2}}$, $\|g - S_n(g)\|_{L^2}$, and $\|g - \mathcal{S}_n(g)\|_{W^{1,2}}$, respectively, when the degree of the corresponding partial sum is $n = 20, 50, 70, 100$.

Table 1. Comparison of errors for $f(x) = e^{-100\left(x - \frac{1}{5}\right)^2}$.

$f(x)$	n	$\|f - S_n(f)\|_{L^2}^2$	$\|f - \mathcal{S}_n(f)\|_{W^{1,2}}^2$
$e^{-100\left(x - \frac{1}{5}\right)^2}$	20	0.0429888570	6.913578532
	50	$0.2910e - 7$	$0.947433e - 3$
	70	$-0.10e - 9$	$0.17e - 7$
	100	$-0.5e - 10$	$-0.4e - 8$

Table 2. Comparison of errors for $g(x) = e^{-10\left(x - \frac{1}{2}\right)^2}$.

$g(x)$	n	$\|g - S_n(g)\|_{L^2}^2$	$\|g - \mathcal{S}_n(g)\|_{W^{1,2}}^2$
$e^{-10\left(x - \frac{1}{2}\right)^2}$	20	$-0.3049602377e - 2$	4.169434664
	50	$-0.3947474069e28$	$0.3921865313e33$
	70	$-0.1956184312e52$	$0.1582890113e58$
	100	$-0.1966428327e92$	$0.9631073012e97$

Finally, the following figures display the approximants of f' corresponding to both the derivatives of the partial sums of the Fourier–Legendre expansion of f and the derivatives of the partial sums of the Legendre–Sobolev Fourier expansion of f, respectively. It is evident that these approximants are very similar and, therefore, both the Fourier–Legendre partial sums and the Legendre–Sobolev Fourier partial sums are poor near the end-points of the interval $(-1, 1)$. For the function g, a similar result is obtained.

Acknowledgments

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